[redacted]

TREASURY DEPARTMENT

U. S. COAST AND GEODETIC SURVEY

W. W. DUFFIELD

SUPERINTENDENT

PHYSICAL HYDROGRAPHY

MANUAL OF TIDES

PART I

By ROLLIN A. HARRIS

APPENDIX No. 8—REPORT FOR 1897

WASHINGTON
GOVERNMENT PRINTING OFFICE
1898

TREASURY DEPARTMENT

U. S. COAST AND GEODETIC SURVEY

W. W. DUFFIELD

SUPERINTENDENT

PHYSICAL HYDROGRAPHY

MANUAL OF TIDES

PART I

By ROLLIN A. HARRIS

APPENDIX No. 8—REPORT FOR 1897

WASHINGTON

GOVERNMENT PRINTING OFFICE

1898

APPENDIX NO. 8—1897.

MANUAL OF TIDES.

Part I.

INTRODUCTION AND HISTORICAL TREATMENT OF THE SUBJECT.

By ROLLIN A. HARRIS.

Submitted for publication November 15, 1897.

PREFACE TO PART I.

When the plan of this Manual of Tides was proposed it was considered best to prepare Part III in advance of Parts I and II. The reasons for this are stated in the preface to Part III, which appeared in the Report for 1894, where a brief outline of the several parts may be found.

Before attempting to point out the contributions of individuals to the subject of the tides, it has seemed best to give, as an introduction: (1) the definitions of terms of common occurrence; (2) a clear idea concerning the movements of fluid particles in simple wave motion; (3) a popular account of the cause of the tides; (4) the general properties of tides, and tidal inequalities together with means of ascertaining them.

In the chapters which then follow no attempt is made to include the histories of those sciences (e. g., astronomy and hydrodynamics) with which all study of the tides is closely connected, nor is it even attempted to give anything like a catalogue of tidal workers, or a full account of their works; but these chapters aim to give, in a nearly chronological order, some account of such results, work, or theories as may seem worthy of notice, generally because they mark some advance in the development of the subject, but sometimes either because they illustrate certain errors into which individuals have fallen or simply because they show the notions which have been entertained in the past.

As a rule direct quotations are taken in preference to comments whenever they seem to serve the purpose in hand.

A vague rule for deciding how far to describe or carry out the work of those individuals who have made extensive or profound investigations in connection with the tides, has been to either omit or to barely state such portions of their work as will probably be resumed or at least referred to in subsequent parts of this manual. But those portions of their work which will probably not recur, or not recur unchanged in form, have been given or described in greater detail, especially if they are well known or useful.

Concerning the work of Thomson and Darwin, it may be said that a large portion of it will of necessity appear as we proceed. When this is not the case, reference will be made at proper times. For this reason it has seemed unwise to here attempt any comprehensive or minute account of their labors.

CONTENTS OF PART I.

Chapter I.

DEFINITIONS.

Chapter II.

DIGRESSION ON PLANE, OR TWO-DIMENSIONAL, WATER WAVES.

Chapter III.

ON THE ORIGIN OF TIDES.

Chapter IV.

GENERAL PROPERTIES OF TIDES AND MODES OF REDUCTION.

CHAPTER V.

TIDAL WORK AND KNOWLEDGE BEFORE THE TIME OF NEWTON.

CHAPTER VI.

NEWTON TO LAPLACE.

CHAPTER VII.

LAPLACE.

CHAPTER VIII.

WORK SINCE THE TIME OF LAPLACE.

APPENDIX NO. 8—1897.

MANUAL OF TIDES—PART I. INTRODUCTION AND HISTORICAL TREATMENT OF THE SUBJECT.

By Rollin A. Harris.

CHAPTER I.

DEFINITIONS.

1. The principal movements of the sea may be divided into three classes: *Ordinary* or *wind waves*, *tidal movements*, and *ocean currents*. The essential feature of any tidal movement is, as the name implies, its periodicity. The period may not be of constant length, but if variable it must follow some conceivable law. In conformity with this notion the word *tide* may be defined as the periodic rising and falling of oceanic and other large bodies of water, due mainly to the attraction of the moon and sun as the earth rotates upon its axis. This rising and falling is accompanied by and depends upon a lateral or horizontal movement of the waters; such movements are called *tidal currents*. Their periodic character distinguishes them from *ocean currents*. Remarkable stages of the water level at a given place, whether due to earthquakes, gales, or other causes which probably have no definite law of recurrence, although popularly known as "tidal waves," can not be regarded as belonging to tidal phenomena. On the other hand, the stages of a river, if periodic in their nature, may with propriety be included in its tides.

2. The tide rises until it reaches a maximum height called *high water*, and then falls until it reaches a minimum height called *low water*. These two phases of the tide may be spoken of as the *tides*. The difference between a high and a low water is called a *range* of tide, and so is independent of absolute heights; its average value is called the *mean range* (Mn). For a few minutes before and after high or low water, it is difficult to observe any vertical motion in the tide. While thus apparently stationary, the tide is said to *stand*. In this connection see § 15.

For reasons to be given later, based upon the fact that the tides are due chiefly to the difference between the moon's attraction upon the enveloping sea and the earth as a whole, one would expect that at most tidal stations two high waters and two low waters would occur each lunar day; in other words, to each transit of the moon (inferior as well as superior) there would correspond one high water and one low water. On an average, the time of high water at a given station follows the time of transit by a certain number of hours and minutes, called the *high-water interval* (HWI), or *high-water lunitidal interval*, or *corrected establishment*. In like manner the *low-water interval* (LWI), or *low-water lunitidal interval*, indicates, unless otherwise specified, the average number of hours and minutes between the time of transit and the time of low water. If intervals at some particular time are meant they should be properly distinguished by name or otherwise; or the average values may, for distinction, have the word *mean* prefixed to their name. The *establishment* or *vulgar establishment* is the (apparent local) time of high water occurring at new or full moon; or, preferably, and what is about the same thing, the high-water interval when the transit (just preceding the tide) occurs at noon or midnight.*

*Cf. Lalande, Astronomie (1771–1781), Vol. IV., pp. 43, 314–319; Whewell, Phil. Trans., 1833, pp. 163, 229, 230; Lubbock, ibid., pp. 19–21; Raper, Navigation (1840), p. 261; Darwin, Admiralty Manual, 5th ed., p. 55; Wharton, Hydrographic Surveying (1882), pp. 145, 149.

An *inequality* in interval, range, or height is a systematic departure of the same from its mean value. The extreme amount of this regular departure is sometimes called the *coefficient of the inequality.*

3. Not long after new or full moon, the tidal effect of the sun is added to that of the moon. When this effect upon the range is greatest the tides are called *spring tides* (*marées de vive eau*). At any given place the *retard*, or interval between new or full moon and spring tides, may be regarded as constant. Not long after the moon is in quadrature, the tidal effect of the sun is taken away from that of the moon, and when the range becomes a minimum from this cause *neap tides* (*marées de morte eau*) occur.* Their retard at a given place may be regarded as constant, and it does not differ much from the retard of the spring tides, unless the water is shallow, in which case the retard (spring or neap), as derived from the high waters alone, will differ from that derived from the low waters.

The inequality, or apparent irregularity, in time or height introduced by the sun, and so dependent upon the moon's phase, is variously styled the *semimenstrual*, *semimensual*, *semimonthly*, or *phase inequality;* the last seems preferable for most purposes, because there are several kinds of month in common use, especially in tidal work, and the word "phase" suggests a connection with the age of the moon.†

When the sun's tidal effect shortens the lunitidal intervals, causing the tides to occur earlier than usual, there is said to be a *priming* of the tide; when, from the same cause, the interval is larger than usual, there is said to be a *lagging.*

A *tidal day* is the variable interval ($24^h\ 50^m$ on an average) between two alternate high or low waters. A more accurate definition is the interval between the mean of four consecutive tides and the mean of the succeeding or preceding group of four consecutive tides.

The amount by which the tidal day exceeds $24^h\ 50^m$ is sometimes called the "lagging of the tide," and the amount which it falls short the "priming."

The amount by which corresponding tides grow later day by day (i. e., the amount by which the tidal day exceeds $24^h\ 00^m$) may be called the *daily retardation.*‡

The retard, especially spring and more especially spring high-water, has been called the *age of the tide.*§ If this term is to be retained, it seems desirable to suppose the age to have one value, and that such as to suit the neap as well as the spring tides, the low waters as well as the high. Moreover, instead of "age of the tide," the expression "age of the phase inequality" will generally be used in what follows. It will subsequently appear that, for deep water at least, the lunitidal interval of such tides as happen to occur as many hours after syzygy as represent the age of the phase inequality, have their mean values. In other words, the spring and neap intervals are about equal to the mean intervals. Because of this fact the times of such tides as give mean intervals may be used in determining the age. Experience has shown that the ages as determined from heights or ranges do not agree with those determined from times (intervals).|| For this reason it seems best, wherever possible, to define it by the value obtained from the harmonic constants of the place and explained in Part III.¶

---

* The spring and neap ranges are conveniently denoted by the symbols Sg, Np. Their connection with Mn is shown by the approximate expressions (83), (84), Part III. When μ_2 has a large shallow-water part (2MS), as at Liverpool, so that $2\,M^\circ_2 - S^\circ_2 - \mu^\circ_2$ is not near zero, it may be worth while to replace μ_2 in these expressions by $\mu_2 \cos (2\,M^\circ_2 - S^\circ_2 - \mu^\circ_2)$.

† There are some objections to the term "phase inequality" inasmuch as particular portions or aspects of the tide, such as high water, low water, or an intermediate time, may be referred to as its phases. Again, according to established usage, the phase of a wave, or oscillation, is the angle upon which the displacements depend; e. g., the phase of the harmonic oscillation

$$y = C \cos (ct + \gamma)$$

is the angle $ct + \gamma$.

‡ Cf. Laplace, Méc. Cél., Bk. IV, §§ 35 et seq.

§ Cf. Whewell, Darwin, Wharton, loc. cit.; Ferrel, United States Coast Survey Report, 1875, p. 209.

|| Airy Tides and Waves, Arts. 541-547; Phil. Trans., 1843, pp. 53, 54. Ferrel, United States Coast Survey Report, 1868, pp. 55, 75, 76; Tidal Researches, pp. 174-199; United States Coast Survey Report, 1875, pp. 209-212.

¶ Age of phase inequality expressed in hours $= 0{\cdot}984\,(S_2^\circ - M_2^\circ)$. Laplace shows that (for Brest) the age obtained from heights near the syzygies ($=1{\cdot}51349$ days) is very nearly equal to the age similarly determined from heights

Since successive transits of the moon occur on an average $12^h\ 25^m$ apart, the age can be approximately expressed by stating the number of the transit preceding the tide to which the lunitidal intervals are to be applied.* The effect of selecting an earlier transit is to increase the lunitidal interval by $12^h\ 25^m$. Of course, by adapting the transits to a suitable terrestrial meridian, any age can be allowed for.

Another way of reckoning the age is by the hour of the moon's transit. The time of transit increases on an average 50^m daily, so that if the transit used for spring tides occur at $0^h\ 50^m$, such transit follows syzygy by 24^h. But the tide follows the transit by the lunitidal interval;

$$\therefore \text{age} = \frac{24 \times 60}{50} \begin{bmatrix} \text{hour of transit for} \\ \text{maximum range} \end{bmatrix} + \tfrac{1}{2}\,(\text{HWI} + \text{LWI}), \qquad (1).$$

Whenever the "hour of transit" exceeds 12^h, 12^h must be rejected. The same formula is adapted to neap tide, by replacing the word "maximum" by the word "minimum," and always discarding 6^h or 18^h from the "hour of transit."†

To infer the age from the time when the interval has its mean value, replace "maximum" range by "mean lunitidal interval," $= \frac{1}{2}$ (HWI + LWI).

Some writers prefer to increase the age or retard, as defined above, by the high-water interval, because of the fanciful notion that they thereby obtain the interval between the transit of the moon and the appearance of the resulting high water.

4. Other things being equal, the range of tide becomes a maximum soon after the moon is in perigee and a minimum soon after she is in apogee. At these times *perigean* and *apogean* tides occur.‡ The amount by which these effects follow their respective causes may be called the *age of the parallax inequality*. Like the age of the phase inequality it may be defined in terms of the harmonic constants.§

If this age be approximately allowed for by selecting a proper transit, the lunitidal interval will, so far as this inequality is concerned, remain nearly constant throughout its period, which is an anomalistic month.

If, however, the intervals be distributed under two arguments, the moon's phase and her parallax or anomaly, the departures from the average values of the intervals will depend upon both arguments. The phase inequality being known for a mean value of the moon's parallax, the tabular values just described give, when diminished thereby, the parallax inequality in interval arranged under two arguments. Even when thus distributed, the parallax inequality in time is small; but the parallax inequality in height is of considerable importance.‖ If a wrong age of the parallax inequality be taken (i. e., if the tides be referred to a wrong transit so far as this inequality is concerned), the inequality in interval will become greater.¶ If the tides are not classified with respect to the moon's phase (i. e., if they are classified with respect to parallax or anomaly only), the value of the parallax inequality in time will, as already stated, be small if the transit used corresponds well with the age of the parallax inequality.**

5. In a similar way the effect of the moon's declination or longitude may be considered. Soon

---

near the quadratures (1.51116): Méc. Cél., Bk. XIII, § 7. Ferrel's constants make the age from heights 1·42 days and from intervals 1·87: United States Coast Survey Report, 1875, pp. 209–212. The harmonic constants make the age 1·63 days.

* Lubbock, Treatise on Tides, pp. 25–29, or Phil. Trans., 1837, p. 97.

† Because of the moon's variation, 50^m should be replaced by 51^m for spring tides and by 49^m for neap tides; but, as both spring and neap tides can generally be used in determining the age, this becomes unnecessary.

‡ The perigean and apogean ranges of tide are conveniently denoted by Pn, An.

§ Age of parallex inequality $= 1{\cdot}837\ (M_2^{\circ} - N_2^{\circ})$ hours.

‖ E. g., Lubbock, Phil. Trans., 1836, pp. 58, 59; 1837, pp. 119, 133. Ferrel, United States Coast Survey Report, 1868, p. 69.

¶ E. g., Lubbock, Phil. Trans., 1834, pp. 144, 163; 1835, p. 286.

** E. g., Ferrel, United States Coast Survey Report, 1868, pp. 60, 76, especially the low-water intervals. Here the implied (parallax) age is about 50^h for the high waters and 57^h for the low waters. The harmonic constants give 58^h for the age of the parallax inequality.

Ibid., 1875, p. 196. Here the implied (parallax) age for the high waters is 8^h and for the low waters $14\frac{1}{2}^h$. The harmonic constants give 33^h. As might be expected, the tabulated inequality (in time or interval) is somewhat greater than in the preceding instance.

after the moon is upon the equator the greatest semidaily range of tide will occur, other things being equal, and soon after the moon's extreme declination, the smallest. If a transit be selected which corresponds well with the age of this inequality, the intervals will be, as in the case of the parallax inequality, little affected by its presence. There is one difference, however, and that is, the sun's declination has a direct effect upon the lunitidal intervals, even if the proper transit has been selected. The period of the sun's declinational inequality in the tides is, in the long run, the same as that of the moon's, viz., a half tropic month, and so the two can not be separated in the treatment of a long series of observations all distributed under one argument—the moon's declination or longitude.* This combined effect may be styled the *declinational inequality.* Its age is pretty nearly equal to that of the phase inequality.

6. Other irregularities in the motions of the moon and sun give rise to corresponding apparent irregularities or inequalities in the tides. Among these may be mentioned one depending upon the longitude of the moon's node, one upon the moon's evection, and one upon the sun's anomaly.†

In ascertaining how the tide of a given day is disturbed by the inequalities, care should be taken to observe whether or not one inequality is involved in another. For instance, if the moon's anomaly is the argument for the parallax inequality, the tabular values (if derived from a sufficiently long series of observations) will be free from the inequality due to the evection, and this latter may be tabulated and used as an independent correction. If, however, the moon's *parallax* be taken as an argument, the inequality due to evection, i. e., having the evectional period, must be small in comparison with its former value, because it is, for the most part, tabulated under the argument "parallax." So if we use the moon's longitude as an argument, the declinational inequality (semidiurnal) will, in the long run, be free from the inequality due to the regression of the lunar node. If, however, the declination of the moon, instead of the longitude, be used as an argument, the nodal inequality will be nearly allowed for.

7. *Diurnal inequality in height* is the difference in height between two consecutive high waters or low waters.‡

Diurnal inequality in time or interval is the difference in length of two consecutive high water intervals or low water intervals.

At most places the high water inequality (in height or time) differs from the low water inequality.

If the greater height inequality be in the high waters, the greater time inequality will be in the low waters, and conversely.

Wherever both of the height inequalities are small in comparison with the (semidaily) range of tide, the inequalities in time (interval) are very small.

These inequalities vary in value throughout a half tropical month, and also a half tropical year.

The portion due to the moon may be computed and tabulated under the argument of the moon's declination (as it was at a time anterior to the time required, determined by the *age of the diurnal inequality*§). The portion due to the sun may be tabulated with the two arguments, the day of the year and the hour of the moon's (upper) transit reckoned from 0 to 24.‖ The combined effect of moon and sun may be made to follow the moon's declination and the day of the year. If this inequality be comparable in size with the phase inequality, the two should be tabulated together, thus necessitating a table of three arguments, the two just mentioned and the hour of transit.

If, however, we disregard the variation in the obliquity of the lunar orbit to the plane of the equator, two arguments suffice for the combined effect, viz., the hour of transit and the day of the year; for these two then infer the moon's right ascension and so her longitude or declination. The number of days from the moon's zero or extreme declination is preferable to the day of the

* E. g., Ferrel, United States Coast Survey Report, 1868, pp. 60, 78; Ibid., 1875, p. 197.

† E. g., Ferrel, United States Coast Survey Report, 1868, pp. 79–82.

‡ At places where the phase inequality is large in comparison with the diurnal, it becomes necessary to compare a given high water, say, with the mean of the immediately preceding and following high waters in order to put in evidence the high water diurnal inequality; similarly for the low water inequality.

§ E. g., Ferrel, United States Coast Survey Report, 1868, p. 97.

‖ Ibid., pp. 100, 101.

year as an argument. But for the variation in the obliquity of the lunar orbit to the plane of the earth's equator, very good predictions could be made from tables having these two arguments.*

The *diurnal wave* is that portion of the tide whose period is approximately one day. Its range varies throughout the half tropical month and half tropical year. The maximum value of this range may be regarded as occurring, in the long run, a constant number of hours (viz., the age of the diurnal inequality) after the moon reaches her extreme fortnightly declination; at such times *tropic tides*† are said to occur, because for most places the moon is then near one of the tropics.‡ The age of the diurnal inequality is such that if the times of zero declination be increased thereby, the range of the diurnal wave will be a minimum. This age, like the ages of other inequalities, may be expressed in terms of the harmonic constants.§

The diurnal inequality is due to the presence of the diurnal wave. At the time of the tropic tides, the diurnal inequality (time or height) may be spoken of as tropic. The inequality in high water heights is then denoted by HWQ and in low water LWQ. The larger one then has its maximum value very nearly; so has the quantity $\sqrt{\mathrm{HWQ}^2+\mathrm{LWQ}^2}$, which is an approximate expression for the tropic range of the diurnal wave, and with greater reason. At places where HWQ, say, is several times smaller than LWQ, the high water inequality when tropic tides occur may not have, even approximately, its maximum value.

Of the four ranges of tide upon a day when tropic tides occur, the greatest is called the *great tropic* (Gc) and the least the *small tropic* (Sc).

The mean range from all four tropic tides is the *mean tropic range* (Mc).‖

The *great [diurnal] range* (Gt) is the difference between the mean of all the higher high waters (HHW) and the mean of all the lower low waters (LLW) of each day during one or more half tropical months.

The *small [diurnal] range* (Sl) is the difference between the mean of all the lower high waters (LHW) and the mean of all the higher low waters (HLW) of each day during one or more half tropical months.

It is sometimes convenient to distinguish between the four ranges, which at most stations occur upon any given tidal day, by means of the following terms: The *great range*, the *small range*, the *high range*, and the *low range*. (See Fig. 1.) At stations where the tide is diurnal there are but two ranges each tidal day, the great and the small.

The great tropic range and the lunitidal intervals connected with it can be observed even if the tide becomes wholly diurnal in its character. So with the great diurnal.

The *sequence of tide* is the order in which the four tides of a day occur, particularly the tropic tides. It may be expressed thus, "higher high to lower low," or "lower low to higher high," as the case may be. The $\genfrac{}{}{0pt}{}{\text{former}}{\text{latter}}$ expression indicates that (tropic) lower low water $\genfrac{}{}{0pt}{}{\text{follows}}{\text{precedes}}$ (tropic) higher high water without the lesser tides intervening. The time between (tropic) $\genfrac{}{}{0pt}{}{\text{higher high}}{\text{lower low}}$ water and (tropic) $\genfrac{}{}{0pt}{}{\text{lower low}}{\text{higher high}}$ water must be taken as less than a half lunar day. At places where HWQ and LWQ are very unequal, the sequence, even of the tropic tides, may be different for different seasons of the year.

The *type of tide* is its characteristic form. It is generally indicated by the sequence of tides, the ratios of the tropic diurnal inequalities, and of the spring range, to the mean range. For shallow waters, however, in rivers especially, the duration of rise or fall may become very important.

---

* E. g., Lubbock, Phil. Trans., 1836, pp. 65–73; 1837, pp. 109–118, 126–130. Bache, United States Coast Survey Reports, 1854–1864, "Tide tables for the use of navigators."

† Cf. Airy, Phil. Trans., 1845, pp. 44–46, where approximate values of the tropic semirange of the diurnal wave are given on the coasts of Ireland. The word *tropic* was officially adopted by the Coast and Geodetic Survey, Dec. 19, 1894.

‡ The coasts of Europe form an exception, the age being from 2 to 6 or more days. Whewell, Phil. Trans., 1837, p. 81.

§ Age of diurnal inequality expressed in hours $= 0{\cdot}911\ (\mathrm{K_1}^\circ - \mathrm{O_1}^\circ)$.

‖ It is equal to $\frac{1}{2}(\mathrm{Gc}+\mathrm{Sc})$, and is somewhat less than Mn, the relation being $\mathrm{Mc}=\mathrm{Mn}-2\mathrm{K_2}\cos[(\mathrm{K_1}^\circ-\mathrm{O_1}^\circ)\sim(\mathrm{K_2}^\circ-\mathrm{M_2}^\circ)]$. Expression (89), Part III, includes semidiurnal constituents only, and so is not exactly equal to $\frac{1}{2}$Mc; in strictness, the cosine factor should there also be appended to $\mathrm{K_2}$.

See eq. (74), Part III

$$+\frac{(K_1+O_1)^2 d_1^2 - kc}{2M_2\ m_2^2}$$

Fig. 1 illustrates the tropic tides and quantities connected with them at San Francisco. In this case the tide is largely diurnal, the sequence is HHW to LLW, and LWQ > HWQ.

8. The *tide curve* or *marigram* is a curve whose abscissæ increase uniformly with the time, and whose ordinates represent the heights of the tide or sea at the corresponding times.

The average value of the ordinates of the tide curve (reckoned from some fixed mark upon the shore) defines the height known as *mean sea level* or *mean water level.*

The average value of the heights of high and low waters (reckoned from some fixed mark upon the shore) defines the height known as *half-tide level.*

Mean sea level and half-tide level do not differ much from each other except at places where the duration of fall differs by a considerable amount from the duration of rise, or where the diurnal inequality is large.*

Mean sea level (MSL) is the most nearly fixed, and therefore the best, of all planes defined by the tide. *Planes used in reducing soundings or in reckoning elevations above the sea should always have a known relation to mean sea level, or at least to half-tide level* (HTL).

The plane of *average* or *mean low water* is one-half Mn below half-tide level. The soundings

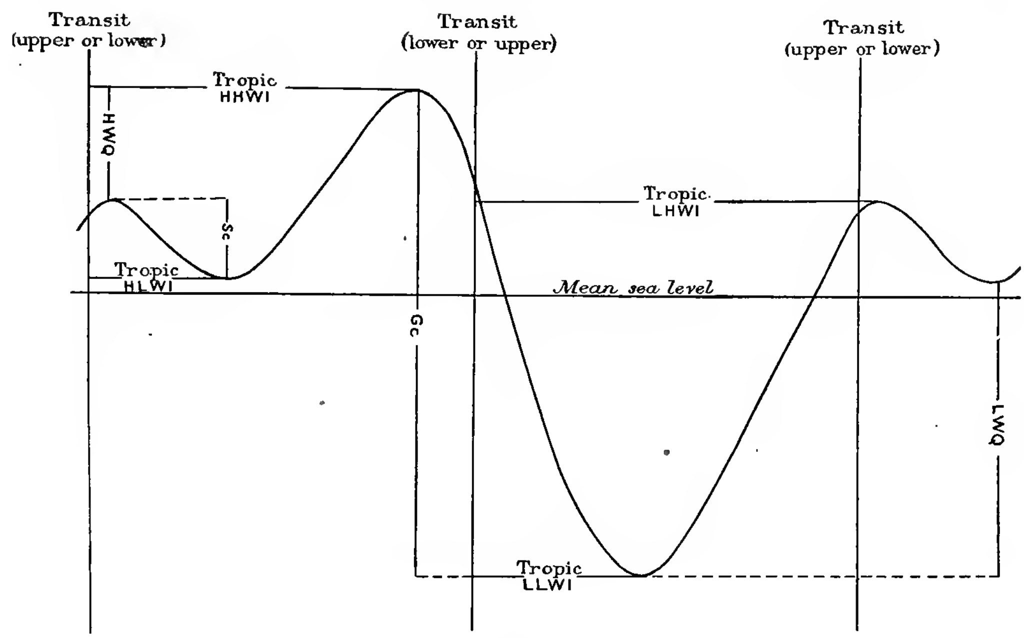


Fig. 1.

on the Coast Survey charts of the Atlantic coast of the United States are reduced to this datum.

The plane of *mean low-water* (ordinary) *springs* is about one-half Sg below half-tide level or mean sea level. This is the plane generally used along the outer coasts of Europe. The soundings upon the charts of the French coast are reduced to the lowest tides observed.

The plane of *average lower low water* or *average daily low water* is used generally upon the Pacific coast of the United States as well as the Gulf of Mexico.

The *Indian* (*harmonic*) *tide plane* or *Indian spring low-water mark* is $M_2 + S_2 + K_1 + O_1$ below mean sea level. These symbols are defined in the next paragraph.

The plane of *equinoctial low-water springs* is the datum sometimes used by the British Admiralty in Indian waters.

In many localities the datum of soundings is an arbitrary, but known, distance below a fixed bench mark. In the establishment of any such plane the hydrographer usually aims at some

* Part III, § 24.

$$\mathrm{HTL} = \mathrm{MSL} + M_4 \cos\left(2M_2^\circ - M_4^\circ\right) - 0{\cdot}04\,\frac{(K_1+O_1)^2}{M_2}\cos\left(M_2^\circ - K_1^\circ - O_1^\circ\right)$$

plane which is capable of definition with respect to the tide; e. g., mean low water, low-water springs, etc.

In a few cases the plane of reference is the height of the lowest observed tides. Of course a datum of this kind cannot be recovered unless through an established bench mark.

Since all datum planes must be connected with mean sea level, it might be advisable to reduce soundings to a plane an integral number of feet below this level, the same number to be used over a considerable area, and to be determined by the lowest tides likely to occur. Such areas or regions could be indicated upon the charts.

Some hydrographers have used the expression "low-water springs" to denote extremely low low waters due to various inequalities in the tide. Such careless usage should be discouraged.

Heights on the land when accompanied by the expression "above tide" or "*A. T.*," usually refer to high water. This is objectionable, because the height of high water depends upon the range of the tide, and this in turn upon the locality of the tidal observations. Above mean sea or mean water level is a much less objectionable signification.

In connection with tidal planes, see § 20, Part II, and §§ 42–45, Part III.

9. *Tidal constants* are certain intervals (angles) and heights (amplitudes) used in describing the tide; they are absolutely constant, or nearly so, at a given place.

Nonharmonic constants are those tidal constants which refer in some way to high and low water instead of to the constituent periodic elements into which, as will be shown in Part II, the tidal wave may be resolved. *Harmonic constants* refer to these periodic elements.

The portion of the tide following any period strictly, can, by Fourier's theorem, be analyzed into one or more simple cosine terms whose angles or arguments (which are proportional to time) go through 360°, and multiples thereof, in the given periodic time. Either the process or the result is spoken of as an *harmonic analysis.** Each such term is called an *harmonic component*, *component tide*, *partial tide*, *simple tide*, or simply a *component*. The (uniform) hourly change in the angle of any component is called its *speed;* the value of its angle reckoned from its high water at any given instant is called its *phase;* its (constant) semirange is called its *amplitude;* the (constant) angular retard of the maximum of any component C behind its astronomical cause or fictitious moon (as assigned by the uncorrected equilibrium theory in Part II) is its *epoch* or *lag*. The amplitude of C will be denoted by C, the epoch by C°, and the speed by c. If C have a period of approximately one day or twenty-four hours, it is said to be diurnal and is written C_1; if it have a period of approximately twelve hours, it is said to be semidiurnal and is written C_2, and so on. At most places the semidiurnal components are so much larger than the diurnals, quarter diurnals, etc., that the tide curve of any particular day approximates toward a sine (or cosine) curve whose period is about twelve lunar hours.

In the analysis of tidal currents C and C° may be replaced by $\dot{C}$ and $\dot{C}^\circ$, $\dot{C}$ denoting the amplitude of the component velocity and $\dot{C}^\circ/c$ being the interval between the transit of the fictitious moon (Table 3) and maximum velocity.

The *principal lunar component*, denoted by M_2, has an hourly speed of 28°·984 1042, and so its period is a half lunar day.

The *principal solar component*, denoted by S_2, has an hourly speed of 30°·000 0000, and so its period is a half solar day.

The *luni-solar diurnal component*, denoted by K_1, has an hourly speed of 15°·0410 686, and so its period is a sidereal day.

The *principal lunar diurnal component*, denoted O_1, has an hourly speed of 13°·943 0356, and so its period exceeds the lunar day by the same amount as the period of K_1 falls short of it.

For a more extended list of components see §§ 15–18, Part II, and Tables 1, 36.

*For analyzing the quality or timbre of a given note, Helmholtz made use of a series of spherical shells or resonators whose periods of vibration were fixed and known. In this way he could pick out the overtones which were present in the note sounded. The object of the harmonic analysis of a series of heights, tabulated and summed according to a given component time, is quite analogous to that of the analysis of a musical note. The harmonic analyzer to be described in Part II may be likened to Rudolph Koenig's combination of resonators (Jamin, Cours de Physique, 4th ed., Vol. III, p. 175; or Ann. de Pogg., Vol. 122 (1868), pp. 666 et seq.), while the tide predictor (Part III) may be likened to the sirens of Seebeck and Koenig (Jamin, Vol. III, p. 172).

Example showing how one simple wave is displaced by another.—The height of the surface of the sea from mean level due to the two components A, B is

$$y = A \cos (at + \alpha) + B \cos (bt + \beta) \tag{2}$$

Here the amplitudes are A, B, the speeds a, b, and the initial phases α, β.

If $a = b$, the resultant wave is harmonic, having as its amplitude

$$\sqrt{A^2 + B^2 + 2\,AB \cos (\alpha \sim \beta)} \tag{3}$$

That is, if we form a parallelogram analogous to the parallelogram of forces regarding α, β as giving the directions of A, B, the resultant amplitude is the diagonal of the parallelogram, setting out from the intersection of A and B. Or, it is the third side of a triangle whose opposite angle is $180° - (\alpha \sim \beta)$.

The phase of the resultant wave is the angle whose tangent is

$$\frac{A \sin \alpha + B \sin \beta}{A \cos \alpha + B \cos \beta} \tag{4}$$

The resultant wave may, therefore, be written

$$\sqrt{A^2 + B^2 + 2\,AB \cos (\alpha \sim \beta)} \cos \left[\left(at + \tan^{-1} \left(\frac{A \sin \alpha + B \sin \beta}{A \cos \alpha + B \cos \beta} \right) \right] \right. \tag{5}$$

In the parallelogram construction just referred to, this angle is the direction of the resultant diagonal. If α (or β) $= 0$, the above angle is the angle between the resultant and A (or B). In fact, it is then the angle adjacent to A (or B) of the triangle referred to above.

10. A *cotidal line* is an assemblage of points on the earth's surface where tides occur at the same absolute time. The number of each such line is usually taken as the lunar time (i. e., the lunar hour after upper or lower transit) at Greenwich when high water occurs at stations along the cotidal line. If solar hours are used—reckoned, of course, from the time of the moon's transit—each period of cotidal lines will consist of 12.42 hour-lines instead of 12. The cotidal lunar hour of a place whose west longitude in time is L is

$$0{\cdot}966 \text{ HWI} + L,^* \tag{6}$$

while the cotidal solar hour is

$$\text{HWI} + 1.035\ L.^\dagger \tag{7}$$

If Greenwich transits be used in making a "first reduction," § 51, the interval so obtained $+ S$, the longitude of the time meridian expressed in time, is the cotidal solar hour. If the meridian over which the moon is assumed to pass have a west longitude in time equal to E, then L must, in all cases, be replaced by $L - E$.

If instead of HWI, we write the vulgar establishment or lunitidal interval at full and change, we have the cotidal lines for full and change ‡ and not for spring tides or for tides of mean lunitidal interval. On the other hand, the retard of the spring tides is not the same the world over, and so the cotidal line

$$0{\cdot}966 \text{ HWI} + L$$

does not represent a series of points along which it is simultaneously and exactly high-water springs; in fact, such lines do not exist.

For limited areas, lines of equal lunitidal interval may be drawn instead of cotidal lines. This amounts to making $L = 0$ in (6) and (7).§

* Cf. Whewell, Phil. Trans. 1836, p. 293 and chart opp. p. 306. Bache, United States Coast Survey Reports: 1854, p. 149 and sketch 26; 1855, p. 339 and sketch 49; 1862, p. 127 and sketch 46.

† Cf. A. S. Christie, The Lady Franklin Bay Expedition, Vol. II, pp. 697 et seq.

‡ E. g., Whewell, Phil. Trans. 1833, pp. 148, 149; 1848, p. 7. Airy, Tides and Waves, plate 6; reproduced in Enc. Brit., Art. "Tides." Haughton, Manual of Tides and Tidal Currents (1870), Plate IV. Berghaus, Physikalischer Atlas (1892). Probably most astronomies and physical geographies adopt this system.

§ E. g., Schott, United States Coast Survey Report, 1854, p. 173, sketch 16.

Intervals referring to the diurnal wave can be used in a similar way for obtaining cotidal lines referring to the diurnal portion of the tide.*

If observations were sufficiently extensive it would be possible to draw a set of cotidal lines for each harmonic component of the tide. Accordingly, a cotidal line for the component A_i is an assemblage of points on the earth's surface where the A_i tide occurs at the same absolute time. Such lines are naturally numbered in A_1 hours after the transit of the A_i fictitious moon across some fixed meridian, as that of Greenwich; but they may be numbered in B_1 hours after the transit of the A_i moon.

The cotidal A hour for the component A is

$$\frac{A_i{}^\circ}{i\,15} + L, \tag{8}$$

while the cotidal B hour for the component A is

$$\frac{b}{a}\left(\frac{A_i{}^\circ}{i\,15} + L\right). \tag{9}$$

The principal tidal components are M_2, S_2, K_1, and O_1. If $A_i = M_2$,† the cotidal *lunar* hour for M_2 is

$$\frac{M_2{}^\circ}{30} + L \tag{10}$$

which is, in deep water, nearly equal to

$$0{\cdot}966\ \mathrm{HWI} + L.$$

If $A_i = S_2$, the cotidal *solar* hour for S_2 is

$$\frac{S_2{}^\circ}{30} + L. \tag{11}$$

If $A_i = K_1$,‡ the cotidal *sidereal* hour for K_1 is

$$\frac{K_1{}^\circ}{15} + L. \tag{12}$$

11. *Tide tables* are ephemeral publications, usually covering a calendar year, showing in tabular form the predicted or computed times and heights of the high and low waters.

For certain principal or typical stations such predictions are given in full, i. e., all tides of the year are predicted; but for most places tidal *differences* and *ratios* are given, which enable the user to obtain his tides from the tides at stations having full predictions.

The following are the principal tide tables:

"Tide Tables by the United States Coast and Geodetic Survey." This publication covers quite thoroughly the coasts of the United States and less thoroughly the world at large.

"Tide Tables for the British and Irish Ports," containing "also the times and heights of high water at full and change for the principal places on the globe," by the British Admiralty.

"Tide tables for the Indian Ports," by authority of the Secretary of State for India in Council.

"Annuaire des Marées des Côtes de France," by the French hydrographic service.

"Gezeitentafeln," by the German admiralty. These include daily predictions for several stations in addition to those of the German coast; also intervals and ranges for the world at large.

* E. g., Bache, United States Coast Survey Report, 1862, p. 127 and sketch 46. Cf. Part III, § 56.

† E. g., Van der Stok, Studiën over Getijden in den Indischen Archipel, XII (1895), p. 23 and Kaart I. The longitude, L, is here reckoned from Batavia.

‡ E. g., ibid., p. 31 and Kaart II.

12. The *velocity* (*drift*) of a current is the rate at which the fluid particles move horizontally. It is usually expressed in knots, i. e., nautical miles per hour, but sometimes in feet per second.* The velocity generally differs for different depths, but its value at the surface may be understood unless otherwise specified. The velocity of propagation of the tidal wave is many times greater than the velocity of the current, and the two must not be confounded.

The *direction* (*set*) of a current is the direction or point of the compass toward which the fluid particles move.

The movement of the fluid in one direction, usually inland, is styled *flood*, and in the opposite direction, *ebb*. The two are not always distinct, and, even if they are, it is not always possible to know which movement should be taken for the flood and which for the ebb. Flow or flood and ebb correspond to the French *flot* and *jusant*, while rise and fall correspond to *montant* or *gagnant* and *perdant*.

The maximum of the flood or ebb current is sometimes called the *strength of flood* or *ebb*.

The effect of the tidal wave in giving rise to currents is obvious in two extreme cases:

(1) Where there is a small tidal basin connected with the sea by a large opening.

(2) Where there is a large tidal basin connected with the sea by a very small opening.

In the first case the velocity of the current in the opening will have its maximum value when the tide or height of sea is changing most rapidly, i. e., at a time about midway between high and low water. In other words, the water level in the basin keeps at about the same level as the surface of the water outside. Flood corresponds to the rising and ebb to the falling tide.

In the second case the velocity of the current in the opening will have its maximum value when it is high water or low water without; for then there is the greatest head of water for producing motion. Flood begins about three hours after low water and ebb begins about three hours after high water, and so slack water occurs at times about midway between the tides.

Many currents in nature lie, in a general way, between these two extreme cases; but see §22.

Slack water denotes the state of the current when its velocity becomes a minimum. It follows high- or low-water stand by intervals ranging from zero to three hours, depending upon the locality.† *Change* or *turn of tide* are expressions sometimes used instead of "slack water."

The velocity and direction of tidal currents are much modified by extremely local causes, while the times and heights of the tides are about the same over considerable areas.

13. *Representation of currents, etc.*

The velocity and direction of a current at any given time are often indicated by an arrow. Usually the arrow indicates direction only, the velocity being written just beyond the point. If the currents corresponding to several phases of the tide, or rather tidal current, are shown upon one sheet, several arrows will usually radiate from the same point upon the map. Their numbers indicate the order of occurrence.‡ A *current station* is a point where currents have been observed. Unless the station happens to be in the channel, it is obvious that the rising tide will generally reach and swell the water in the channel before its effect is felt at the station. Similarly the falling tide begins to lower in the channel earlier than in the shallower regions. Hence the order, in time, for the pointing of the radiating arrows is—

Shoreward, upstream, offshore, downstream.

This is evidently clockwise for stations upon the right-hand bank (looking downstream) and counter clockwise for stations upon the left.

For an instantaneous representation of the condition of the currents in a given harbor or region it is customary to make use of *lines of flow*. A line of flow is such a line that at all of its points the motions of the fluid particles coincide with it. In other words, if we draw at each point of the fluid a very short arrow, whose direction indicates the direction of the current at the given instant, then a curve, coinciding with a series of them, is a line of flow.§ At any given

---

*To change velocities given in feet per second to knots per hour, multiply by $\frac{45}{76}=0·5921=\frac{1}{1·689}$; to statute miles, multiply by $\frac{15}{22}$.

† *Étales de flot et de jusant* vs. *étales de pleine mer et de basse mer*.

‡ For examples see Coast Survey charts, also the Reports; for instance 1879, opp. p. 175 and p. 181.

§ A set of charts for the Irish Sea and English Channel, by Beechy (q. v.), is given in the Philosophical Transactions for 1848.

instant the motion of surface of the water will be represented by a system of curves covering it. If the motion be *steady*, i. e., independent of the time, lines of flow are usually known as *stream lines.*

If across the lines of flow we draw a line cutting the system everywhere at right angles, the line is a *line of equal velocity potential*, or an *equipotential line.* We may assign to one such line any number we please; but having done so, the numbers belonging to other such lines become fixed. Suppose the numbers upon two adjacent lines of equal velocity potential to differ by a small constant quantity $d\phi$. The distance apart of the two lines (ds) becomes known when the velocity is known (and conversely) through the equation

$$ds = -d\phi \div v,$$

or

$$\frac{d\phi}{ds} = -v. \tag{13}$$

Lines of equipotential form a system of curves cutting the system of lines of flow everywhere at right angles.

When the motion is steady and the body of water uniform in depth, the two systems are not only *orthogonal*, but also *isothermal;* that is, if we construct stream lines and equipotential lines, as above directed, we can select stream lines as far apart as are the equipotential lines in the same vicinity, and so divide the whole surface of the water into elementary squares. If the real part of a function of a complex variable represents one of these systems, the purely imaginary part will represent the other. Such a function is defined by the boundary conditions; that is, the function must be such that the stream lines coincide with the fixed boundaries of the fluid.*

Another pair of systems which might be used to represent the motion of the water are *lines of equal velocity* and *lines of equal bearing.* All along a line of equal velocity, the velocity of a current is constant; all along a line of equal bearing the direction or set of the current is constant. These two sets of curves often intersect orthogonally or nearly so. For uniform depth and steady motion the systems are also isothermal.

The map of a body of water may have drawn upon it a series of lines, along any particular one of which the current turns at a given lunar hour. Such lines are called *cocurrent lines;* they are quite analogous to cotidal lines, and, like them, admit of numerous varieties.†

If we are concerned with the current at one station only, the velocities may conveniently be taken as the (positive) ordinates of a curve, the times being the abscissæ. The directions may be written at the feet of the ordinates. At a station where the flood and ebb are distinct, the velocities of the one may be taken as positive and the other negative. This representation is quite analogous to the tidal sheet or marigram.

14. For explaining the origin, propagation, and properties of the tide wave, numerous *theories* are required, according to the circumstances. When the explanation is less complete, or when observation is called in to supplement certain of its defects, the underlying and so-called theory is really only a *working hypothesis.* However, since nobody can hope for theories covering all cases, and since the same theory at one place may serve to explain the tides, while at another place it can serve only as a working hypothesis, we shall follow usage and make the word "theory" cover the expression "working hypothesis."

The (uncorrected) *equilibrium* or *statical* theory assumes that at each instant the surface of the sea is a prolate elipsoid whose longest diameter points at the tidal body. This hypothetical surface is often defined as being one which is everywhere normal to the direction of gravity as perturbed by the tidal body alone.

If the ocean covered the entire earth, the effect upon the direction of gravity of the layer of water constituting the hypothetical tide could be computed. The eccentricity of the equilibrium spheroid, and so the range of tide, would then be somewhat increased.

---

* Bache, United States Coast Survey Report, 1851, pp. 136, 137, and Sketch A, No. 3; the latter represents the currents of Boston Harbor by making the distance between the lines (of flow) inversely proportional to the velocity at the given point. In case of steady motion and uniform depth such lines would be continuous.

† Schott, United States Coast Survey Report, 1854, pp. 168–179, discusses the currents of Long Island Sound. His cocurrent lines are really lines of equal lunicurrent interval. His "lines of direction" differ from lines of flow in that they are taken not exactly simultaneously but at the time of greatest velocity.

The *corrected equilibrium theory* differs from the former in assuming the earth not wholly covered by water, and so the surface of even a deep sea cannot actually coincide with the spheroid of the uncorrected theory, but will be parallel to it the distance therefrom at any given point varying with time. This is necessitated by the incompressible property of water.

The difference between these two theories can be illustrated by means of a small body of water, a lake, or even a pail of water. The uncorrected theory implies that the whole surface of the small body of water rises and falls by the same amount, twice each lunar day. Moreover, this range of tide would be the same as the range of tide in the same latitude were the surface of the earth covered by an ocean. The corrected theory involves the fact that, on the whole, the surface of the small body of water always has the same height, and so its tides are caused by the water at one side being slightly elevated while in another portion it is slightly depressed. But the surface is normal to disturbed gravity, or parallel to the uncorrected tidal spheroid. The cotidal lines will radiate from a *no-tide point* instead of being arcs of terrestrial meridians as in the case of the uncorrected theory.

The equilibrium theories assume that the water surface arranges itself in each locality normal to the force of disturbed gravity, but they do not explain how this arrangement is made, nor whether it is possible with the known properties of water. They avoid altogether the question of depth of the water, the surface alone being considered.

Laplace attempted a theory in which not only the disturbing or tidal forces, but also the motion of the water regarded as a heavy or inert body, are taken into account. Such theories are known as *dynamic* or *kinetic*. They should take account of the viscosity or internal friction of the water as well as the friction at the bounding surfaces.

The *wave theory* considers the tide as a wave and develops the properties of such motions. From the nature of the case it is a kinetic theory.

The uncorrected equilibrium theory is useful as a working hypothesis in tidal analysis because it enables one to infer suitable forms of expression for the tidal disturbances, knowing the laws of the forces to which they are due. The principle that the disturbances are coperiodic with the forces, whether the tide approach its equilibrium condition or not, is a deduction of dynamics.

The corrected equilibrium theory applies to small, deep bodies of water.

The kinetic theory enables one to infer that the amplitude of tidal oscillations having sensibly equal periods are to one another as are their forces, and that their epochs are equal.

The wave theory is applicable to canals or tidal rivers.

MISCELLANEOUS TIDAL PHENOMENA.

15. In shallow estuaries where the range of tide is considerable, the high water is propagated inward faster than the low water, for at high water the greater depth prevails. The high water thus gaining upon the low water causes the duration of rise of tide to become shorter as the wave proceeds; and so the farther the wave goes without breaking, the more abrupt its front becomes. Finally, it becomes so steep that the top of the wave falls forward (not in the middle of the stream but near the shelving shores) something like the crest of a breaker. This phenomenon, usually accompanied by much noise, is called a *bore*. Other names for the bore, boor, or boar's head are *eager* (England), *mascaret* or *barre* (France), *prororoca* or *pororoca* (Brazil). The following rivers and arms of the sea have bores: The Amazon,* Tsien-tang,† Brahmaputra, Ganges, Hooghly, Indus, Garonne, Dordogne, Seine, Trent, Severn, and Wye rivers, Solway Frith; arms or bays at the head of the Bay of Fundy, and perhaps Magellan Strait and Cook Inlet.‡

An *agger* is a double-headed tide; that is, a tide having two maxima or two minima instead of the usual high or low water.§ This gives to the tide a long "stand," and so may be of much practical value. At Southampton there is a double high water, at Portland a double low water,

* J. C. Branner, Pop. Sci. Monthly, Vol. 38 (1890), pp. 208–215.

† See figure 19.

‡ For tidal diagrams of French rivers, see Comoy, Étude Pratique sur les Marées Fluviales.

See Airy, Tides and Waves, Art. 514; also this manual under Alexander the Great, Strabo, Hakluyt, and Sturmy.

§ Cf. Airy, Tides and Waves, Art. 518, and Ferrel, Tidal Researches, § 254.

and at Havre an almost double high water.* This peculiarity of the tide does not generally persist throughout the lunation. It is usually, but not always, the most pronounced at spring tide.

At some places the high and low waters may be very sharp, thus making a stand of short duration. The high waters at Ipswich and the low waters at Philadelphia may be mentioned as cases of this kind.†

Whenever the tidal water has to pass through a rather narrow or shallow channel to fill a tidal reservoir beyond, a strong current is necessitated. This is sometimes called a *race*; e. g., the Race at the eastern entrance to Long Island Sound. Of course each of the two bodies of water connected by the strait may be tided; e. g., the Hell Gate, Messina Strait. But a tidal race is more properly defined as a strong current caused by the meeting of two wave systems from different or opposite directions. The effect will be the greatest where the range of tide is most diminished by this meeting; that is, at a place where trough meets crest, as can be seen by a brief study of the water particles in wave motion (Chapter II). The Portland Race, the Maelstrom, and Seymour Narrows may be instanced.

16. *Seiches* (sāsh) are short-period oscillations (usually from about 10 to 60 minutes) existing at times in many (if not all) lakes and landlocked bays. They represent oscillations in which usually the whole body of the liquid swings to and fro. They are caused by sudden changes of atmospheric pressure, or winds which sweep over its surface. The period of such a seiche is

$$\frac{\text{twice length of lake}}{\sqrt{gh}} \tag{14}$$

where g denotes the acceleration due to gravity, and h the depth of the lake or bay.

Seiches may not always be uninodal, as supposed above, nor does the nodal line always run transversely to the body of water.

This phenomenon has been observed on Lakes Geneva, Constance, Ontario, Michigan, Cazenovia (N. Y.), and others; on bays in India; at Swansea (Wales), Malta (Greece), and Bristol (R. I.).‡

The effect of an earthquake upon the sea is known as an *earthquake wave*, an *earthquake sea* (or *ocean*) *wave*, or a *great sea wave*. The last term serves to distinguish it from the corresponding oscillation in the solid earth which is known as a *great earth wave*. The great sea wave may sometimes, perhaps, be due to a tumbling down of submarine cliffs instead of to an earthquake proper.

The effect of these waves is often transmitted to distant shores, where it is recognized (although not always with absolute certainty) by the peculiar oscillations which it adds to the record of an automatic or self-registering tide gauge. Peru, Japan, and Malay Archipelago have furnished notable instances of this phenomenon.

---

*See figure 18.

For tidal diagrams, Phil. Trans., 1843, opp. p. 46, and Comoy, op. cit.

To ascertain from the harmonic constants the natures of the high and low waters at a given place, draw a curve, as in § 63, Part III, consisting of M_2 and its harmonics M_4, M_6, M_8, ...

In computing the mean range of tide the second portion of formula (65), Part III, should not, perhaps, be used, but rather a value obtained from the drawing just mentioned.

†Phil. Trans., 1843, opp. p. 52. See figure 11.

‡The following are a few references to this phenomenon:

C. B. Comstock, Annual Report of the Survey of the Northern and Northwestern Lakes, 1872, pp. 14–16 and Pl. VI.

Airy, Phil. Trans., 1878, pp. 136–138.

Günther, Geophysik, Vol. II (1885), pp. 373–376.

Nature, Vol. 14 (1876), p. 164; Vol. 17 (1878), pp. 234, 281; Vol. 18 (1878), pp. 100, 101; Vol. 19 (1879), p. 446; Vol. 21 (1880), pp. 397, 443; Vol. 33 (1885), p. 184.

Science, Vol. 7 (1886), p. 412; Vol. 15 (1890), pp. 99, 117.

CHAPTER II.

DIGRESSION ON PLANE, OR TWO-DIMENSIONAL, WATER WAVES.

17. *Fundamental equations.*

A more exhaustive account of fluid motion will be given in Part IV. In the present chapter the assumptions made are few and simple. The main object is to give an introduction to the study of wave motion, which shall clearly indicate how the water particles behave according to theory, and which shall also show some applications of the results obtained to the water movements in nature.

By taking the displacement equations (26), (27), and the equation (28) for granted, several of the paragraphs on wave motion can be understood without reading the present paragraph.

In any motion of a fluid, the entire volume taken into consideration must not be altered. That is, if we assume any small mass of the fluid bounded by an imaginary surface to be slightly displaced in the motion, its volume will remain as before; or if we assume an imaginary surface fixed in the fluid and inclosing a small mass or volume of it, the amount contained in this surface will be constant, whatever the motion of the fluid, provided only that the surface remain entirely submerged.

We shall assume that all motions take place in or parallel to the vertical plane xy, and, for convenience, that the thickness (z) of the body of water treated is unity. Then, considering an imaginary rectangular boundary whose edges are dx, dy, in length, and letting u, v denote velocities along x, y, the difference between the entire quantity flowing into and out of this boundary which is supposed to be stationary is, obviously,

$$\left(u + \frac{\partial u}{\partial x}\,dx\right)dy - u\,dy + \left(v + \frac{\partial v}{\partial y}\,dy\right)dx - v\,dx.$$

This is equal to zero, because as much flows in as out;

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (15)$$

is the equation of continuity.

In using this equation it is to be remarked that x, y are the true coördinates of the particle, whereas in the work about to be given x, y are the coordinates of the particle when in its undisturbed condition. The true coordinates are $x + \mathbf{x}$, $y + \mathbf{y}$.

To find the equation of continuity in terms of small displacements $\mathbf{x}, \mathbf{y}$ instead of u, v, assume that the elementary rectangle whose corners had originally the coordinates

$$0,\,0 \quad dx,\,0 \quad dx,\,dy \quad 0,\,dy$$

becomes so altered by the motion that the coordinates of the corners are

$$\begin{cases}\mathbf{x}\\ \mathbf{y}\end{cases} \quad \begin{cases}\mathbf{x} + \frac{\partial \mathbf{x}}{\partial x}dx + dx\\ \mathbf{y} + \frac{\partial \mathbf{y}}{\partial x}dx\end{cases} \quad \begin{cases}\mathbf{x} + \frac{\partial \mathbf{x}}{\partial x}dx + \frac{\partial \mathbf{x}}{\partial y}dy + dx\\ \mathbf{y} + \frac{\partial \mathbf{y}}{\partial x}dx + \frac{\partial \mathbf{y}}{\partial y}dy + dy\end{cases} \quad \begin{cases}\mathbf{x} + \frac{\partial \mathbf{x}}{\partial y}dy\\ \mathbf{y} + \frac{\partial \mathbf{y}}{\partial y}dy + dy.\end{cases}$$

Let the small change among neighboring particles be such that the elementary rectangle becomes a parallelogram whose sides are approximately parallel to those of the rectangle; its area is approximately equal to

$$\left(1 + \frac{\partial \mathbf{x}}{\partial x}\right)\left(1 + \frac{\partial \mathbf{y}}{\partial y}\right)dxdy.$$

Since this must be equal to $dxdy$, it follows that

$$\left(1+\frac{\partial \mathbf{x}}{\partial y}\right)\left(1+\frac{\partial \mathbf{y}}{\partial y}\right)=1, \tag{16}$$

or

$$\frac{\partial \mathbf{x}}{\partial x}+\frac{\partial \mathbf{y}}{\partial y}=0, \tag{17}$$

provided $\frac{\partial \mathbf{x}}{\partial x}\cdot\frac{\partial \mathbf{y}}{\partial y}$ may be neglected, as is the case when each factor is small. Either (16) or (17) is the equation of continuity, the former being, of course, the more accurate;

$$\therefore \quad \mathbf{y}=-\int_{y=b}^{y=y}\frac{\partial \mathbf{x}}{\partial x}dy+\text{a function of } x, \tag{18}$$

where b denotes the height of the bottom. If Ξ denotes the value of the displacement $\mathbf{x}$ at the bottom, the corresponding value of $\mathbf{y}$ is

$$\mathbf{y}=\Xi\frac{db}{dx}; \tag{19}$$

and since at the bottom where $y=b$ the integral is zero, it follows that this is the required function of x;

$$\therefore \quad \mathbf{y}=-\int_{y=b}^{y=y}\frac{\partial \mathbf{x}}{\partial x}dy+\Xi\frac{db}{dx}. \tag{20}$$

The last term becomes zero for a horizontal bottom.

If the motion be such that all particles once in a vertical line always remain so, $\mathbf{x}$ can be replaced by ξ, which is its surface value, and we may take as elementary area a rectangle whose length is $dx+\frac{\partial\xi}{\partial x}dx$ and whose height is $h+\eta$, a much larger quantity. The area must remain $h\,dx$.

$$\therefore \left(1+\frac{d\xi}{dx}\right)\left(1+\frac{\eta}{h}\right)=1 \tag{21}$$

is the equation of continuity, in this case.

The dynamical or pressure equation is

$$\frac{\partial^2\mathbf{x}}{\partial t^2}=X+\frac{\partial}{\partial x}\left\{-g\eta-\int_{y=y}^{y=h}\frac{\partial^2\mathbf{y}}{\partial t^2}dy\right\}, \tag{22}$$

in which η denotes the value of $\mathbf{y}$ where $y=h$, the undisturbed depth, and X denotes the intensity of any impressed force acting in the x-direction.

Since x does not vary with t, the value of $\frac{\partial^2(x+\mathbf{x})}{\partial t^2}$ is $\frac{\partial^2\mathbf{x}}{\partial t^2}$, which is the acceleration (or effective force per unit mass due to the horizontal motion) in the x-direction. This must be the result or the equivalent of the x-component of all other forces connected with the motion.

$g\eta$ is the disturbing pressure due to height reckoned from the undisturbed surface, and so the partial x-derivative of $-g\eta$ is the corresponding accelerating force in the x-direction.

$\frac{\partial^2\mathbf{y}}{\partial t^2}dy$ is, since weight or mass is proportional to dy, an element of the pressure due to the vertical velocity of an elementary mass above the point x, y. The aggregate pressure is the same integrated up to the surface, and the corresponding accelerating force is minus the partial

x-derivative. In this integration it is allowable to take the upper limit as h, instead of the slightly different value $h+\eta$, because the vertical acceleration $\frac{\partial^2 \mathbf{y}}{\partial t^2}$ is assumed to be a moderately small quantity.*

If $X=0$, the motion of the body is "free," not "forced;" i. e., the body is left to itself.

If the vertical acceleration can be omitted, the water must so move as to keep all particles which lie in a given vertical line, always in a vertical line. If the area be divided into elementary vertical strips of length dx and height $h+\eta$, the elementary volume of water, $dx \times (h+\eta) \times 1$, varies with the instantaneous height; and so the force equivalent to the effective force in the moving element must likewise vary,

$$\therefore \frac{\partial^2 \mathbf{x}}{\partial t^2} = \frac{h+\eta}{h}\left(X - g\frac{\partial \eta}{\partial x}\right) \tag{23}$$

making use of the corresponding equation of continuity and putting $X=0$, we have

$$\frac{\partial^2 \mathbf{x}}{\partial t^2} = gh \frac{\frac{\partial^2 \mathbf{x}}{\partial x^2}}{\left(1+\frac{\partial \mathbf{x}}{\partial x}\right)^3}\,; \tag{24}$$

which becomes, if $\frac{\delta \mathbf{x}}{\partial x}$ is small, or if the relative displacement of two neighboring elements of the fluid is small in comparison with the distance between them,

$$\frac{\partial^2 \mathbf{x}}{\partial t^2} = gh \frac{\partial^2 \mathbf{x}}{\partial x^2}\,. \tag{25}$$

WAVES IN A CANAL OF UNIFORM DEPTH AND INDEFINITE LENGTH.

18. It is here proposed to give an interpretation of the wave motion defined by the following equations, and to point out how the long or tidal wave differs from the short, oscillatory, or surface wave.

Let us assume that the horizontal and vertical displacements of the fluid are of the respective forms, †

$$\mathbf{x} = A \cosh ly \sin (at - lx + \alpha) \tag{26}$$
$$\mathbf{y} = A \sinh ly \cos (at - lx + \alpha), \tag{27}$$

where A, a, α, l are constant throughout the canal and for all time; x, y are independent of the time but vary from point to point, x being measured horizontally from an arbitrary origin, and y vertically from the bottom of the canal. These evidently satisfy the equation of continuity (17); they also satisfy the dynamical equation (22) provided

$$\tanh lh = \frac{a^2}{gl}, \tag{28}$$

or

$$a^2 = gl \tanh lh. \tag{29}$$

Equations (17), (22) imply that $\mathbf{x}$ and $\mathbf{y}$ are small in comparison with the wave's length and the depth of the water, respectively. The motion defined by (26), (27) is periodic in time and distance. Any increase of time, accompanied by a proper increase of distance, leaves $\mathbf{x}, \mathbf{y}$ unaltered, showing that the wave motion represented advances uniformly along x increasing, the velocity being $\frac{a}{l}$. The motion represented is evidently such that similar terms involving $2at$, $3at$, etc., may be disregarded.

$$* \frac{\partial^2 \mathbf{x}}{\partial t^2} = X - \frac{1}{\rho}\frac{\partial p}{\partial x},$$

where p denotes the intensity of pressure per unit area at a given point; ρ the density of the fluid, i. e., its mass per unit volume, and which may be taken as unity.

† For definitions and numerical values of hyperbolic functions, see Table 46.

19. *Deductions from* (26), (27).

The horizontal and vertical component oscillations (displacements) of any given fluid particle are each simple harmonic functions of the time, and of like periods. Eliminating the angle involving t, we have

$$\frac{\mathbf{x}^2}{A^2\ \cosh^2\ ly}+\frac{\mathbf{y}^2}{A^2\ \sinh^2\ ly}=1, \tag{30}$$

showing that any particle whose (undisturbed) height above the bottom is y describes an ellipse whose major and minor semi-axes are A cosh ly and A sinh ly. Consequently the foci are distant $A\sqrt{\cosh^2 ly-\sinh^2\ ly},=A$, from the center of the ellipse. As this distance is independent of both x and y, it is the same for the orbit of *any* particle in the fluid mass; i. e., the two foci of any ellipse are $2A$ apart and lie in a horizontal line.

[It may be noted, although it is not important for the present purpose, that the law of description is precisely the same as that of a body revolving about a central force whose intensity increases directly with the distance of the body from the center.]

Let x be constant in equations (26), (27). Since the angle $at-lx+\alpha$ does not involve y, it is obvious that particles originally in the same vertical line are, at any given instant, in the same phase of either the vertical or the horizontal oscillation (displacement). In this respect the motion of a vertical filament of water somewhat resembles that of a stalk of wheat swaying to and fro in the wind.

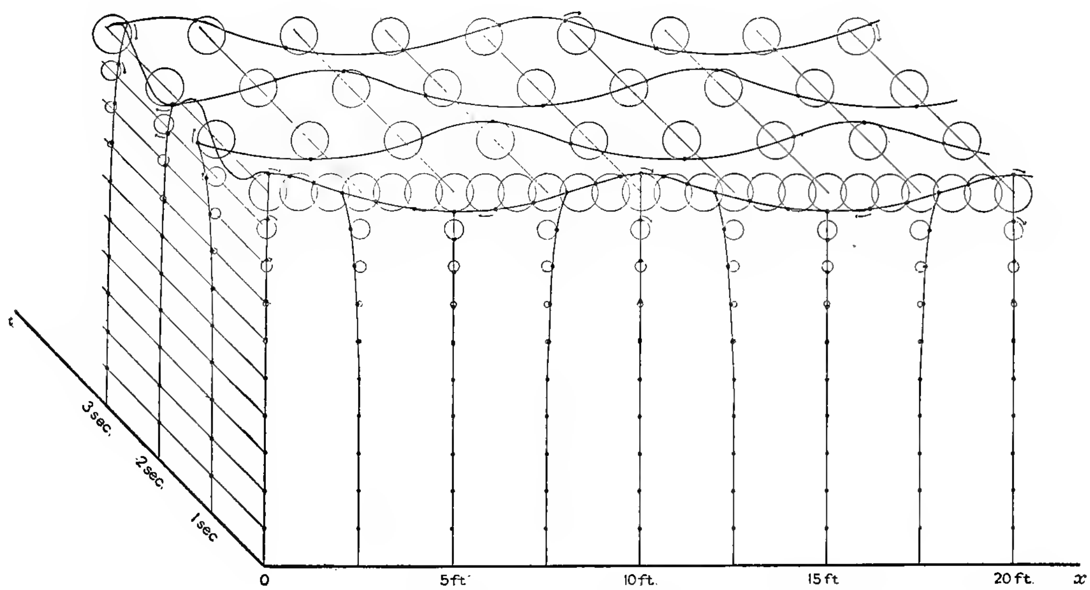


Fig. 2. For illustrating wave motion.

20. Figure 2 illustrates the wave motion in a vertical section of water, the wave being propagated in the direction Ox. This is not a view of a three dimensional volume of water, but consists of a series of instantaneous views of the same plane; the times at which the views are supposed to be taken are, as indicated upon an arbitrary time axis t, 0, 1, 2, and 3 seconds, respectively.

The orbit of any given particle is fixed; i. e., is the same for all values of t; but the particle itself occupies different positions as t varies. In other words, it describes the orbit and in the direction (clockwise) indicated by the arrow. The orbits which are sufficiently far from the bottom to be shown in the figure, are, very nearly, circles. The wavy line having its axis parallel to the t-axis is a view showing how the height of the surface (at $x=0$) changes as t varies. In other words, it is a view of what would be traced upon a self-registering apparatus at the locality $x=0$.

Of course, the scale in which time along the t-axis is reckoned is arbitrary. The instantaneous wave profiles are the wavy lines parallel to the x-axis; they approximate closely to curves known as curtate cycloids.

To obtain such a curve, let a series of circles be uniformly distributed along a horizontal line; take a point on each a constant angular distance from the position of the point on the adjacent circle to the left, say; join the points thus obtained.

In order that the cycloid be curtate, it is necessary that the common distance between the centers of the circles be greater than the arc subtending the constant angular distance just referred to.

The wavy line parallel to the t-axis differs from the wave profiles only that the abscissæ may be drawn to a different scale.

21. Let us now return to equations (26), (27). If in these two equations we assume two of the coordinates t, x, y to be constant while one is variable, equations (26), (27) are the two equations of a displacement curve, the non-constant coördinate being the variable parameter. (Or, if we eliminate this variable parameter, an equation in $\mathbf{x}$, $\mathbf{y}$ is obtained.) This locus must be such that if we proceed along the t-, x-, or y-axis, as the case may be, the successive displaced particles must fall upon it; the ($\mathbf{x}$, $\mathbf{y}$)-origin is supposed to coincide with the undisturbed position of any particle along the axis in question.

By supposing t and x constant, equations (26), (27) represent an hyperbola; or, eliminating y, we obtain the single equation

$$\frac{\mathbf{x}^2}{A^2 \sin^2 (at-lx+\alpha)} - \frac{\mathbf{y}^2}{A^2 \cos^2 (at-lx+\alpha)} = 1. \tag{31}$$

By supposing y and t constant, equations (26), (27) represent an ellipse. As x increases the ellipse is described counterclockwise.

These equations likewise represent an ellipse when y and x are constant. As t increases the ellipse is described clockwise. In either case the resultant equation is

$$\frac{\mathbf{x}^2}{A^2 \cosh^2 ly} + \frac{\mathbf{y}^2}{A^2 \sinh^2 ly} = 1. \tag{32}$$

Now regarding x (or t) as the parameter of a system of curves, equation (31) represents a system of confocal hyperbolas—the foci being $2A$ asunder. Similarly regarding y as the parameter, equation (32) represents a system of confocal ellipses. These ellipses and hyperbolas are bi-confocal and constitute a pair of orthogonal and isothermal systems. When the ellipses are circles, the hyperbolas become radiating straight lines.

To see the system of displacement ellipses (circles) in the figure, drop all orbits which are in the same vertical line to the bottom of the canal; they will then have a common center. The nearly vertical line joining any originally vertical series of particles becomes an hyperbola (radial line) cutting the ellipses (circles) at right angles.

[If we put

$$x' = at - lx + \alpha, \quad y' = ly,$$
$$z' = x' + iy', \quad Z = \mathbf{x} + i\,\mathbf{y},$$

then (26), (27), are equivalent to the single equation

$$Z = A \sin z' \tag{33}$$

But if the x' y' plane or the x y-plane be divided into a system of squares by means of lines parallel to the coördinate axes, they become in the $\mathbf{x}$ $\mathbf{y}$-plane by the transformation (33), the confocal system of ellipses and hyperbolas already described.]

22. A wave whose length is several or many times the depth of the water, is called a *long wave*. Such waves form a limiting case of wave-motion in water defined by the displacements (26) and (27). The other limiting case being that of *surface* or *short* waves, whose character is shown in Fig. 2. For a long wave, ly is a small quantity, and so $\cosh ly \doteq 1$, $\sinh ly \doteq ly$,

$$\mathbf{x} = A \sin (at - lx + \alpha), \tag{34}$$
$$\mathbf{y} = A\, ly \cos (at - lx + \alpha). \tag{35}$$

From (34) we see that, to quantities of the second order, the horizontal displacements are the same for all depths of the liquid inasmuch as y is not involved. That is, the water must move to and fro as if divided up into vertical slices. The expression for $\mathbf{y}$ shows that the vertical displacements increase as the distance from the bottom increases.

The orbits of the particles are the extremely elongated ellipses having as their equation

$$\frac{\mathbf{x}^2}{A^2}+\frac{\mathbf{y}^2}{A^2 l^2 y^2}=1. \qquad (36)$$

They have a constant major axis at all depths, but the minor is proportional to the depth taken. At the surface the amplitude (η) of the rise and fall (tide) is lh times the amplitude of horizontal displacement (current).

The *velocity* of the fluid particles is

$$\dot{\xi}=\frac{d\,\xi}{dt}=Aa\cos\,(at-lx+\alpha); \qquad (37)$$

$$\therefore\ \dot{\xi}/\eta=\frac{a}{lh},=\sqrt{\frac{g}{h}}, \qquad (38)$$

as follows from (28), (34), and (35).

Example.—When the height of the (rising) tide from mean water level is 2 feet, in a long tidal river 30 feet deep, the velocity is $2\times\sqrt{\frac{g}{h}}$ or 2·07 feet per second (flood).

The maximum flood velocity occurs at the time of high water, and the maximum ebb velocity at the time of low water. Slack water occurs at the time of mean water level. From Fig. 2 it is readily seen that the particles of water in a wave surface may, at certain portions of the wave period, be actually flowing up hill. This is one of the most obvious ways of detecting wave motion.

Experience shows that the motion of the water in tidal rivers which are not abruptly terminated, is well represented by the wave motion here considered. For in such cases reflection can alter the wave but slightly.

If two waves of like periods and moving in opposite directions be superposed, the result will depend upon the manner of the incidence. If high water falls upon high water the range of the wave will be increased, while the velocity of the current may be reduced to almost zero (see Fig. 2). For, the particles at high water in each wave move in the direction of wave propagation, and so the resultant motion is perhaps zero. If a high water fall upon a low water, the range of the wave may be almost reduced to zero while the current will have its velocity increased.

West of the Isle of Man the cotidal hour is about ten, whether the tide comes from the north or from the south. The consequence is that the velocity of the current is small.

23. a denotes the number of degrees by which the phase of the component displacements of any particle is altered in a unit of time. When the orbit of the particle is circular, a denotes its angular velocity.

$$\therefore\ \frac{360^\circ}{a}=\tau \qquad (39)$$

where τ is the periodic time of the particle or of the wave.

l denotes the number of degrees by which the phases of the component displacements of two particles differ—the centers of their orbits being unit distance apart. (See Fig. 2.)

$$\therefore\ \frac{360^\circ}{l}=\lambda,\ \text{or}\ l=\frac{360^\circ}{\lambda} \qquad (40)$$

where λ is the length of the wave (in feet). 360° should of course be replaced by 2π if we wish to reckon l in radians.

$$\therefore\ \frac{a}{l}=\frac{\lambda}{\tau}=\text{velocity of the wave.} \qquad (41)$$

This is independent of the amplitude.

From (29) we have

$$\tau^2 = \frac{2\pi\lambda}{g} \Big/ \tanh\frac{2\pi h}{\lambda}.^{*} \tag{42}$$

When $\frac{h}{\lambda} \overset{=}{>} 1$,

$$\text{Period (seconds)}, \tau, \doteq \sqrt{\frac{2\pi\lambda}{g}}, \doteq \frac{4}{9}\sqrt{\lambda},^{\dagger} \tag{43}$$

$$\text{Velocity (feet per second)}, \frac{\lambda}{\tau}, \doteq \sqrt{\frac{g\lambda}{2\pi}}, \doteq \frac{9}{4}\sqrt{\lambda}, \text{ or } \frac{g\tau}{2\pi}. \tag{44}$$

$\frac{9}{4}$ feet per second $= \frac{4}{3}$ nautical miles per hour $= 1.53$ statute miles per hour. [The period of a wave whose length is λ is $(2\pi)^{\frac{1}{2}}\sqrt{\frac{\lambda}{g}}$, while the (complete) period of a pendulum whose length is λ is $2\pi\sqrt{\frac{\lambda}{g}}$.]

When $\frac{h}{\lambda}$ is several times smaller than unity

$$\text{Period}, \tau, \doteq \frac{\lambda}{\sqrt{gh}}, \doteq 0{\cdot}176\frac{\lambda}{\sqrt{h}}. \tag{45}$$

$$\text{Velocity}, \frac{\lambda}{\tau}, \doteq \sqrt{gh}, \doteq 5{\cdot}67\sqrt{h}. \tag{46}$$

5·67 feet per second $= 3{\cdot}36$ nautical miles per hour $= 3{\cdot}87$ statute miles per hour. The equation $v = \sqrt{gh}$ is known as Lagrange's formula.

From (43) and (44) it follows that the period and velocity of short waves in deep water vary as the square root of the wave length and are independent of depth of the water.

From (46) it follows that the velocity of a wave very long compared to the depth of the water (as is the free tidal wave) varies as the square root of the depth, and is independent of the wave length.

24. *Equations of the wave profile.*

Let ν denote the number of wave lengths (not necessarily an integral number) from the origin of coördinates to the undisturbed point; then for the x we have

$$x = \nu\lambda;$$

and for the true x or the x of the disturbed point,

$$x = \nu\lambda + \mathbf{x} = \nu\lambda + A\cosh lh \sin(at - 2\pi\nu + \alpha); \tag{47}$$

also, for the true y,

$$y = h + \mathbf{y} = h + A\sinh lh \cos(at - 2\pi\nu + \alpha). \tag{48}$$

By so taking the origin that $at + \alpha = 0$ and writing θ for $2\pi\nu$, we have

$$x = \frac{\lambda}{2\pi}\left(\theta - \frac{2\pi}{\lambda}A\cosh lh \sin\theta\right), \tag{49}$$

$$y - h = \frac{\lambda}{2\pi}\left(\frac{2\pi}{\lambda}A\sinh lh \cos\theta\right). \tag{50}$$

---

* See Tables I, II, III of Airy's Tides and Waves; or, Tables 47, 48, 49 this manual.

† Cf. Newton's Principia, Bk. III, Props. 44–46.

Now $l = \frac{2\pi}{\lambda}$; and so for water deep in comparison with λ, cosh lh and sinh lh are sensibly equal ($\doteq \frac{1}{2}e^{lh}$).

For this reason we may write

$$x = \frac{\lambda}{2\pi}(\theta - m \sin \theta), \tag{51}$$

$$y - h = \frac{\lambda}{2\pi} m \cos \theta. \tag{52}$$

These are the equations of a curtate cycloid (or a trochoid) whose generating wheel, of radius $\frac{\lambda}{2\pi}$, rolls below a line distant $\frac{\lambda}{2\pi}$ above the surface of repose, or $h + \frac{\lambda}{2\pi}$ above the bottom; m is the fraction of the radius from the center to the tracing point. For curves below the surface, h expressed or implied in the above equations should be replaced by y' where y' denotes the height of any surface of repose above the bottom.

In Fig. 2, $\lambda = 10$ feet, $h = 10$ feet, A cosh $lh = 0{\cdot}5$ foot, $\frac{\lambda}{2\pi} = 1{\cdot}6$ feet, which is the radius of the generating circle, and $m = 0{\cdot}314$.

For waves which are long in comparison with the depth, cosh $lh \doteq 1$, and sinh $lh = lh$, so that the equations for the wave profile are

$$x = \frac{\lambda\theta}{2\pi} - A \sin \theta, \tag{53}$$

$$y = h + Alh \cos \theta. \tag{54}$$

Now if the amplitude of the x-displacement (A) be small in comparison with λ, these two equations represent a very flat cosine curve.

25. *Distinction between ordinary and tidal waves.*

The following characteristics are deductions from the preceding paragraph on wave motion in canals.

Short waves.	*Long or tidal waves.*
[The depth of the water is supposed to exceed the length of the wave, and the rise to be several times less than the wave length.]	[The depth of the water is supposed to exceed, by a considerable amount, the rise and fall of the tide, and the length of the wave to much exceed the depth of the water.]
Particles move in ellipses which are very nearly circles at the surface.	Particles move in ellipses approaching horizontal straight lines.
The horizontal and vertical displacements of the particles diminish rapidly below the surface.	The horizontal displacements of the particles are about the same at the bottom as at the surface; the vertical displacements are proportional to the heights of the particles above the bottom.
Particles originally in the same vertical line are, at any given instant, in the same phase of oscillation.	Particles once in the same vertical line remain so for a long time.
The wave profile approaches a curtate cycloid.	The wave profile approaches a cosine curve.
The marigram, or record of a self-registering gâuge, is a curtate cycloid or a projection of one.	The marigram, or record of a self-registering gauge, is a cosine curve.
The period or wave length assumed, the other becomes fixed, regardless of the depth of the water or the rise and fall of the surface.	Two of the quantities period, wave length, and depth of water, assumed, the remaining one becomes fixed, regardless of the rise and fall of the surface.
The velocity of propagation depends upon the wave length only.	The velocity of propagation depends upon the depth only.

Some characteristics not deduced from the preceding paragraph, but rather from observation, are added here:

Wind waves.	*Tidal waves.*
The period of a short wave at a given place depends upon the velocity, continuance, and (in limited bodies of water) direction of the wind.	The period of a tidal oscillation does not depend upon the given place, but upon the astronomical forces to which it is due.
The amount of rise and fall at a given place depends upon the velocity, continuance, and direction of the wind.*	The amount of rise and fall of the tide at a given place depends upon, or rather varies with, the direction, and intensity of the astronomical forces to which the tide is due.
Wind waves do not arise unless the velocity of the wind exceed a certain value — 0·45 miles per hour for capillary waves, 2 miles for gravity waves.†	Tidal oscillations of like periods are very nearly proportional to the disturbing causes, however small these latter may be.
The period, as well as the amount of rise and fall, may vary rapidly from place to place, as can be seen in passing around a breakwater.	The period is fixed the world over; the amount of rise and fall changes slowly from place to place.
Wind waves are much confused, and their period uncertain.	Tidal waves recur with remarkable regularity.
Wind waves are soon destroyed by the viscosity of the water.	Tidal waves move on as free waves through long distances.
Storm waves at sea (wind 30 or 40 knots) have a rise and fall of 15 or 20 feet, a period of about 10 seconds, and, by (43), a length of about 500 feet. ‡	The rise and fall of the tide at sea is, by the equilibrium theory about 1·8 feet × (cosine of latitude)^2 § and the length of the tidal wave is hundreds, or even thousands, of miles.

26. *Ordinary water waves compared with polarized light.*

Imagine the orbits of the surface particles to be not in the plane of the paper as shown in Fig. 2, but perpendicular to it, the centers of the orbits still occupying their former positions, and so lying upon the same horizontal straight line.

Suppose these orbits circular, and suppose the particles to move clockwise if we look from 0 toward $+x$, the polarization is circular and right-handed; if the particles move in the opposite direction, it is left-handed. In either case the particles will lie upon a helix or screw. The shadow of these points upon the horizontal plane, or upon a plane parallel to the plane of the paper, will represent a beam of plane polarized light. A circularly polarized beam is well illustrated by sticking large-headed pins into a wooden pencil so that their heads lie upon a helix, and then rotating the pencil uniformly upon its axis. The shadow of this shows the wave motion in plane polarized light.

When the orbits of the particles are ellipses, they will represent elliptically polarized light, while the shadows or projections represent plane polarized light.

27. *Long water waves compared with sound.*

The horizontal displacement ξ now represents the longitudinal displacement of the particles constituting the medium through which sound is propagated. Because η is proportional to $-\frac{\partial\xi}{\partial x}$, it is proportional to the variation in pressure at a given time at any given cross-section (and so to the variation in density) due to the motion. Where $\frac{\partial\eta}{\partial x}$ (or $\frac{\partial^2\xi}{\partial x^2}$) $= 0$, there is a maximum or a minimum. In water waves the values of x satisfying $\frac{\partial\eta}{\partial x} = 0$ are evidently the points of high and

* According to an item published in Van Nostrand's Eng. Mag., Vol. 24 (1881) p. 36, rise and fall in meters $= \frac{3}{4}v^{\frac{2}{3}}$, v being the velocity of the wind in meters. The rule is probably due to Coupvent Desbois. See Günther, Geophysik, Vol. II, p. 378.

† Lamb, Hydrodynamics, §§ 246, 303. Russell, Report B. A. A. S., 1837, p. 455.

‡ This length is probably too great. Russell, Report B. A. A. S., 1837, pp. 446 et seq. W. Walker, ibid., 1842 (II), pp. 21, 22. Captain Stanley, ibid., 1848, pp. 38, 39. W. Scoresby, ibid., 1850 (II), pp. 26–31. C. W. Merrifield, ibid., 1869, pp. 32, 33. Günther, Geophysik, Vol. II, p. 378. R. Abercromby, Phil. Mag., Vol. 25, 1888, pp. 263–269. Theodore Cooper, Trans. Am. Soc. of Civil Engineers, Vol. 36 (1896), pp. 139 et seq.

For velocity of propagation, see Stokes, Mathematical and Physical Papers, Vol. II, pp. 239, 240.

§ See § 47. At Honolulu, Hawaiian Islands, the mean range is 1·2 feet; at Easter Island, 2·2 feet; at St. Helena Island, 2·1 feet; at Ascension Island, 1·4 feet. But it is not to be inferred from these values that the tides in extended oceans are in any way explained by the uncorrected or corrected equilibrium hypothesis.

low water at the assumed time. To see how the horizontal projections of fluid particles are crowded together (like a condensation of a sound wave) at the high waters, and drawn apart at the low waters, we may even make use of Fig. 2. Take the front row of surface particles and project them upon the x-axis, remembering that the displacements (ξ) must be regarded as small in comparison with the length of the wave.

28. *On the reflection of plane water waves.*

The displacements in an infinite fluid are

$$\mathbf{x} = A \cosh ly \sin (at - lx + \alpha),$$
$$\mathbf{y} = A \sinh ly \cos (at - lx + \alpha).$$

Now interpose a vertical barrier where $x = L$, and the displacements of the reflected wave will be

$$\mathbf{x}_r = -A_r \cosh ly \sin [at - l(2L - x) + \alpha], \tag{55}$$
$$\mathbf{y}_r = A_r \sinh ly \cos [at - l(2L - x) + \alpha]. \tag{56}$$

That is, horizontal motions after reflection change their direction, while vertical motions do not. This can be easily seen because the reflection is really due to the horizontal motion. $\mathbf{x}_r$, $\mathbf{y}_r$ being simple harmonic displacements of the same periods as $\mathbf{x}$, $\mathbf{y}$, they combine with the latter to alter the phase of the wave at a given time and place (cf. § 9).

Let us now suppose a complete reflection to take place so that the amplitude of the reflected portion should, if conditions permitted, be as great as the amplitude of the original wave.

Let $x' = x - L$ and $\alpha' = \alpha - lL$; then

$$\mathbf{x} = A \cosh ly \sin (at - lx' + \alpha'),$$
$$\mathbf{y} = A \sinh ly \cos (at - lx' + \alpha'),$$

$$\mathbf{x}_r = - A \cosh ly \sin (at + lx' + \alpha'),$$
$$\mathbf{y}_r = A \sinh ly \cos (at + lx' + \alpha');$$

$$\therefore\ \mathbf{x} + \mathbf{x}_r = -2A \cosh ly \cos (at + \alpha') \sin lx' = -2A \cosh ly \cos (at + \alpha - lL) \sin [l(x - L)] \tag{57}$$
$$\mathbf{y} + \mathbf{y}_r = 2A \sinh ly \cos (at + \alpha') \cos lx' = 2A \sinh ly \cos (at + \alpha - lL) \cos [l(x - L)]. \tag{58}$$

29. *Wave motion propagated up a canal closed at one end.*

Now in order that the displacements just written apply to the case in hand, two conditions must be fulfilled besides the equation of continuity (17) and the dynamical equation (22).

First. Where $x = 0$,

$$\mathbf{y} = A \sinh ly \cos (at - lx + \alpha);$$

for otherwise, an abrupt change in height would take place as we pass from open water into the mouth of the canal.

Second. Where $x = L$,

$$\mathbf{x} = 0;$$

for, at the head of the canal no horizontal motion can take place.

If we write

$$\mathbf{x} = \frac{A \cosh ly}{\cos lL} \sin [l(L - x)] \cos (at + \alpha), \tag{59}$$

$$\mathbf{y} = \frac{A \sinh ly}{\cos lL} \cos [l(L - x)] \cos (at + \alpha), \tag{60}$$

all of these conditions are fulfilled.

30. *Application to long or tidal waves.*

The expressions (59) and (60) now become, since $\cosh ly \doteq 1$, $\sinh ly \doteq ly$,

$$\xi = \frac{A}{\cos lL} \sin [l(L - x)] \cos (at + \alpha), \tag{61}$$

$$\mathbf{y} = \frac{A ly}{\cos lL} \cos [l(L - x)] \cos (at + \alpha). \tag{62}$$

From these we see that for all values of x, ξ and y or η are simple harmonic functions of the time; their amplitudes, however, depend upon the value given to x. Throughout the canal the tide rises and falls simultaneously; in like manner, it ebbs and flows. Moreover, it is slack water throughout the canal at the time of high or low water. If $lL = 90°$, or any other odd multiple of $\frac{1}{4} l\lambda$, the value of η becomes very great, especially as x approaches L, while for the even multiples it becomes zero for all values of t. λ is the length of a free tidal wave in a canal not obstructed by a barrier; when expressed in angular measure, $l\lambda$ is, of course, 2π or $360°$.

The average depth of the Bay of Fundy along its axis is 40 or 50 fathoms. Table 50 gives about 800 statute miles for the corresponding λ. $\therefore \frac{1}{4}\lambda$ is about 200 miles. Now it happens that the length of the bay is about 150 miles, so that these particular dimensions may in part account for the large tides near its head.* But the wave progresses at about the rate due to depth according to Lagrange's formula, and so it is reasonable to suppose that the effect of the barrier is scarcely felt because of the gradual shoaling in the upper part of the bay.

The Gulf of Maine, whose length inward is about 200 miles and whose depth about 75 fathoms, is, by Table 50, nearly $\frac{1}{4}\lambda$ in length. Hence the stationary character of the wave and the increase in range.†

Portland Canal, forming a part of the boundary between Alaska and British America, furnishes a good illustration of a nearly stationary wave. Its width and depth are quite uniform and its termination is sudden. Simultaneous observations show that the tide at Somerville Bay is simultaneous with the tide at Halibut Bay, 30 miles farther up the canal. Also that the tide at Ford's Cave, 60 miles above Somerville Bay, is but five minutes later. Now the depth of the canal is about 125 fathoms on an average along its axis, and so the time required for a wave to be transmitted 60 miles would be about half an hour, instead of five minutes. The range of tide is nearly constant, being on an average 13 or 14 feet. For a depth of 125 fathoms λ = about 1 300 miles, Table 50.

31. *Forel's seiche period.*

Let it now be required to find the period in which a body of water, as a lake, whose length is $2L$ and whose depth is h, will swing when disturbed from its position of equilibrium by a sudden vertical force acting near either end, or a longitudinal horizontal force acting upon intermediate points.

Taking the middle point as origin, either half may be treated as a canal closed at one end, and affected at the mouth with a periodic disturbance whose period is determined by its length and depth. We have just seen that L should be $\frac{1}{4}\lambda$, in order to bring about the greatest rise and fall at the closed end. But the wave-length and depth being fixed, the periodic time becomes fixed by the equation

$$\tau = \frac{\lambda}{\sqrt{gh}}. \tag{63}$$

Replacing λ by $4L$, we have

$$\tau = \frac{4L}{\sqrt{gh}} = \frac{\text{twice length of lake.}}{\sqrt{gh}} \tag{64}$$

It is an easy matter to test this formula experimentally. Suppose we have a rectangular tray of water $2L$ inches in length and h inches in depth. Now, suddenly raise one end or otherwise disturb the equilibrium of the fluid. The free wave immediately traverses the length of the tray, returns, sets out again, and so continues to go back and forth until the equilibrium is gradually restored. Next, suppose that as soon as the wave returns to the end of the tray where it was produced a similar disturbance is repeated. The wave will this time set out increased in size. Let the slight disturbance be repeated periodically, according to the period thus determined, until finally the water simply swings, as it were, there being no progressive character of the motion to be seen. Formula (64) gives the period of the oscillation in seconds, provided we express g in inches (= 386). Of course the period will generally be altered when the depth ceases to be uniform.

Ferrel's explanation‡ of the abnormally large semidiurnal tides of the North Atlantic Ocean is based upon the fact that a tray or canal closed at both ends, extending from Europe to America, having the average depth of the ocean along the parallel of about 52° north, would have about

*Cf. Airy, Tides and Waves, Art. 506.

†Cf. Mitchell, U. S. Coast and Geodetic Survey Report, 1879, pp. 175–190.

‡Tidal Researches, pp. 237 et seq.

twelve lunar hours for its complete period of oscillation. Possibly the periods of free oscillation of certain zones of the Pacific have considerable influence upon the size of the diurnal oscillations in that ocean.

It is possible that component tides, whose periods are some fractions like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . . of a half tidal day, may be due in part to stationary oscillations of the kind just referred to; that is, they may owe their size to the length and depth of the body of water in which they occur, rather than to the water being so shallow that the range of tide is a considerable fraction of the depth.

32. Returning to equations (61), (62), we have

$$\mathbf{y} = \frac{ly}{\tan\left[l\,(L-x)\right]}\xi. \tag{65}$$

That is, the surface particles, or particles originally occupying the same horizontal line, execute simple harmonic oscillations along fixed rectilinear paths, the tangent of whose inclination to the horizontal is

$$\frac{ly}{\tan\left[l(L-x)\right]}. \tag{66}$$

Near the head of the canal this becomes

$$\frac{y}{L-x}. \tag{67}$$

Let us now suppose the displacements of particles in the same horizontal row to take place about the same point; that is, let the distance x be eliminated, then

$$\xi^2 + \frac{\mathbf{y}^2}{(ly)^2} = \left[\frac{A\ \cos\,(at+\alpha)}{\cos\,lL}\right]^2. \tag{68}$$

This shows that if the middle points of all the rectilinear paths be placed at one point, the extremities of these paths will define an ellipse. For different values of t, the size of this ellipse will vary, but its shape will be unaltered. At the time of mean sea level the ellipse becomes a point. For different values of y the major axis of the ellipse will remain unaltered, but the minor will be proportional to the depth taken.

Confining ourselves to the horizontal motion, we may liken it to the horizontal motion of the particles in an elastic body fixed at one end, that is where $x = L$, and to the other end of which a force is applied. If $L - x$ is small, the body is one of rectangular xy-section, and the displacement of the particles will be proportional to the distance from the fixed end. When the length $L - x$ is not small, the xy-section is bounded by the lines

$$\begin{aligned} x &= L \\ x &= x \\ y &= 0 \\ y &= \pm\, k\ \sec\,[l(L-x)] \end{aligned} \tag{69}$$

where k is a constant.

33. *How the range of tide may be increased when the cross section of the canal becomes smaller.*

The energy contained in a long wave can be shown to be directly proportional to its length, breadth, and the square of the amplitude of its vertical oscillation. Now, if the cross section varies so slowly that the wave is not disintegrated by reflection, and if other dissipating causes are ignored, the energy will remain constant.

$$\therefore \lambda \times b \times \eta^2 = \text{a constant} \tag{70}$$

But λ is proportional to $\sqrt{gh}$, and so

$$\eta = \frac{\text{a constant}}{b^{\frac{1}{2}}h^{\frac{1}{4}}}. \tag{71}$$

The effect of gradual shoaling and converging shore lines, is an increase in the amplitude of the tide wave.* Having once been so increased, it is possible for it to be propagated along the shore as a free wave, virtually governing the tide for considerable distances.

* See Lamb, Hydrodynamics, §§ 171, 181, 182.

SOME HYDRAULIC CONSIDERATIONS.

34. *A bay, harbor, or tidal river with but one opening.*

Let us suppose that the tide is known at a sufficient number of places to enable one to ascertain approximately the height of tide at any given place in the harbor. We are now not concerned with what takes place outside, but simply with the ever-changing tidal volume within.

When the volume is a maximum it is clearly "slack-before-ebb" at the opening or mouth of the harbor; when a minimum, "slack-before-flood."

If, in a short canal of uniform width closed at one end, the depth be such that the range and shape of the tide are constant, then it will be slack water at a given cross section when the crest or trough of the tide wave is midway between the given cross-section and the head of the canal;* for, the average depth of the water will then be a maximum or a minimum.

Supposing the cross section (F) at the mouth of a harbor to be constant, we have for the velocity

$$v = \frac{1}{F}\frac{dV}{dt} \tag{72}$$

where dV denotes the change in volume during the short interval of time dt (say ten or twenty minutes).

If the area of the harbor is the same at high as at low water, and if the rise and fall of the average surface be denoted by

$$y = A \cos (at + \arg_o A - A^\circ) + B \cos (bt + \arg_o B - B^\circ) + \quad . \quad . \quad . \quad . \quad , \tag{73}$$

then the velocity at the mouth of the harbor is evidently proportional to $\frac{dy}{dt}$, and so to

$$Aa \sin (at + \arg_o A - A^\circ) + Bb \sin (bt + \arg_o B - B^\circ) + \quad . \quad . \quad . \quad . \tag{74}$$

In other words, the amplitudes of the various current components compare among themselves, not as the amplitudes of the corresponding tidal components of the harbor, but as these latter multiplied by their respective speeds. Hence, the diurnal inequality in the current velocities is less striking than in the heights of the tide. If, for the sake of form, we write cosines in the place of sines, we must apply $\pm 90^\circ$ to the above angles.

Example.—The area of San Francisco Bay and tributaries being about 430 square miles, the width of the Golden Gate at Fort Point 1 mile, and the average (mean sea level) depth at this section 30 fathoms, required, the velocity of the current when the height of the bay is changing at the rate of $1\frac{1}{2}$ feet per hour.

Here the hourly change of volume is

$$430 \times 5280 \times 5280 \times 1\tfrac{1}{2} \text{ cubic feet,}$$

and so the change per second is about 5 000 000 cubic feet. The area of the cross-section is about 950 000 square feet.

$$\therefore v = \frac{500}{95} = 5{\cdot}3 \text{ feet per second} = 3{\cdot}1 \text{ knots.}$$

In a body of water as large as this, ranges of short duration can not conveniently be used with accuracy for estimating the hourly change in height, unless the tide is known at several points in the bay.

35. *On the steady flow of streams.*

The well-known formula due to Brahms and Chézy is

$$v = c\sqrt{RS}, \tag{75}$$

or

$$\text{velocity} = \text{empirical constant} \times \sqrt{\frac{\text{area of cross-section}}{\text{length of wetted-perimeter}} \times \frac{\text{head or fall}}{\text{length}}},$$

$$= \text{a coefficient } \sqrt{\text{hydraulic radius} \times \text{slope.}} \tag{76}$$

---

* Cf. L. d'Auria, Jour. Franklin Institute, Vol. 131 (1891), p. 267.

Experiments show that c, in a measure, depends upon the roughness of the wetted-perimeter, upon the value of R, and of S. The value of c is often round about 90, when the foot unit is used, and about 50 when the meter; but the values vary widely.* The best known of the more elaborate formulæ is the one generally styled Kutter's.† Because of the inertia of the water, it is obvious that no general formula can be consistent for various sections of a large river like the Lower Mississippi.

36. *On the flow through a small opening connecting two large bodies of water.*

Suppose we have two large tanks of water connected by a very short horizontal pipe; also suppose the difference in level $(y_m - y_n)$ of the surfaces of the fluid to remain constant. All particles in this pipe, at whatever depth it may be situated, and whatever may be its dimensions, provided only it is moderately small, should, by Torricelli's theorem, move with a velocity

$$v = \sqrt{2g\,(y_m - y_n)}. \tag{77}$$

If the dimensions of the pipe have to be taken into account, because of friction between it and the water, we have

$$v = \sqrt{\frac{2g\,(y_m - y_n)}{1 + \zeta \dfrac{L}{R}}} \tag{78}$$

where ζ is a coefficient supposed constant for a given material.

37. *Short tidal river or strait connecting two large bodies of water, one or both of which are tided.*

We shall suppose that the horizontal motions of the two bodies is so small that their influence upon the velocity of the water in the strait may be neglected. Considering only one component of the tide, the respective heights at any given time are

$$y_m = A_m \cos\,(at + \arg_0 A - A_m{}^\circ), \tag{79}$$

$$y_n = A_n \cos\,(at + \arg_0 A - A_n); \tag{80}$$

$$\therefore\ y_m - y_n = [A_m \cos A_m{}^\circ - A_n \cos A_n{}^\circ] \cos\,(at + \arg_0 A)$$

$$+\ [A_m \sin A_m - A_n \sin A_n{}^\circ] \sin\,(at + \arg_0 A),$$

or

$$= \sqrt{A_m{}^2 + A_n{}^2 - 2A_m A_n \cos\,(A_m{}^\circ \sim A_n{}^\circ)}\, \cos\,(at + \arg_0 A + \delta), \tag{81}$$

where

$$\tan \delta = -\frac{A_m \sin A_m{}^\circ - A_n \sin A_n{}^\circ}{A_m \cos A_m{}^\circ - A_n \cos A_n{}^\circ},$$

showing that the difference in level of the two surfaces is a simple harmonic function of the time. Supposing the motion steady for a limited time, the horizontal velocity in the strait should be, at a given point, proportional to

$$\sqrt{y_m - y_n}.$$

Now it can be shown by § 58, Part II, that

$$\pm\ \sqrt{|\sin \theta|} = 1{\cdot}112 \sin \theta + 0{\cdot}155 \sin 3\,\theta + 0{\cdot}066 \sin 5\,\theta + \ .\ .\ .\ .\ , \tag{82}$$

$$\pm\ \sqrt{|\cos \theta|} = 1{\cdot}112 \cos \theta - 0{\cdot}1\,15 \cos 3\,\theta + 0{\cdot}066 \cos 5\,\theta - \ .\ .\ .\ . \tag{83}$$

consequently the velocity of ebb and flow is not a simple harmonic function, although the tide in either body of water rises and falls according to such law; that is, there are terms whose periods are $\frac{1}{3}$, $\frac{1}{5}$, . . . part of the period of the fundamental. Their effects upon the current curve, is to give it a less pointed appearance than a curve of sines, i. e., to render it more like a semicircle. An illustration of such a condition is to be found in the East River, which connects New York Bay with Long Island Sound. Diagrams of the currents in this river, off Twenty-third street, New York, are given upon pp. 423, 425 of the Report of the United States Coast and Geodetic Survey for 1886.

* For numerical values under various conditions, see Hering and Trautwine's translation of Ganguillet and Kutter's work, A General Formula for the Uniform Flow of Water in Rivers and in Other Channels, pp. 39, 160–223, 233–236; also Church, Mechanics of Engineering, pp. 758–761. The symbols R, S, and c are here special or temporary notation.

† Ganguillet and Kutter, loc. cit., pp. 24 et seq., p. 129. Church, loc. cit., p. 759.

Example.—Given, at Governors Island, $M_2 = 2{\cdot}1$ feet, $M_2{}^\circ = 231^\circ$; at Willets Point, $M_2 = 3{\cdot}6$, $M_2{}^\circ = 330^\circ$; required, the times of slack water and of maximum velocities in East River.

Reckoning from the time of transit, we have, for the difference in level,

$$y_m - y_n = 2{\cdot}1 \cos (m_2t - 231^\circ) - 3{\cdot}6 \cos (m_2t - 330^\circ).$$

This becomes zero when

$$m_2t = 88^\circ \text{ or } 268^\circ;$$

and a maximum or a minimum

$$m_2t = 178^\circ \text{ or } 358^\circ.$$

But when $y_m - y_n$ is zero, so is v of (77) or (78); likewise for a maximum or a minimum. When it is high water at Governors Island, $m_2t = 231^\circ$; and when low water, $m_2t = 51^\circ$. $88^\circ - 51^\circ = 37^\circ = 1{\cdot}3$ hours as the time which slack (before flood) in East River should follow the time of low water. The strength of flood (easterly current) should occur when $m_2t = 178^\circ$; this means $231^\circ - 178^\circ$ or 53° or 1·8 hours before high water at Governors Island. Similarly, the slack before ebb in East River occurs 1·3 hours after, and the strength of ebb 1·8 hours before, the time of high water at Governors Island. These statements conform well with observed values, the results of which are given in the Coast Survey Tide Tables.

That the velocities of the fluid particles inherent in the wave motions do not account for the currents in East River, appears from the fact that at two hours before high water at Governors Island, the current off Old Ferry Point (a few miles west of Willets Point) is flowing westerly at the rate of 1·5 knots, which is nearly its maximum value at that place. In the Lower Hudson, off Thirty-ninth street, the maximum velocities are about simultaneous with the tides at Governors Island, as the theory of wave propagation in an indefinite canal would require. Two hours before high water the velocity is small (0·7 knots) and in a northerly direction. Now, had the wave been propagated up East River in a similar manner the velocity would be small at this hour. As a matter of fact it has very nearly its maximum value throughout the narrow portions of the river. Moreover, it can not be due to the wave motion from the east, because off Old Ferry Point the velocity is small and in an opposite direction.

Thus it is seen that the rapid currents of East River, and particularly around Blackwells Island, are not due to the superposition of two horizontal motions of the water as in simple wave motion, but to the difference in head between New York Bay and the western portion of Long Island Sound.*

* Cf. Mitchell, United States Coast and Geodetic Survey Report, 1887, p. 311.

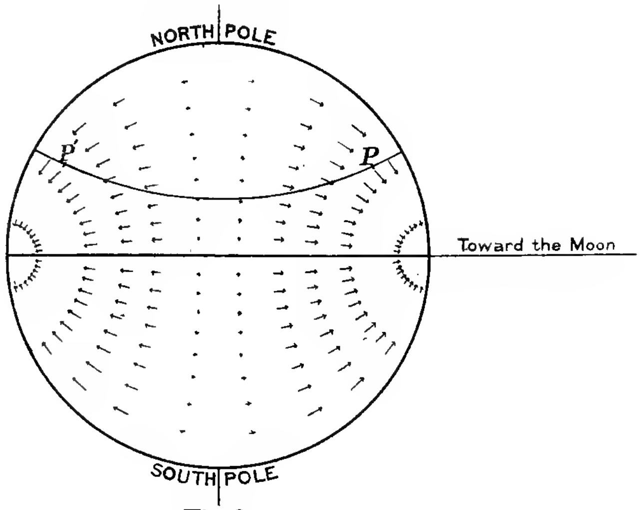


Fig. 3

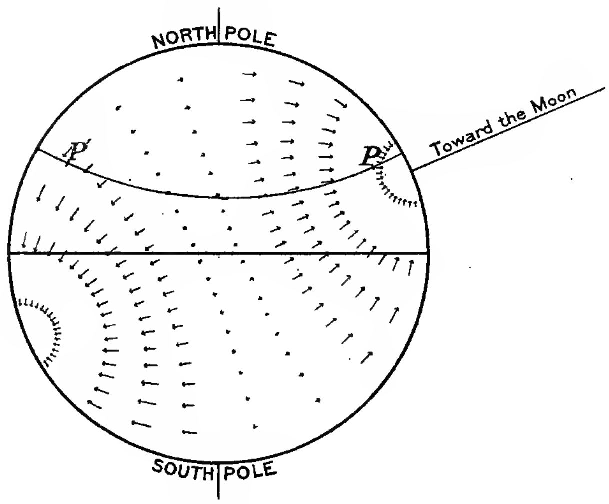


Fig. 4

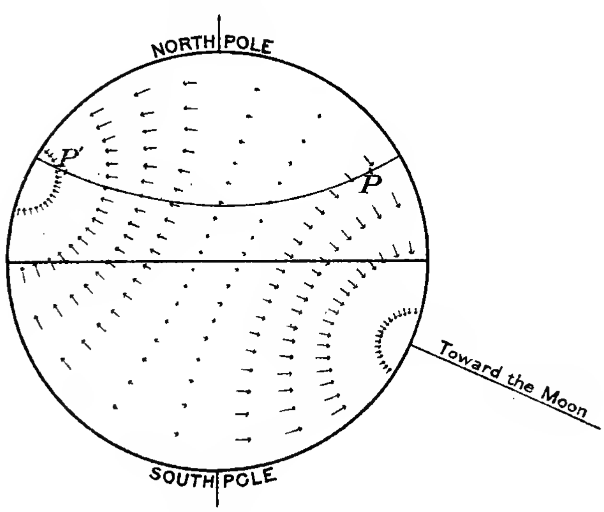


Fig. 5

THE NORRIS PETERS CO., PHOTO-LITHO., WASHINGTON, D. C.

CHAPTER III.

ON THE ORIGIN OF TIDES.

38. In the preceding chapter, the tide wave has been regarded as an existing phenomenon without special reference to its astronomical cause. It is proposed to here briefly consider the origin of the tide under several conditions. This will indicate some of the difficulties with which a general theory has to contend in the case of nature.

All particles of the earth (including the seas) will occupy positions fixed relatively to one another if no other forces act upon them than the following: the earth's attraction, its centrifugal force of axial rotation, and an extraneous force acting upon all of its particles alike. If the extraneous force does not act upon all particles alike, then motions will be set up in the yielding parts.

Suppose the earth to consist chiefly of a spheroidal nucleus either rigid throughout or rigid in its outward layers. Suppose this nucleus to be covered in whole or in part by one or more seas either shallow or not, in comparison with the earth's radius. The attraction of the moon upon any given particle near the surface, say, is along a line drawn (at any given instant) from the particle to the moon's center; its intensity, which is inversely proportional to the square of the distance, and local direction (i. e., direction with respect to the earth's surface) continually change as the earth rotates upon its axis. The attraction of the moon upon a particle at the earth's center is along a line drawn from the earth's center to that of the moon; its intensity is independent of the earth's axial rotation.

The difference between these two forces may be called the *tide-producing force* at the surface point in question.

Just what this force will do to the water as the earth rotates upon its axis, cannot be clearly seen except for very special cases. This force being very small in comparison with the earth's attraction, its vertical component, which slightly alters the intensity but not the direction of terrestrial gravity, cannot set up in seas shallow in comparison with the earth's radius any considerable motion amongst the fluid particles. The horizontal component of the tide-producing force may, however, impart a sensible horizontal motion to the waters of an extended sea, and, because the fluid is incompressible and continuous within a given basin, indirectly create a slight rising and falling of the surface, whether or not the period of the earth's axial rotation were sufficiently long to enable the surface to approach a level surface; i. e. to arrange itself normal to disturbed gravity at each point.

39. *The tide-producing force.*

The system of arrows in Figs. 3, 4, and 5 are intended to represent the horizontal component of the moon's tide-producing force at various places on the earth's surface. The arrows located upon the same small circle (isodynamic line) are supposed to be of equal length, and all arrows are supposed to lie in a system of great circles which meet in a point directly under the moon and, of course, in a point 180° therefrom. At these two points the length of the arrows is zero; for, the horizontal component of the moon's disturbing force must there vanish—the force itself being vertical. The length of the arrows is likewise zero along a great circle midway between these two points; for, all points along this circle are very nearly as far from the moon as is the earth's center.

The system of arrows is fixed with respect to the moon, and so sweeps over the surface of the earth as the moon performs her apparent daily revolution, or shifts somewhat as she declines north or south from the celestial equator. At any point P on the earth's surface, the moon being upon the equator, the horizontal forces are equal in magnitude and direction to the horizontal forces at P', a point upon the same parallel of latitude as P, but 180° distant in longitude; or, what amounts to the same thing, they repeat themselves at any given point P every half lunar day. But when the moon is not upon the equator, the forces are not generally the same at P and P', either in magnitude or in direction, and so do not exactly repeat themselves every half lunar day. This alternation of the forces gives rise to a diurnal inequality in the tides.

It will be noticed that for places situated upon either side of the equator, the forces have, when the moon is upon the equator, a meridional component directed from the poles toward the equator, and that this component never points from the equator toward the poles; consequently the existence of the moon causes the water (half-tide level) at the equator to be higher than it would otherwise have been (cf. § 47).

The moon's movement in declination causes a fortnightly fluctuation in half-tide level.

The magnitude of the tide-producing force can be shown graphically by means of the following obvious construction: *

Let M denote the moon, or its mass; E, the earth's center, or the earth's mass; P, any point whose distance is ρ from E; r, the distance EM, and D, the distance PM.

Locate a point O on PM produced through P such that $OM = r \times \frac{r^2}{D^2}$. Consequently if we let EM represent the attraction of the moon, M, upon unit particle at E, which attraction is $\frac{\mu M}{r^2}$, OM will represent, upon the same scale, the attraction of M upon unit particle at P, which is $\frac{\mu M}{D^2}$. μ denotes the attraction between two unit particles unit distance apart. Join O and E; then OE represents in both magnitude and direction the disturbing force of M upon P. The projection of OE upon PE, produced when necessary, is the vertical component of the disturbing force, and the perpendicular line from O to PE is the horizontal component.

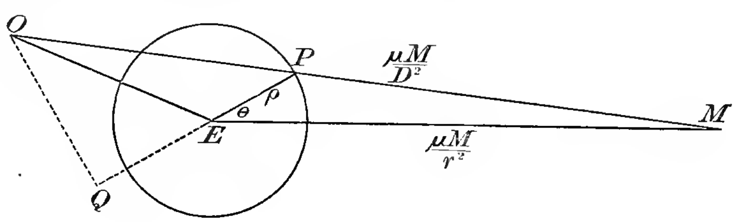


Fig. 6.

1. To find the magnitude of OE when P lies 90° from M. In this case O coincides with P and OE becomes equal to ρ. But if the length EM or r represents the force $\frac{\mu M}{r^2}$, the length OE or ρ must represent a force $\frac{\mu M}{r^2} \times \frac{\rho}{r}$. That is, the tide-producing force at a point 90° from M acts vertically downward, and its magnitude is one $\frac{\rho}{r}$th, or about one-sixtieth part of the direct attraction of the moon upon unit mass of the earth. To express this in feet and seconds units, as g is usually expressed, we note that

$$g = \frac{\mu E}{a^2}, \tag{84}$$

or

$$\mu = \frac{a^2}{E} g, \tag{85}$$

where a is the mean radius of the earth and E its mass.† Since $\rho \doteq a$, the compressing force at P becomes

$$\frac{M}{E}\left(\frac{a}{r}\right)^3 g, \tag{86}$$

or

$$\frac{1}{81{\cdot}07}\left(\frac{1}{60{\cdot}34}\right)^3 g, \tag{87}$$

$$\begin{aligned} &= 0{\cdot}000\ 000\ 056\ 15\ g \\ &= 0{\cdot}000\ 001\ 806 \text{ feet per second.} \end{aligned} \tag{88}$$

* Cf. Newton's Principia, Bk. I, Prop. 66.

† $a = \sqrt[3]{(\text{equatorial radius})^2\ (\text{polar radius})} = 20\ 902\ 000$ feet $= 3958{\cdot}7$ miles.

2. To find the magnitude of OE when P lies on the line EM. Here $D = r \mp \rho$; and so $OM \left(= r \times \frac{r^2}{D^2}\right) = r \pm 2\rho$, and $OE = \pm 2\rho$. But upon the scale adopted OE must represent a force whose magnitude is

$$2\frac{\mu M}{r^2} \times \frac{\rho}{r}$$

$$= 2\frac{M}{E}\left(\frac{a}{r}\right)^3 g. \tag{89}$$

It is directed toward the moon when P lies between E and M and from the moon when beyond E. The horizontal component is zero. The entire range of the disturbing force, as the angle between M and ρ varies from 0° to 90°, is

$$3\frac{M}{E}\left(\frac{a}{r}\right)^3 g = 0{\cdot}000\ 000\ 168\ g. \tag{90}$$

3. The above construction shows that for most positions of P the vertical and horizontal components of the tide-producing force are not very unequal in magnitude; for, OE is inclined at all angles according to the positions of P. The general expressions for these are not so conveniently derived from the above construction as from Proctor's, given in § 31, Part II, or from differentiating the tide-producing potential along the direction of the required force. The vertical and horizontal component forces are

$$\frac{\mu M\rho}{r^3}\left(3\cos^2\theta - 1\right), \tag{91}$$

$$3\frac{\mu M\rho}{r^3}\sin\theta\cos\theta, \text{ or } \frac{3}{2}\frac{\mu M\rho}{r^3}\sin 2\theta. \tag{92}$$

Since the horizontal acceleration $\left(\frac{\partial^2\xi}{\partial t^2}\right)$ is $\frac{3}{2}\frac{\mu M\rho}{r^3}\sin 2\theta$, or $0{\cdot}000\ 002\ 71 \sin 2\theta$ feet per second, we have upon integration over 90° or three lunar hours,

$$\xi = 0{\cdot}000\ 002\ 71 \times \frac{1}{m_2^2} = 137 \text{ feet}, \tag{93}$$

where m_2 is 0·000 1405 radian per second, instead of 28·984 degrees per hour. This gives 137 feet for the maximum excursion of a particle at the equator east or west from its mean position, due to the moon.

In order to see that the vertical force can have little or nothing to do with the tides, let us suppose that, in a sea of uniform depth, the density of the water be increased or decreased from place to place in such proportions as the force of gravity is altered by the vertical disturbing force of the moon. But the extreme variation in density over the globe would then be only 0·000 000 168. And so, returning to the consideration of water of constant density, it follows that the extreme variation in the height of the free surface of a sea of uniform depth would be but a 0·000 000 168 part of the depth.

The vertical force being generally about the same magnitude as the horizontal, and acting nearly perpendicularly to the free surface of the fluid, cannot create a horizontal motion comparable with that created by the horizontal force.

The deviation of the plumb line is evidently due wholly to the horizontal force. It is supposed to be practically independent of the depth or mass of the water, the topography of the continents, etc. Any surface normal to the disturbed plumb line is a *level surface.*

If a liquid surface coincide with an instantaneous level surface, while the latter undergoes changes, the forces responsible for the behavior of the liquid must be horizontal and not vertical.

40. *A small but not extremely shallow body of water.*

In this case the motion of the fluid hardly need be considered. The only thing necessary to be done is to find at any given instant how the direction of terrestrial gravity—that is, the direction of the plumb line—is perturbed because of the moon's attraction, and to then assume that the instantaneous surface of the water is perpendicular to this direction.

The direction of the plumb line at a given place, but for the presence of a tide-producing body, would remain fixed with respect to the solid earth, although the latter rotate upon its axis. In general, the horizontal component of the moon's tide-producing force will cause the plumb bob to deviate slightly from its undisturbed position. The point of suspension of the plumb line being fixed with respect to the earth's nucleus, cannot be altered with respect thereto, no matter what the tide-producing force may be. But the plumb bob is acted upon by the tide-producing force for the particular place (which is the difference in the moon's attraction for this place and for the earth's center) and is free to move horizontally. This force combined with the force of terrestrial gravity shows the deviation in the direction of the plumb line.

If the surface of a small body of water arrange itself normal to the disturbed plumb line, it must perform two similar oscillations each lunar day when the moon is on the equator. The tides in such a body are necessarily small because this deviation of the plumb line is only $0''{\cdot}017$ ($= 0{\cdot}000\ 000\ 084$ radian) either way from its mean position.

When the moon is not upon the equator, the tide-producing force at a given time differs somewhat from the force twelve lunar hours before or after. For this reason the two high waters and the two low waters of a day will generally be somewhat unequal.

The Levant, or part of the Mediterranean Sea east of Sicily, can be taken as an illustration. It extends approximately east and west about 1 100 miles, with a depth ranging from one to two thousand fathoms. [The velocity of a free wave in this depth is three or four hundred miles per hour, and so the surface can, as will be explained in §42, keep itself approximately perpendicular to the direction of perturbed gravity.] Because this sea is in not very high latitude, and extends approximately east and west, it should be high water on the coast of Syria about three hours before the moon's transit over the middle of the sea, and low water at Malta and the eastern coast of Sicily at the same time. In other words, it should be high water at the latter places about three hours after transit. These conclusions agree with the results of observation. The range at the ends should be roughly

$$11\ 000 \times 5\ 280 \times 0{\cdot}000\ 000\ 084 = 0{\cdot}5 \text{ feet},$$

which is somewhat smaller than observed values.

Suppose that at the center of gravity of a lake's surface we draw a plane normal to the plumb line; it will cut the surface in a line which may be called a "line of strike" and whose direction is that in which the deviating force is zero. The deviating horizontal force acts in a direction (an azimuth) perpendicular to this line. The height of the tide at any instant, or the volume of water in an elementary area, will evidently be proportional to the distance of the given elementary area from this line. Therefore if the volume of water in the lake remain constant, the statical moment of the entire lake surface must be zero, and so the line must pass through the center of gravity as we have assumed. This point is evidently the point of no-tide. Being a point of no-tide, the times of high water are anything whatever; in other words, all cotidal lines radiate from this point. In ascertaining the tidal forces at such a point, only its latitude need be considered, and any diagram for the purpose will apply anywhere upon the same parallel, regardless of longitude.

The horizontal deviating force is

$$\tfrac{3}{2}\,\frac{\mu M\rho}{r^3}\sin 2\,\theta = 0{\cdot}000\ 000\ 084\ g \sin 2\,\theta. \tag{94}$$

$$\therefore\ 0{\cdot}000\ 000\ 084 \sin 2\,\theta \tag{95}$$

is the angle of deviation (expressed in radians) or the slope of the surface of the water due to the moon's disturbance.

For a given latitude (λ) and hour-angle ($\psi - l$), we can suppose θ known through the equation

$$\cos\theta = \cos\lambda\cos\delta\cos(\psi - l) + \sin\lambda\sin\delta, \tag{96}$$

δ being the moon's declination; and the local direction of force through the equation

$$\sin z\sin\theta = \cos\delta\sin(\psi - l). \tag{97}$$

In this way for a given value of δ and of λ, a set of force arrows radiating from the assumed no-tide point and equal to sin 2 θ can be constructed for various values of $(\psi - l)$. Their heads will define a certain curve. To ascertain the height of the tide at any point, not too far distant, at a given lunar hour, compute or observe the value of sin 2 θ on the force diagram; also ascertain the distance in feet from the given point to the line of strike (which is perpendicular to the force arrow); multiply these together and this product by 0·000 000 084; the result is the height of the tide in feet reckoned from the lake surface as it would be if no moon existed. The required distance can be conveniently ascertained by letting the given point and the no-tide point define the diameter of a circle. Then produce the force arrow until it meets the circumference. From this intersection to the given point is the distance required.

To determine the time of high or low water, ascertain the point on the force curve where the normal is parallel to the diameter of the circle defined by the given point and the no-tide point. The hour-angle belonging to that force arrow is the angle required. Of course the time used in reckoning is that belonging to the longitude of the no-tide point.

When $\delta=0$, the force curve is an ellipse whose equation is

$$y^2 + (\sin^2 \lambda)\, x^2 = (\sin \lambda \cos \lambda)\, y \tag{98}$$

where y is reckoned southward.

In practice it is more convenient and accurate to make use of the horizontal disturbing force resolved into two directions (north-and-south, east-and-west). These are each developed, § 49, Part II, into semidiurnal, diurnal, and long period terms with either constant or somewhat variable coefficients.

The results obtained in § 49, Part II, or which may be obtained as above, for the tide at Duluth, Lake Superior, agree well with observed values. The computed equilibrium tides at Chicago and Milwaukee, Lake Michigan, have ranges considerably smaller than the observed values and their intervals are not as satisfactory as the interval obtained for Duluth. In fact it seems that almost all bodies of water have portions of their coast lines where the range of tide is unreasonably large, indicating that a wave movement has been propagated over a sloping bottom.

The tides of the Gulf of Mexico can be explained by aid of certain assumptions which are based upon observations.

Assuming that the semidiurnal wave does not exist (or is very small) south of the Yucatan Channel, then there is no derived semidiurnal tide from that source. A cross-section of Florida Strait is small in comparison with a cross-section of Yucatan Channel. The semidiurnal wave is not large in any portion of Florida Strait; and so from that source the Gulf can hardly have any considerable derived tide, the eastern part excepted.

At Vera Cruz, which is near deep water, the observed interval ($2^h\ 49^m$) and range (0·4 feet) of the semidiurnal wave approximately agree with the theoretical values obtained by considering the Gulf a closed sea. In the northeastern portion of the Gulf which has broad shallows, the range of tide is greatly increased. At Port Eads, near the mouth of the Mississippi River, the range is nearly zero as we might expect it to be, on account of the proximity of the no-tide point.

Now observation shows that the diurnal wave does exist south of the Yucatan Channel. The latter being broad and deep, permits enough water to enter the Gulf to raise its whole surface almost simultaneously. Observation shows that the tropic high-water interval of the diurnal wave for Gulf stations is generally between 19 and 22 hours, while the tropic range of the diurnal wave is generally between 1 and 2 feet.

41. *An hypothetical equatorial canal of uniform depth surrounding the earth.*

The present illustration is given for the purpose of showing that the surface of the sea does not of necessity arrange itself normal to the plumb line as disturbed by the moon; and here also it is necessary to consider the force tending to deviate the same, that is, the horizontal component of the moon's tide producing force. All particles of the canal in the hemisphere toward the moon have imparted to them horizontal accelerations, urging them toward a point of the canal where the moon is on the meridian. All particles in the other hemisphere are at the same time urged toward a point 180° distant in longitude. There is no acceleration (east or west) at these two points, or at points 90° distant where the moon is in the horizon. Consequently, at any given

place, from moonrise to upper local transit, the acceleration is eastward, because the particle is continually approaching the moon; from transit to moonset the acceleration is westward; from moonset to lower transit it is eastward; and from lower transit to moonrise it is westward. Now, if the fluid be heavy and frictionless, the maximum eastward velocity will occur after all the eastward acceleration has been imparted, that is, at lunar noon or midnight; the greatest westward velocity, at moonrise or moonset; and zero velocity at the third, ninth, fifteenth, and twenty-first lunar hours.

All the time from moonrise till transit, more water flows toward the east than enters from the west at a given place, because the particles of moving water are continually acted upon by a force imparting to them an eastward acceleration. The reverse is true from transit to moonset. Similarly for the other half of the lunar day. During one of these periods the tide must be continually falling, and during the other continually rising. Low and high waters occur at the close of these periods, that is, at the transits and when the moon is in the horizon. The tides having a fixed position with respect to the moon, it follows that ~~the~~ two wave-forms travel~~s~~ westward around the earth ~~twice during~~ each lunar day. Of course even the horizontal displacements of the fluid particles are small in comparison with the earth's radius. But it is obvious that if the wave-form advance westward, the orbital direction of the fluid particles must be such that the particles at high water are moving westward and at low water eastward; and so, as just stated, high water must occur when the moon is in the horizon, and low water when on the meridian.

Let x denote the distance of the place east from the meridian of Greenwich, or $\frac{x}{a}$ be its east longitude (in radians); let α or $\arg_0 \mathrm{M}_2$ denote twice the hour angle of the moon at the time $t=0$.

$$\therefore 2\,\theta = \mathrm{m}_2 t + \frac{2x}{a} + \alpha \tag{99}$$

is twice the angle reckoned westward between the point and the moon.

The eastward and upward displacements of the forced waves are

$$\xi = -\tfrac{1}{4}\frac{f\,a^2}{c^2 - \mathrm{m}_1{}^2 a^2}\sin 2\,\theta, \tag{100}$$

$$\eta = \tfrac{1}{2}\,\frac{c^2\,H}{c^2 - \mathrm{m}_1{}^2 a^2}\cos 2\,\theta, \tag{101}$$

wherein are put temporarily

$$f = \tfrac{3}{2}\,\frac{\mu\,M\,a}{r^3},$$

$$c^2 = g\,h,$$

$$H = \frac{a\,f}{g}.$$

These satisfy the equation of continuity (17) and the dynamical equation [cf. (23), (25), (92)]

$$\frac{\partial^2 \xi}{\partial t^2} = c^2\,\frac{\partial^2 \xi}{\partial x^2} - f\sin 2\,\theta. \tag{102}$$

The eastward accelerating force due to the rate of change in pressure due to height of the free surface is $c^2\,\frac{\partial^2 \xi}{\partial x^2}$. But the moon exerts a retarding (westward) force equal to $f\sin 2\,\theta$. Hence the above equation. Frictional resistance can be taken into account by subtracting from the right-hand member of this equation a term proportional to some power of the velocity $\left(\frac{\partial \xi}{\partial t}\right)$. The forms of ξ and η must then be altered to correspond.*

*Airy, Tides and Waves, Arts. 322 et seq.; Abbott, Elementary Theory of the Tides (1888), pp. 33 et seq.; Lamb, Hydrodynamics, §§ 178, 276 et seq.

When the water is shallow

$$\xi = \tfrac{1}{4} \frac{f}{m_1^2} \sin 2\,\theta = 137 \sin 2\,\theta \text{ feet,} \tag{103}$$

$$\eta = -\tfrac{1}{2} \frac{c^2 H}{m_1^2 a^2} \cos 2\theta = -\tfrac{1}{2} \frac{hf}{m_1^2 a} \cos 2\theta = -0{\cdot}\,000\,013\, h \cos 2\theta \text{ feet.} \tag{104}$$

The amplitude of the tide in a shallow equatorial canal is therefore but a small fraction of a foot and is proportionate to the depth. The tide is evidently *inverted* since low water occurs when $2\theta = 0^\circ$.

When the depth is such that the free wave travels as fast as the moon or the forced tidal wave, then

$$c^2 - m_1^2 a^2 = 0; \tag{105}$$

and so the displacements tend to become infinite. Such a value of h is 67 000 feet. For greater depths the tide would be *direct*, i. e. high water would occur when $2\theta = 0^\circ$.*

When the depth is very great the displacements approach the values

$$\xi = -\tfrac{1}{4} \frac{a^2 f}{c^2} \sin 2\theta, \tag{106}$$

$$\eta = \tfrac{1}{2} H \cos 2\theta. \tag{107}$$

It will be seen from § 47 that this value of η is equivalent to the tide of the equilibrium theory, H denoting the range of the lunar portion, which is about 1·8 feet.

It is important to note that in a shallow canal (and no others exist on the earth) the displacements depend upon the period of the disturbing force, or the "speed" of the disturbing body. The greater the speed, the smaller the amplitude. Hence, the ratio of the amplitude of the solar to that of the lunar tide ought to be less than the ratio as derived from the equilibrium theory. In nature this is generally found to be the case.

It should also be noted that when friction is taken into account, not only does the amplitude or coefficient of the displacement depend upon the "speed" of the tidal body, but the angle or phase of the displacement does likewise depend upon it.†

42. *The tides of the ocean do not produce a level surface.*

In order to be convinced that the instantaneous surface of a large body of water cannot be a *level surface*, let us inquire what takes place when a vessel or trough of water is slightly disturbed—say slightly elevated at one end. If the disturbance be sudden, evidently a wave will be seen passing back and forth across the surface; finally it will disappear and equilibrium will be restored. If the disturbance be gradual no wave may be noticeable, but equilibrium will be restored at each instant. Now suppose the trough to be so long that the gradual elevation and depression of one end of it takes place in a period of time less than the time required by the free wave to traverse its length. As long as the disturbance continues equilibrium cannot be restored. Hence it follows that the length of the trough and the depth of the water must be such that the free wave travels the length of the trough several times during the period of the disturbance if the surface is to remain nearly level. From this we conclude that the surface of an ocean cannot be a level surface, because too much time is required for the free wave to cross and recross—in other words, for the surface to arrange itself normal to the plumb line.

For example, the period of the (semidaily) tidal forces is about equal to the time required for a free wave to be transmitted from Japan to California.

43. Tidal observations have been confined to such portions of the sea as have comparatively small depths. For this reason it has seemed impossible to ascertain what actually takes place in

* This usage of the word "direct" is due to Dr. Thomas Young. Perhaps it might be well to substitute therefor the word "erect," and to use the word "direct" in antithesis to "derived."

† Airy, loc. cit.

deep waters, causing the tides observed along the shores of continents and islands. If it shall become possible to detect, even roughly, the rising and falling of the tide at sea by means of the barometer,* this much-needed information will be supplied, and charts of cotidal lines may then be constructed with some degree of certainty.

In case of a deep but small inclosed sea, the surface of the water undoubtedly keeps itself perpendicular to the direction of the earth's gravitational force as perturbed by the tidal body. In other words, the maximum excursions of fluid particles take place when the deviating force is a maximum. As already noted, this implies that for an equatorial lake or sea the surface is most inclined to its undisturbed level at the third hour before or after its semidaily transit across the no-tide meridian. At the third hour before the transit the eastern portion should have high water and the western low water, and *vice versa* for the third hour after its transit.

It is reasonable to suppose that the particles which constitute a very large body of water, like an ocean, may have a tendency, because of their inertia, to reach their greatest displacements in a given direction not when the force acting in that direction becomes a maximum, but rather when the force in the opposite direction becomes a maximum or nearly so, as in the case of the equatorial canal already considered; for instance, extreme eastward elongation at a given place may for this reason occur about three hours after the time of the moon's transit across the local meridian. But, as shown in § 41, the range of tide in a self-returning equatorial canal is small in comparison with the range of tide in even a moderately large sea where the (corrected) equilibrium theory must still approximately apply. From this we are led to believe that in producing the tide the effect of the horizontal motion alone in a boundless shallow ocean is probably small. It is the boundary conditions (shores and bottom) which come in to enable the astronomical forces to produce tides of considerable magnitude. In fact these conditions make an approximate equilibrium tide possible in a sea not too extended. Other effects of boundaries have been noted in §§ 30, 33.

A large body of water approximately surrounded by lands and shoals is set into some kind of oscillation by the tide-producing forces. The manner in which it will be divided into vibrating masses by nodal lines depends upon the extent, shape, and depths of the body of water.

Probably off the eastern and western boundaries would be found the greatest direct† effect of this action, although right at the shores themselves the range of tide must generally be still greater, owing to the propagation of the free wave over shallow areas. Whether the eastern or western edge have the greater rise and fall of tide will depend upon the bounding lands and shoals, also upon the extent of the water.

If the tide at one point be considerably greater than that at others, the wave there generated may be propagated far as a free wave and partly control the positions of the cotidal lines.‡

In large oceans there may be traces of waves, which move westward at the rate of 360° in a lunar day. But whatever waves go to build up the tide, the combined effect probably makes cotidal lines generally real.

In regard to latitude it may be said that direct equatorial tides should have large semidiurnal and small diurnal constituents. The semidiurnals should decrease as the latitude increases, and the diurnals increase up to latitude 45°, after which they decrease.

44. If a bay communicates with the ocean, its tide is almost wholly derived from without. That is, a free wave is propagated up the bay, its velocity depending chiefly upon the depth of the water (§ 23). Numerous reflections from the sides and the head of the bay may be sufficient to alter the phase of the tide (at any given instant) somewhat, and so the velocity of the resultant crest. The same cause may likewise alter the amplitude. It is obvious that the corresponding reflections of the diurnal wave will not, in general, have the same accelerating or retarding effect upon its crest as had those pertaining to the semidiurnal. In such regions the type of tide may vary rapidly from point to point; that is, the diurnal and semidiurnal waves do not travel with the same velocity as is the case in a canal of uniform depth, the friction being for simplicity assumed to be zero. Speaking more generally, the particular form and size of a bay, in conjunction

* See R. Abercromby, Phil. Mag., Vol. 25 (1888), pp. 263–269.

† I. e., as opposed to "derived" or "propagated." This usage of the word "direct" has nothing in common with Dr. Young's usage of the same word mentioned in § 41.

‡ E. g., Gulf of Panama.

with the fixed periods of the tides, govern or define its tidal movements or vibrations. For example, the Tidal Survey of Canada has shown that the semidiurnal tides at St. Peters, Prince Edward Island, are about two hours earlier, and those at Miramichi Bay more than three hours earlier, than the tides at St. Paul Island at the entrance to the gulf.

The statements made concerning the propagation of a wave up a bay, must not be taken too literally. The fact is that even if we have so simple a case as a sea or bay (whose own or direct tide is negligible) communicating with the ocean by one or more comparatively small and definite openings, and if at such mouths or openings the tides and currents are known from observation so that they could be predicted at any future time, we are, in the present state of our knowledge of wave motion, quite unable to predicate the motion in various parts of the sea or bay, although all depths and boundaries are known. It is probable that cases of this kind will sooner or later be attacked by mathematicians, and with some measure of success.

Such considerations as the above, but more especially the fact of interference due to the water approaching from more than one direction, explain in part the great variety in the types of tide found in island regions.

The reason why diurnal tides should succeed better than semidiurnals in passing barrier reefs, and therefore becoming conspicuous in island regions, is obvious. In fact, the shorter the period of the wave, the more a reef or other great obstruction will affect it; and this because the amount of water which passes it in the flow and the ebb (the ranges being equal) increases when the length of the period is increased. A tide of long period and an almost complete barrier, so far as tides of short periods are concerned, serve to illustrate what is here intended.

45. Perhaps the only ocean region where the tide approaches its normal* form is the North Pacific Ocean. For, so far as the semidiurnal wave is concerned, both North and South Pacific form one region, and so there is no interfering from a neighboring region. The diurnal wave naturally becomes small as the equator is approached, and so the derived diurnal wave from the South Pacific can hardly be felt far north of the equator, where the direct diurnal wave is large. The western coast of America, the eastern coast of Japan, and perhaps the Sandwich Islands, seem to be most favorable localities for normal tides. Next to these regions is, perhaps, the eastern portion of South Pacific—say along the western coast of South America and around the southeastern islands of Polynesia. The East Indies naturally constitute the most complicated tidal region, and the West Indies probably come next.

So far the type of tide has been governed by the size and position of the diurnal wave with respect to the semidiurnal wave, and these are quantities most subject to variation in short distances.† But when long distances are considered, the fact that each wave is composed of a solar as well as a lunar portion becomes important. The oscillations being of slightly different periods, it is easy to suppose (because the wave is generally neither direct nor derived, but in part both) that one oscillation may gain in phase somewhat upon the other, and that the ratio of the two amplitudes may vary. In a region where there may be reflections or interferences of the tidal wave, the relative amplitudes and epochs of two oscillations of different periods must generally differ from point to point. The two portions of semidiurnal wave may thus follow their apparent causes (sun and moon) by different intervals in certain localities (thus causing the "age" to vary from point to point), and the ratio of their amplitude may likewise vary. So for the diurnal wave.

This is substantially Laplace's‡ explanation of how the speeds of components may alter their relative amplitudes and epochs. Airy§ found that with fewer assumptions fluid friction would accomplish a like result. It is probable that amplitude ratios and the ages of the corresponding inequalities depend upon both of these causes.

Vague considerations like these lead to inferences like the following:

Large diurnal or semidiurnal waves are approximately normal, and small ones abnormal.

---

*I. e., a tide in which the $\frac{\text{diurnal}}{\text{semidiurnal}}$ components have such relations to one another as are implied in the equilibrium theory.

†E. g., the coast of Ireland, Phil. Trans., 1845, p. 45.

‡Méc. Cél. Bk. IV, sec. 18.

§Tides and Waves, Art. 329; Lamb, Hydrodynamics, Ch. XI.

By large and small are here meant large and small in comparison with their equilibrium values, or even in comparison with their values in the surrounding regions.

Small ages indicate normal tides; large ages, abnormal.

Diurnals are generally abnormal in localities having abnormal semidiurnals; but observation shows that abnormal diurnals are not always accompanied by abnormal semidiurnals.

The following are references to articles giving general explanations in regard to the causes of tides: Airy, Tides and Waves, Section VIII; Ferrel, Tidal Researches, Chapter VIII and sections 145–149; Lentz, Fluth und Ebbe (1879), Chapters I, II; Günther, Geophysik, Volume II, Chapter IV, sections 5–9; Darwin, Encyclopædia Britannica, Article "Tides," section 3.

CHAPTER IV.

GENERAL PROPERTIES OF TIDES AND MODES OF REDUCTION.

46. *General properties of tides.*

Confining one's attention to a particular station, the following properties common to most tides are usually revealed by means of a few days' observations:

(1) Two high waters and two low waters occur during each twenty-four or twenty-five hours.

(2) The alternate high or low waters are more or less unequal.

(3) The heights of corresponding tides vary from day to day.

(4) The lunitidal intervals (high or low water) are different for alternate tides.

(5) The lunitidal intervals for corresponding tides vary from day to day.

(6) The inequality in height or interval referred to in (2) or (4) becomes greater as the moon's declination, either north or south, increases. This does not apply, because of the sun's tidal effect, to the lesser inequality at stations where the high and low waters are affected by quite unequal amounts.

(7) The range of tide (as determined from all four tides of the day) is $\begin{array}{c}\text{greater}\\ \text{less}\end{array}$ than usual near the time of $\begin{array}{l}\text{new or full moon.}\\ \text{the moon's quadrature.}\end{array}$

(8) The range of tide is $\begin{array}{c}\text{greater}\\ \text{less}\end{array}$ than usual near the time when the moon is in $\begin{array}{l}\text{perigee.}\\ \text{apogee.}\end{array}$

(9) The lunitidal intervals are $\begin{array}{c}\text{shorter}\\ \text{longer}\end{array}$ than usual near the times of the $\begin{array}{l}\text{first and fifth}\\ \text{third and seventh}\end{array}$ octants.

The above statements do not usually apply to the tides at stations where but one high and one low water occur daily. The readily observable properties of such tides are:

[1] But one high and one low water occur daily when the moon is far from the equator.

[2] Two high and two low waters, both comparatively small, may occur daily when the moon is near the equator.

[3] The moon being far from the equator, the (diurnal) range of tide is $\begin{array}{c}\text{increased}\\ \text{decreased}\end{array}$ near the time of either $\begin{array}{l}\text{solstice.}\\ \text{equinox.}\end{array}$

47. *The equilibrium theory of tides.**

The uncorrected equilibrium theory begins by assuming—

(1) That the nucleus of the earth is comparatively rigid (or that at least its outer layer is a rigid shell), and that it is composed of concentric spherical layers, each layer having a constant density.

(2) That the nucleus is covered by a fluid of uniform depth, shallow as compared to the radius of the nucleus, but deep as compared to the rise and fall of tide.

(3) That this fluid has neither inertia nor viscosity, nor is there friction between the fluid layer and the nucleus or the enveloping atmosphere.

As these conditions are far from being realized in the case of nature, observations will show at best only certain approximations toward ideal values. Before introducing the modifications

* I. e. *theory* in the sense of a working hypothesis, not as a basis of explanation.

necessary to adapt the theory to the tides, it seems desirable to ascertain what the tendencies are in the ideal case.

Since the angular velocity of the moon in her orbit and the rotary motion of the earth's surface are finite, while the particles of fluid are supposed to respond *immediately* to the forces acting upon them, we may consider the earth's surface as stationary during any given instant, and treat the surface assumed by the water as a case of static equilibrium.

Because of hypothesis (1), the attraction of the moon upon the nucleus is the same as it would have been had the entire mass been concentrated at the earth's center.

At any given place the tide-producing tendencies depend chiefly upon the distance and direction of the disturbing body, and are governed by what may be referred to as Laws I and II.

Law I.—The tendency to produce tides at a given station varies directly as the mass of the disturbing body and inversely as the cube of the body's distance from the earth's center.

In consequence of this law the amplitude of the solar tide ought to be about 0·458 times that of the lunar tide. For, the mass of the sun $=331\,000$, and the mass of the moon $=1/81$, the mass of the earth being unity, while the sun's distance $=92\,800\,000$ miles and the moon's distance $=239\,000$ miles, so that

$$\text{solar tide} : \text{lunar tide} = \frac{331\,000 \times 81}{(92\,800\,000)^3} : \frac{1}{(239\,000)^3}; \tag{108}$$

$$\therefore \text{solar tide} = 0{\cdot}458 \text{ lunar tide.} \tag{109}$$

The eccentricity of the lunar orbit being 0·055, this law gives

$$\text{perigean range} : \text{mean range} = \frac{1}{(1-\text{eccentricity})^3} : 1, \tag{110}$$

$$\text{apogean range} : \text{mean range} = \frac{1}{(1+\text{eccentricity})^3} : 1. \tag{111}$$

$$\therefore \text{perigean range} = 1{\cdot}17 \text{ mean range,} \tag{112}$$

$$\text{apogean range} = 0{\cdot}84 \text{ mean range.} \tag{113}$$

Law II.—The tendencies to produce tide for various relative positions of the tide-producing body are proportional to

$$3\cos^2\theta - 1, \tag{114}$$

where θ is the zenith distance of the body corrected for parallax. In other words, θ is the angle at the earth's center defined by the given station and the center of the disturbing body.

If u denote the height of tide expressed in terms of the earth's radius, a, then it is proportional to $3\cos^2\theta-1$, or equal to say $\alpha'(3\cos^2\theta-1)$. The equation of the surface of the sea at any given instant is

$$\rho = a\,(1+u), \tag{115}$$

or

$$\rho = a + a\,\alpha'\,(3\cos^2\theta - 1), \tag{116}$$

which is the equation of an ellipsoid whose semiaxes are

$$a\,(1+2\,\alpha'),\ a\,(1-\alpha'),\ a\,(1-\alpha'). \tag{117}$$

That is, forces acting according to this law cause the surface of the sea to assume the form of an ellipsoid of revolution whose longest axis points toward the tide-producing body.

It will be observed that when the moon, say, is in the zenith (or nadir) the elevation of the sea is $2\,a\,\alpha'$ higher because of the existence of the moon; but when in the horizon, the elevation of the sea is $a\,\alpha'$ lower.

For a given place the height of the tide will vary from hour to hour of the day chiefly on account of the variations in θ; but, as already noted, it varies somewhat on account of the variation in r, the moon's distance.

For a given place the angle θ depends upon the moon's hour angle and its declination, both of which are functions of time. From spherical trigonometry,

$$\cos\theta = \cos\lambda\cos\delta\cos(l-\psi) + \sin\lambda\sin\delta \tag{118}$$

where

λ = geographic latitude of the station,
l = longitude of the station (W. from Greenwich),
δ = moon's declination,
$\psi = mt$ = moon's hour angle (W. from the meridian of Greenwich).

$$\begin{aligned} \therefore a\,\alpha'\,(3\cos^2\theta - 1) &= \tfrac{3}{2}\,a\,\alpha'\cos^2\lambda\cos^2\delta\cos 2\,(\psi - l) \\ &\quad + 3\,a\,\alpha'\sin\lambda\cos\lambda\sin 2\,\delta\cos(\psi - l) \\ &\quad + \tfrac{1}{2}\,a\,\alpha'\,(3\sin^2\lambda - 1)\,(3\sin^2\delta - 1) \\ &= \text{height of tide according to the uncorrected equilibrium theory.} \end{aligned} \tag{119}$$

For the lunar tide,

$$a\,\alpha' = \tfrac{1}{2}\,\frac{\text{mass of moon}}{\text{mass of earth}} \times \frac{a^4}{(\text{moon's distance})^3} = 0{\cdot}59 \text{ feet;} \tag{120}$$

and for the solar tide,

$$a\,\alpha' = \tfrac{1}{2}\,\frac{\text{mass of sun}}{\text{mass of earth}} \times \frac{a^4}{(\text{sun's distance})^3} = 0{\cdot}27 \text{ feet.} \tag{121}$$

(i) The height of the semidiurnal portion of the lunar or solar tide at a given station is proportional to the cosine of twice the local hour-angle of the moon or sun multiplied by the square of the cosine of its declination. The factor depending upon the declination is always near unity.

(ii) The height of the diurnal portion of the lunar or solar tide at a given station is proportional to the cosine of the local hour-angle of the moon or sun multiplied by the sine of twice its declination. The factor depending upon the declination varies almost directly with the declination.

(iii) There is a portion of the lunar or solar tide which depends, at a given station, wholly upon the declination of the moon or sun. The height of this portion is proportional to $3\sin^2\delta - 1$ where δ represents the declination of the moon or sun. The period of this expression is a half tropical month or year as the case may be.

The height of the entire tide, or of the surface of the sea, at any given time and place is the sum of the six terms just referred to—three belonging to the moon and three to the sun.

*The corrected equilibrium theory.**—To approximately adapt the foregoing theory to the case of nature, we may write the height of the lunar or solar tide in the form

$$\begin{aligned} &R_2\cos^2\delta\cos\left[2\,(\psi - l) - \varepsilon_2\right] \\ &+ R_1\sin 2\,\delta\cos\left[\psi - l - \varepsilon_1\right] \\ &+ R_0\left[3\sin^2\delta - 1\right] \end{aligned} \tag{122}$$

where R and ε must be determined from observations at the given stations. Statements (i), (ii), and (iii) require no modification except that for "hour-angle" we must write "hour angle diminished by a constant appropriate for the station in question" and so for "twice the hour angle".

This correction is theoretically necessary (even if the water have neither inertia nor friction) because the earth's surface is not wholly covered with water, and the equation of continuity can not generally be satisfied when the rise and fall is as given by equation (119) unless we continually alter the plane of reference.

The R's, as did the α''s, involve the factor

$$\left(\frac{\text{mean distance of moon}}{\text{actual distance of moon}}\right)^3, = \left(\frac{c}{r}\right)^3, \text{ or } \left(\frac{\text{actual parallax}}{\text{mean parallax}}\right)^3. \tag{123}$$

In practice the inertia and friction of the water produce important modifications in the R's and ε's from their equilibrium values. Nevertheless, the *form* (122) is capable of approximately representing the rise and fall of the tide in nature. This is especially true, if we make the further

* See preceding footnote.

modification of taking δ and r at times anterior to the time of tide. Such times, as well as the R's and ε's, must be determined from observations made at the given station.*

48. *Explanation of phenomena noted in § 46 by the equilibrium theory.*

The tides in (i), § 47, are semidiurnal, while those in (ii) are diurnal. Each may, for any particular day, be represented by a cosine curve of proper length (period) and amplitude. Now, it is obvious that the superposition of a diurnal curve upon a semidiurnal, will, in general, cause the alternate maxima or minima of the semidiurnal curve to become more or less unequal in height and unequally displaced in time. These statements account for (1), (2), and (4) of § 46. As noted in (ii), § 47, the amplitude of the diurnal curve (lunar or solar) is nearly proportional to the declination of the moon or sun. This explains property (6), § 46.

The superposition of a semidiurnal curve or wave upon another of nearly equal period, but of greater amplitude, simply increases or decreases the amplitude of the latter when approximately like or opposite phases coincide; but when the phases differ by approximately 90° or 270°, the principal wave is displaced in time by the subordinate one—accelerated or retarded according as

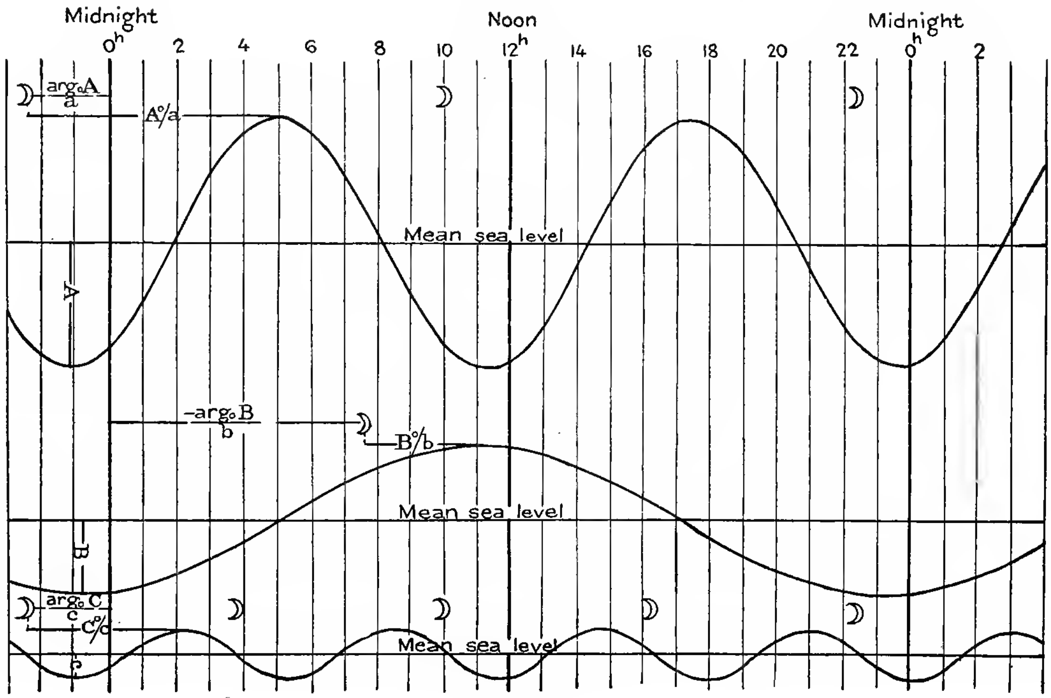


Fig. 7.

the maximum, say, is 90° in advance or in retard of the maxima of the principal wave. This accounts for properties (3), (5), (7), and (9), § 46. Property (8) has been explained in § 47 where the values of the perigean, apogean, and mean ranges are compared. This amounts to varying the a' or the R's inversely as the cube of the moon's distance from the earth's center.

At a station where observation shows that R_1 is several or many times as great as R_2, expression (122), the number of maxima and minima of a curve composed of diurnal and semidiurnal parts will usually depend upon the number of maxima and minima of the diurnal part when the moon's declination is great; but when the moon is near the equator the number may be governed by the semidiurnal part. This accounts for properties [1] and [2], § 46. The moon crosses the equator and reaches its extreme declination at nearly the same points in the heavens as does the sun. This accounts for property [3].

A still more perfect form or expression for the equilibrium theory is obtained by developing the tide-producing potential (the principal part of which is inversely proportional to the cube of

* Cf. Thomson and Tait, Natural Philosophy, §§ 804–811.

the disturbing body's distance from the earth's center, and directly proportional to $3 \cos^2 \theta - 1$, § 47), into a series of cosine terms. For considerable periods of time the coefficients of these terms remain sensibly constant and their angles or arguments increase uniformly with the time. Having found from the development of the potential what are the more important terms, one then assumes that by leaving all amplitudes and epochs arbitrary the series is, by the principle of forced oscillations,* capable of representing the tide at any given station. The harmonic analysis, § 49, has for its object the determination of these amplitudes and epochs from tidal records.

49. *Harmonic analysis.* †

Since the tide is periodic in its character, and since the periods of its causes are known from astronomical considerations, it ought to be possible to represent the height at any given time by means of the Fourier series, or rather an aggregation of such series

$$y = A \cos (at + \alpha) + B \cos (bt + \beta) + \quad . \quad . \quad . \tag{124}$$

where y is reckoned from main sea level.

For aiding the imagination, we may suppose that any given term in this series represents the oscillation caused by a fictitious star, or moon, moving uniformly in the celestial equator around the earth, and at a constant distance therefrom, having the property of producing a maximum of the oscillation, or component tide, a certain number of hours after its meridian passage.

If a denote the hourly speed of the component A, or the apparent angular velocity of its fictitious moon, and A° its epoch or lag expressed in degrees, A°/a is the lag expressed in hours. Also if $\arg_0 A$ denote the hour-angle of the fictitious moon at local mean midnight, $at + \arg_0 A$ is its hour-angle at any subsequent hour t. Consequently the time of high water of the component A is

$$t = \frac{A^\circ}{a} - \frac{\arg_0 A}{a}, \tag{125}$$

and the height at any time t is

$$A \cos (at + \arg_0 A - A^\circ), \tag{126}$$

so that

$$\alpha = \arg_0 A - A^\circ. \tag{127}$$

By replacing A, A°, a, and α by B, B°, b, and β, the corresponding quantities for any other component, B, are obtained.

The heights due to any components may be shown graphically thus (see Fig. 7):

Lay off the hours of the day according to any convenient scale. Draw cosine curves of amplitudes A, B, . . . and of periods $\frac{360}{a}$, $\frac{360}{b}$, . . . hours in length. The first maxima are located upon the hour lines

$$\frac{A^\circ}{a} - \frac{\arg_0 A}{a}, \quad \frac{B^\circ}{b} - \frac{\arg_0 B}{b} \quad . \quad . \quad . \; ; \tag{128}$$

the succeeding maxima are then fixed by the lengths of the several periods. The symbol ☽ may be used to indicate the time of transit of any fictitious moon.

To combine these curves, add the ordinates for each hour, thus obtaining the resultant tidal curve from which the times and heights of high water and low water may be obtained.

The object of the harmonic analysis is to resolve the observed tide, i. e., observed heights of the surface of the sea, into simple elements or component tides, consisting of simple harmonic oscillations. The quantities a, b, . . . and $\arg_0 A$, $\arg_0 B$, . . . are known from astronomical considerations, so that the analysis of the tide at a given place implies only the determination of the amplitudes A, B, . . . and the epochs A°, B°, ...

To harmonically analyze a given tide, let its height be given at each hour of the day, for a year, say. Sum these ordinates, as nearly as may be, at the component hours of each component (its harmonics excepted). The sums belonging to each component will be 24 in number and

* See Laplace, Méc. Cél., Bk. IV, § 16; or see under Laplace. Also, Part II, § 14. † See Part II.

represent sums corresponding to each of the twenty-four hours into which the component day is supposed to be divided. As the summation in each case is made with reference to the component hours, the effect of the other components upon these 24 sums will, in the long run, approach zero. Having found the 24 heights corresponding to these sums, they may be plotted as hourly ordinates; such a plotting would represent the required component tide combined with its harmonics. To analyze these 24 heights, $h_0, h_1, h_2, \ldots h_{23}$, assume each to be of the form

$$h = H_0 + \overline{A}_1 \cos at + \overline{\overline{A}}_1 \sin at + \overline{A}_2 \cos 2at + \overline{\overline{A}}_2 \sin 2at + \ldots + \overline{A}_8 \cos 8at + \overline{\overline{A}}_8 \sin 8at, \tag{129}$$

where $at = 0°, 15°, 30°, \ldots 345°$.

It is not difficult to show that the most probable values of H_0, $\overline{A}$, $\overline{\overline{A}}$ are given by the equations

$$\begin{aligned}
24\,H_0 &= h_0 + h_1 + h_2 + \ldots + h_{23}.\\
12\,\overline{A}_1 &= h_0 \cos 0° + h_1 \cos 15° + h_2 \cos 30° + \ldots + h_{23} \cos 345°,\\
12\,\overline{A}_2 &= h_0 \cos 0° + h_1 \cos 30° + h_2 \cos 60° + \ldots + h_{23} \cos 330°,\\
12\,\overline{A}_3 &= h_0 \cos 0° + h_1 \cos 45° + h_2 \cos 90° + \ldots + h_{23} \cos 315°,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ;\\
12\,\overline{\overline{A}}_1 &= h_0 \sin 0° + h_1 \sin 15° + h_2 \sin 30° + \ldots + h_{23} \sin 345°,\\
12\,\overline{\overline{A}}_2 &= h_0 \sin 0° + h_1 \sin 30° + h_2 \sin 60° + \ldots + h_{23} \sin 330°,\\
12\,\overline{\overline{A}}_3 &= h_0 \sin 0° + h_1 \sin 45° + h_2 \sin 90° + \ldots + h_{23} \sin 315°,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{130}$$

From these values of $\overline{A}$, $\overline{\overline{A}}$, we find A and α by the relations

$$A = (\overline{A}^2 + \overline{\overline{A}}^2)^{\frac{1}{2}}, \qquad \tan\alpha = -\frac{\overline{\overline{A}}}{\overline{A}}. \tag{131}$$

$A°$ then becomes known by the equation

$$A° = \arg_0 A - \alpha, \tag{132}$$

$\arg_0 A$ being known from astronomical considerations.* So for components B, C, etc.

It may be added that because the hourly heights are tabulated in solar time, most of the amplitudes as brought out in the analysis must be increased by a factor a little greater than unity, known as the augmenting factor; also that most of these amplitudes must be corrected for the longitude of the moon's node by the application of a suitable factor. For series less than about a year in length, still other corrections must be applied.

NONHARMONIC REDUCTIONS, ETC.

59. The object of these reductions is to determine nonharmonic quantities from observations made upon high and low waters. Of these quantities the most important are the lunitidal intervals (HWI and LWI) and the mean range of tide (Mn). The times and heights of the individual tides depart more or less from the values which would be obtained from applying the mean lunitidal intervals to the times of transits and using the mean range. The departures have a certain kind of periodicity depending upon the inequalities to which they are due, and they may be tabulated with reference to each inequality known to exist in the tide at all places; that is, so tabulated as to follow some known astronomical argument. A set of tables thus formed may be regarded as a series of corrections to be applied to the mean tides because of the several inequalities. In case of the largest of the inequalities, it is more convenient for some purposes to have tabulated the actual values of the intervals and heights throughout the period of the inequality instead of the departures from their mean values. In case of the phase inequality the "time of tide" may be tabulated directly instead of the lunitidal intervals.†

After such tables have been obtained from the observations, one may, if he choose, analyze the tabular values, thus obtaining a set of coefficients (intervals or epochs and amplitudes) from which these tables could be reproduced.

* The arguments for January 1 of each year from 1850 to 1950 are given in Table 3.

† Cf. Lubbock, Phil. Trans., 1831, pp. 398, 399. Here the time of the year (i. e., the month) is used for one of the arguments and so no correction is necessary because of the equation of time.

For examples of curves of phase inequality, see Phil. Trans., 1868, last plate.

The disadvantage of the nonharmonic treatment lies in the fact that when several considerable corrections are successively applied to the mean tides, the resulting effect is not the same as it would have been had all been applied simultaneously, as in the case of nature. It therefore becomes important to tabulate each inequality not only with respect to its own argument, but also with respect to the argument of the principal or phase inequality. This necessitates a table of double entry. In some cases it might be advisable to have tables of triple entry.

For the present we shall assume that the length of the series of observations treated is sufficient for separating the various inequalities sought. Then it is obvious that, grouping the observations with respect to the argument (uniformly varying with the time, or nearly so) of any particular inequality, we shall obtain in the long run certain departures from the mean values of the lunitidal intervals and range, which may be tabulated *without making any assumption as regards the age of the inequality sought*, and so using, say, the immediately preceding transit.

Such results are of interest for theoretical purposes. For actual prediction, however, where all important inequalities should be properly applied, it is desirable to form tables of double entry.

51. *First reduction.*

The principal object of this reduction is to determine the lunitidal intervals and the mean range of tide. A specimen of first reduction is given in § 60. This shows that, for Tybee Island Light, Georgia,

$$HWI = 7^h\ 11^m,\ LWI = 1^h\ 00^m,\ Mn = 6{\cdot}8 \text{ feet.}$$

To roughly predict the tides for Tybee Island Light (in local time), copy down the local times of the moon's transits across the meridian of Tybee Island Light, and add the intervals, $7^h\ 11^m$ for the high waters and $1^h\ 00^m$ for the low waters. To obtain predictions in eighty-first meridian time, using Greenwich transits, add the uncorrected intervals, $7^h\ 21^m$ for high waters and $1^h\ 10^m$ for low waters. Having thus determined the times, the heights with reference to half-tide level may be roughly determined by calling the high-water heights $\frac{1}{2}$ of 6·8, or $+$ 3·4, and the low-water heights, $-$ 3·4; or, referred to the particular staff used, they are on an average 9·4 and 2·6 feet, respectively, as found in the reduction.

Another specimen of first reduction, with some additional matter not important for the present purpose, is given in § 30, Part III.

52. *Determination of the periods of tidal inequalities.*

The existence of several tidal inequalities can be inferred from very simple considerations; of these may be mentioned the phase, parallax, and declinational inequalities. Others would hardly be suspected without developing the potential of the tide-producing forces. For the present we may assume such a development made in terms strictly periodic, and also assume that terms of like period are to be found in the tide. Such terms, or component tides, as go through their respective periods twice a day, pretty nearly, having the subscript 2 added to the letters used to designate them, are, as already stated, called semidiurnals; and those once a day, having the subscript 1, are called diurnals.* The number of degrees per hour, i. e. the speed, may be denoted in each instance by the corresponding small letter. Looking now at the column headed "Synodic period," Table 1, it will be seen that there are, opposite the semidiurnals, the synodic periods 14·76529, 27·55456, 13·66079, 31·81193, 15·38734, days. These are a half synodic month, an anomalistic month, a half tropical month, the moon's evectional period, etc. To this list we should add such synodic periods as would be obtained by means of the lunar nodal components differing in period very little from M_2 and K_2, respectively.† The synodic period for the first is the node-equinox period, or about 18·6 years, and the corresponding variation in the tide is called the lunar *nodal inequality.* In regions where the water is shallow and subject to annual changes, the year should be used as one of the periods. In this way may be obtained the periods of all sensible inequalities affecting the semidaily tide, i. e., those affecting both high waters of the day alike, also both low waters.

The diurnal inequality in the tide is really made up of several partial diurnal inequalities.

* See Part II for further explanations of the symbols used to designate these components.

† In Ferrel's notation $M'_{(1,2)}$ and $M'_{(3,2)}$, United States Coast and Geodetic Survey Report 1878, p. 270. The speed of the former should be 28·981898, and not 2 × 14·489846, and that of the latter 30·084344, not 2 × 15·043275.

They affect in nearly opposite ways the two high waters of a day and also the two low waters. The periods in which they should be tabulated are those synodic periods given in Table 1 which stand opposite the diurnal components. To this list of diurnal components should be added the lunar nodals having speeds nearly equal to the speeds of K_1 and O_1, respectively.* The synodic periods just referred to are such that the same transit (upper or lower) should be adhered to throughout.

It has been supposed that the inequalities have periods of fixed lengths; but the arguments which divide up the periods may not vary exactly uniformly with the time throughout these periods. Such are styled *circular arguments.* Most inequalities in the moon's motion have little effect upon the lunitidal intervals, especially if the ages of the corresponding tidal inequalities be properly allowed for in the selection of transits. The reason for this is quite obvious. With the amplitude of the tide it is otherwise. By considering the effects upon amplitude only, and disregarding the effects upon interval, it is sometimes possible, when *non-circular arguments* are used, to throw two or more inequalities into one. For instance, if the moon's parallax be the argument (preferably noting whether it is increasing or decreasing, thereby avoiding the consideration of age), the parallax and evectional inequalities referred to above will be embraced in the tabulated height or amplitude corrections. The time corrections will be somewhat uncertain. A similar remark applies to the declinational and nodal inequalities.

If we take a month's observations about the equinoxes, the diurnal inequality will be due to the moon, and so the lunar part of K_1 is involved instead of the entire K_1.

If groups of observations are taken about the times of zero declination of the moon, the diurnal inequality will be due to the sun, and so, instead of K_1, O_1, OO, the involved components are the solar part of K_1 and P_1.

In grouping observations with respect to these arguments or periods, the transits should, as a rule, be kept distinct; for, it is an easy matter to subsequently combine values belonging to the upper and lower transits if desired. For instance, when tabulating for the inequality whose period is a tropical month, we should have, opposite the moon's right ascension or longitude, two classes of high-water lunitidal intervals (approximately equal) derived from upper and lower transits, respectively, or two such classes of intervals (differing by approximately 12 hours) derived from one transit; so for the low-water lunitidal intervals. Each of these four kinds of intervals should be accompanied by a corresponding height. In this case the diurnal and semidiurnal inequality, whose period is a tropical month or a half tropical month, are capable of being tabulated together.

The half-tide level is subject to small inequalities having the periods of the preceding tidal inequalities, or some simple fractions or multiples of such periods. These fluctuations in half-tide level may be due to long-period components (which are astronomical, meteorological, or shallow-water) or to fixed speed relations in the components of short period.† The greatest of these inequalities is, at most places, the annual; it involves the components Sa, Ssa. If high water only be observed, these inequalities in half-tide level cannot be separated from the inequalities in mean high-water heights.

* In Ferrel's notation $M'_{(3,1)}$ and $M'_{(6,1)}$; l. c. ante. Their speeds are 15·043275 and 13·940829.

† For numerical examples, see United States Coast Survey Report, 1868, pp. 80, 81, column headed $\frac{1}{2}(H_1' + H_2')$.

53. *List of inequalities following a single circular argument.*

Name.	Argument.	Hourly variation of argument.	Period.	Components involved.*	Numerical examples.
	In both high waters or both low waters.	°			
Phase.	Hour of transit.†	1·0158958	½ synodic mo.	S_2, μ_2.	Lubbock, Phil. Trans., 1831, pp. 400–403, cols. A, B. Bache, United States Coast Survey Reports, 1853–1864. Ferrel, United States Coast Survey Report, 1868, p. 74, Table V.
Parallax.	Moon's anomaly.‡	0·5443746	Anomalistic mo.	N_2, L_2, 2N.	Ibid., p. 76, Table VI.
Declinational.	Moon's longitude.§	1·0980330	½ tropical mo.	K_2.	Ibid., p. 78, Table VII, left side.
Evectional.	Phase arg.-parallax arg.	0·4715212	Monthly evectional period.	ν_2, λ_2.	Ibid., p. 82, Table IX.
Lunar nodal.	Long. of node.	0·0022064	Node-equinox period.	$M'_{(1,2)}$. [See footnote, ante.]	Ibid., p. 81, Table VIII.
Annual.	Day of year.	0·0410686	Tropical year.	Component‖ having a speed $m_2 \pm \frac{360°}{\text{No. hours in a yr.}}$	Ibid., p. 80, Table VII bis.
Solar parallactic.		0·9748272	Synodic period M_2, T_2.		
	In alternate high waters or alternate low waters.				
	Moon's longitude.§	0·5490165	Tropical mo.	K_1, O_1, OO.¶	Ferrel, l. c., p. 78, Table VII, right side.
		0·4668793	Synodic period M_1, P_1.	P_1.	
		1·0933912	Synodic period M_1, Q_1.	Q_1, J_1.	
		0·5512229	Synodic period M_1, $M'_{(3,1)}$ or $M'_{(6,1)}$.	$M'_{(3,1)}$, $M'_{(6,1)}$.	

* I. e., besides the mean tide, which consists chiefly of M_2. (See Tables 1, 2.)

† Strictly speaking, the apparent local time. For a table of single entry, and also a table of double entry where the season of the year is not one of the arguments, mean time can be used in reductions instead of apparent, and without corrections when the series is long. This argument is for fixing the difference between the right ascension of sun and moon. An equivalent argument is the age or phase of the moon.

‡ True or mean.

§ True or mean, or her right ascension. For short series the component 2N disturbs this inequality, because $2m_2$ is very nearly equal to $2n + k_2$.

‖ Shallow water meteorological.

¶ If we take a month's observations about the equinoxes, the diurnal inequality will be due to the moon, and so the lunar part of K_1 is involved instead of the entire K_1. If groups of observations are taken about the times of zero declinations of the moon, the diurnal inequality will be due to the sun, and so instead of K_1, O_1, OO, the involved components are the solar part of K_1 and P_1.

54. *Analysis of tidal inequalities following a single circular argument.*

If the lunitidal intervals and heights be classified according to an argument x, whose period is that of some tidal inequality, the resulting interval and amplitude may, according to Fourier's theorem, be written

$$B_0+M'_2 \sin x+N'_i \cos x+M'_{ii} \sin 2x+N'_{ii} \cos 2x+ \quad . \quad . \quad . \quad , \tag{133}$$

$$\tfrac{1}{2}\,\mathrm{Mn}+M_i \cos x+N_i \sin x+M_{ii} \cos 2x+N_{ii} \sin 2x+ \quad . \quad . \quad . \quad , \tag{134}$$

where B_0 denotes the mean lunitidal interval, Mn the mean range of tide, and i the characteristic of the inequality. These expressions may be written in the form

$$B_0+B_i \sin (x-\varepsilon_i)+B_{ii} \sin (2x-\varepsilon_{ii})+ \quad . \quad . \quad . \quad , \tag{135}$$

$$\tfrac{1}{2}\,\mathrm{Mn}\,[1+R_i \cos (x-\alpha_i)+R_{ii} \cos (2x-\alpha_{ii})+ \quad . \quad . \quad . \quad]. \tag{136}$$

$$B_i = \pm\sqrt{M'^2_i + N'^2_i} = \frac{M'_i}{\cos\varepsilon_i},\ B_{ii} = \pm\sqrt{M'^2_{ii} + N'^2_{ii}} = \frac{M'_{ii}}{\cos\varepsilon_{ii}},\ \ldots, \tag{137}$$

$$\tan\varepsilon_i = \frac{-N'_i}{M'_i},\ \tan\varepsilon_{ii} = \frac{-N'_{ii}}{M'_{ii}},\ \ldots; \tag{138}$$

$$\tfrac{1}{2}\,\mathrm{Mn}\ R_i = \pm\sqrt{M_i^2 + N_i^2} = \frac{M_i}{\cos\alpha_i},\ \tfrac{1}{2}\,\mathrm{Mn}\ R_{ii} = \pm\sqrt{M_{ii}^2 + N_{ii}^2} = \frac{M_{ii}}{\cos\alpha_{ii}}\ \ldots, \tag{139}$$

$$\tan\alpha_i = \frac{N_i}{M_i},\ \tan\alpha_{ii} = \frac{N_{ii}}{M_{ii}}\ \ldots \tag{140}$$

In reducing tides, the observations are taken in groups. For this reason the coefficients B_i, R_i, and B_{ii}, R_{ii}, as determined above, should be multiplied by the factors (a little greater than unity)

$$\frac{x_q - x_p}{2\sin\frac{1}{2}(x_q - x_p)} \text{ and } \frac{x_q - x_p}{\sin(x_q - x_p)} \tag{141}$$

where x_p and x_q are the values of x at the two limits of the group of observations.

B_i or R_i may be spoken of as the *coefficient of the inequality* whose characteristic is *i*.

To find B_0, M_i', N_i'; M_{ii}', N_{ii}', . . . from the tabulated or classified intervals, we suppose x taken to, say, each 15°. Then we have 24 observation equations for determining B_0, M_i', etc. Let expression (133) be denoted by y_x'; then these 24 intervals are y'_0, y'_{15}, y'_{30}, . . . y'_{345}. It is not difficult to show that the most probable values of the required quantities are given by the equations

$$\begin{aligned}
24\,B_0 &= y'_0 + y'_{15} + y'_{30} + \ \ldots\ + y'_{345},\\
12\,M'_i &= y'_0 \sin 0° + y'_{15}\sin 15° + y'_{30}\sin 30° + \ \ldots\ + y'_{345}\sin 345°,\\
12\,M'_{ii} &= y'_0 \sin 0° + y'_{30}\sin 30° + y'_{60}\sin 60° + \ \ldots\ + y'_{330}\sin 330,\\
12 &= M'_{iii} y'_0 \sin 0° + y'_{45}\sin 45° + y'_{90}\sin 90° + \ \ldots\ + y'_{315}\sin 315,\\
&\ldots\ldots\ldots\ldots;\\
12\,N'_i &= y' \cos 0 + y'_{15}\cos 15° + y'_{30}\cos 30° + \ \ldots\ + y'_{345}\cos 345°,\\
&\ldots\ldots\ldots\ldots
\end{aligned} \tag{142}$$

For finding $\frac{1}{2}$ Mn, M_i, N_i, M_{ii}, N_{ii}, . . . , we have the above equations with all accents dropped, sines and cosines interchanged, and $\frac{1}{2}$ Mn in the place of B_0. (See §§ 49, 55.)

If we suppose that the inequality has been analyzed and certain epochs and amplitudes obtained, it is to be noted that the epochs (with single subscript) of the inequalities should be proportional to the respective ages of the latter. This implies that if an astronomical argument be taken out, on an average, a constant amount earlier or later than the particular phase of the tide (generally midway between high and low water) to which the analysis refers, the epochs must be altered by the respective variations in the arguments during this constant interval; i. e., the position of the tidal inequality is required at the time at which the astronomical argument is taken; or, what amounts to the same thing, the value of the astronomical argument is required at the phase of the tide with respect to which the analysis is carried out. Suppose high-water heights to be analyzed; they belong, on an average, to a time a constant number of hours, HWI (uncorrected), after the moon's transit. Suppose the ranges analyzed; they belong to a time $\frac{1}{2}$ (HWI + LWI) after the moon's transit. For the convenience of many places, all astronomical arguments can be taken out at the times of the moon's transit across, say, the meridian of Greenwich, and the epochs afterwards altered accordingly. The epochs require no alteration if the astronomical arguments be taken out at the particular phase of the tide to which the analysis refers. This latter method, however, seems particularly undesirable, if not practically impossible, in the case of the diurnal inequalities.*

*The fourth sentence from the end of § 46, Part III, is meaningless, and should be replaced by something like the following, viz.: The epochs α_i, α_{ii}, or ε_i, ε_{ii}, belonging to a tidal inequality whose characteristic is *i*, and which depend, in a measure, upon the phase of the tide analyzed, must be so modified as to suit the time at which the astronomical argument is taken; e. g., the time of transit across some given meridian, or the time of high water.

55. *Example.*

Required R_1 and α_1 for Sitka, from the month of observations tabulated in § 30, Part III.

Copy down the transits and the high and low waters, marking, for distinction, the heights which go with the lower transits. Leave no vacancies, but bring consecutive transits into consecutive positions, i. e., upon consecutive lines; similarly for the heights. This necessitates at times a splitting up of the date at left-hand margin. Combine the ranges in pairs, for the purpose of eliminating the diurnal inequality, and copy down the time of transit corresponding to each four tides so taken.

Thus:

Date.	Transit.		HW	LW	Range.
July 1	1 34	1 34	(14·6)	3·4	7·8
1	(13 59)	1 59	12·8	(8·4)	7·4
2	2 24	2 24	(14·4)	3·9	7·6
2	(14 48)	2 48	13·0	(8·3)	7·4
3	3 11	3 11	(14·0)	4·0	7·5
3	(15 34)	3 34	12·8	(7·8)	7·0
4	3 37	3 37	(13·2)	4·3	7·1
4	(16 19)	4 19	12·9	(7·6)	6·4
5	4 41	4 41	(12·5)	5·0	6·8
5	(17 04)	5 04	13·2	(7·2)	6·0
6	5 26	5 26	(12·0)	5·9	6·6
6 and 7	(17 48)	5 48	13·6	(6·6)	....

Distribute these combined ranges according to their hour of transit, using always the nearest whole hour and counting the hour from 0 to 12. Take the sums and means of the values so distributed and determine R_1 and α_1 by equations (136), (139), and (140); or, for convenience, make use of the form labeled "Harmonic analysis of tides," Part II, § 61. The angle (α_1 or ζ) thus determined approximately represents, when converted into time at 1°·016 per hour, the time which must elapse between new or full moon and spring tides. This must be corrected for the mean lunitidal interval, and, when the series is short, for the equation of time.

For Sitka, $\frac{1}{2}$ (HWI + LWI) = $9^h\ 44^m{\cdot}8$; and uncorrected Mn = 7·31 feet, from "first reduction." The analysis gives $\alpha_1 = 26°{\cdot}7$; the hourly variation in the angle of the semimenstrual inequality is 1°·016, ∴ 1°·016 × 9·747 = 9°·9; and α_1 becomes 36°·6. The correction for not having used apparent time is — 0·483 (equation of time). $\left[0{\cdot}483 = \dfrac{24}{\text{lunar day} - \text{solar day}} \times 1{\cdot}016.\right]$ For July 1–29 the equation of time is $+ 5^m{\cdot}4$; and so the correction to α_1 is — 2°·6. This leaves 34°·0 for the true value of α_1. In the harmonic notation, Part III, § 47,

$$\alpha_1 = S_2° - M_2°; \tag{143}$$

∴ Age of the phase inequality = 0·984 × 34 = 33·5 hours.

Upon applying the augmenting factor 1·0211, the $R_1\ \frac{1}{2}$ Mn from the analysis is 0·968 feet. The factor $F_2 = 1{\cdot}36$, Table 33, will very nearly reduce $R_1\ \frac{1}{2}$ Mn to its mean value. In the harmonic notation

$$R_1\ \tfrac{1}{2}\,\text{Mn} = S_2 + \mu_2. \tag{144}$$

Now, μ_2 is 0·01 Mn, nearly; ∴ uncorrected S_2 = 0·968 — 0·073 = 0·895 foot, and corrected S_2, 0·895 × 1·36, = 1·217 feet. To correct $R_1\ \frac{1}{2}$ Mn, add to 0·968, (1·217 — 0·895). ∴ $R_1\ \frac{1}{2}$ Mn = 1·290, R_1 = 1·278 ÷ 3·655 = 0·350.

If we use an argument varying uniformly with time, it is not necessary to have recourse to a nautical almanac. The (constant) period of the inequality is divided into 12 or 24 equal parts, and 0^h of the first day of the series is taken as origin; periods and twelfths or twenty-fourths of periods are laid off thereafter. The times of transits corresponding to the various ranges are dis-

tributed among the parts as accurately as possible. The results are then brought together and analyzed, thus giving an amplitude (R_iMn) and angle (α_i); α_i should then be altered by the speed of the inequality times the mean (uncorrected) interval. The corrected α_i denotes, when divided by the speed of the inequality, the time elapsing between 0^h of the first day of the series and the time when the inequality becomes maximum. This time, diminished by the time when the corresponding forces become a maximum, is the age of the inequality in question.[a]

56. *List of inequalities following a pair of circular arguments.*

Name.	Arguments.	Components involved.*	Numerical examples.
	In both high waters or both low waters.		
	{Hour of transit† (upper and lower) / Moon's anomaly	N_2, L_2, $2N$, ν_2 λ_2	Ferrel, United States Coast Survey Report, 1868, pp. 69–71.
	{Hour of transit (upper and lower) / Day of year ‡	K_2, T_2, R_2, and comp'ts, having speeds $m_2 \pm \frac{360°}{\text{No. hours in yr.}}$	Lubbock, Phil. Trans., 1831, pp. 400–403, 412 Ferrel, United States Coast Survey Report, 1868, pp. 61–68.
	{Hour of transit (upper and lower) / Long. of node	$M'_{(1,2)}$, $M'_{(3,2)}$	
	In alternate high waters or alternate low waters.		
	{Hour of transit (upper or lower) / Day of year ‡	K_1, O_1, P_1, OO	Lubbock, Phil. Trans., 1836, pp. 65–73, 245–254. Bache, United States Coast Survey Report, 1854–1864. Ferrel, United States Coast Survey Report, 1868, pp. 61–68, 100, 101.
	{Hour of transit (upper or lower) / Moon's anomaly	Q_1, J_1, M_1, $2Q$, ρ_1 §	
	{Hour of transit (upper or lower) / Long. of node.	$M'_{(3,1)}$, $M'_{(6,1)}$	

* I. e. besides the mean tide with phase inequality or M_2, S_2, and μ_2. If a component have a synodic period with M_2, S_2 (for semidiurnals, M_1, S_1 for diurnals), or any component involved in the inequality defined by either separate argument, equal to the period of either separate argument or the synodic period of the two arguments or the synodic period with M_2, S_2 (or M_1, S_1) of any other involved component, such component is involved in the tabular values.

† See second footnote, preceding list of inequalities.

‡ Or moon's right ascension, longitude, the number of days from extreme declination of the moon, longitude from intersection of orbit and equator, etc.

§ 2 Q, ρ_1 are here used to denote components whose speeds are 12·8542862, 13·4715144, respectively. See Table 1.

57. *On the use of noncircular arguments.*

Tidal inequalities depend upon the difference in right ascension of sun and moon (which is usually given by the hour of transit), their parallaxes, and their declinations. Unless the ages of the inequalities be properly allowed for by selecting suitable transits, it is necessary to make a distinction between increasing and decreasing parallax or declination. These arguments are naturally suggested by the equilibrium theory of tides.

A set of tables each having the hour of transit as one argument and parallax or declination of the sun or moon for the other, is given in the Tide Tables for the British and Irish Ports, issued by the commissioners of the admiralty; the daily predictions therein published are obtained by

---

[a] E. g., suppose $i=2$ denotes the parallax inequality. Then

$$\text{corrected } \alpha_2 + \arg_0 M_2 - \arg_0 N_2 = M_2^\circ - N_2^\circ, \tag{145}$$

where L_2 is disregarded. From the amplitude

$$\frac{R_2\text{Mn} \times 1\text{·}0138}{f(N_2) - \frac{1}{7} f(L_2)} = 2\, N_2. \tag{146}$$

aid of this set of tables.[a] The tides around the British Isles have very little diurnal inequality, and so are susceptible of this simple mode of treatment. See §§ 4–7.

58. *Inference of tidal inequalities from observed nonharmonic constants, etc.*

Owing to the great amount of labor involved in completely deducing tidal inequalities from observations, it does not seem advantageous to work along that line for obtaining practical results, although it may be desirable for certain theoretical purposes. In fact the results of discussions covering a long period of time (say a node-equinox period) are always instructive.

If, on the other hand, we confine our attention to determining a few quantities which fix the size and position of each important inequality, a general knowledge of tides ought to enable us to form tables, based upon these determinations, which are sensibly true at most places. For instance, if the spring, mean, and neap ranges, also the age of the phase inequality are known, the phase inequality in interval and range become approximately known at any given time by means of the following general table. The two columns at the right enable one to approximately introduce the declinational and solar parallax inequalities. The second table shows the lunar parallax inequality.[b]

Table of phase effects.

Time.	d.	h.	Increase in lunitidal intervals. m.	Increase in semirange of tide.	Time.	d.	h.	Increase in lunitidal intervals. m.	Increase in semirange of tide.	Date.	Factor p.*
After spring tides.	0	00	$0 \frac{Sg-Np}{Mn \times q}$	$+.23p(Sg-Np)$	After neap tides.	0	00	$0 \frac{Sg-Np}{Mn \times q}$	$-.29p(Sg-Np)$	Jan. 1	0.82
	0	06	−5 〃	+.23 〃		0	06	+13 〃	−.29 〃	11	0.88
	0	12	−10 〃	+.23 〃		0	12	+25 〃	−.28 〃	21	0.96
	0	18	−14 〃	+.22 〃		0	18	+35 〃	−.27 〃	31	1.04
	1	00	−19 〃	+.21 〃		1	00	+44 〃	−.25 〃	Feb. 10	1.13
	1	06	−23 〃	+.20 〃		1	06	+52 〃	−.23 〃	20	1.20
	1	12	−28 〃	+.19 〃		1	12	+58 〃	−.21 〃	Mar. 2	1.25
	1	18	−32 〃	+.18 〃		1	18	+62 〃	−.18 〃	12	1.27
	2	00	−37 〃	+.17 〃		2	00	+65 〃	−.16 〃	22	1.28
	2	06	−41 〃	+.15 〃		2	06	+66 〃	−.13 〃	Apr. 1	1.26
	2	12	−44 〃	+.13 〃		2	12	+67 〃	−.10 〃	11	1.22
	2	18	−49 〃	+.11 〃		2	18	+67 〃	−.08 〃	21	1.14
	3	00	−52 〃	+.09 〃		3	00	+66 〃	−.05 〃	May 1	1.06
	3	06	−56 〃	+.07 〃		3	06	+64 〃	−.02 〃	11	0.96
	3	12	−59 〃	+.04 〃		3	12	+62 〃	.00 〃	21	0.87
	3	18	−61 〃	+.02 〃		3	18	+60 〃	+.03 〃	31	0.77
	4	00	−63 〃	−.01 〃		4	00	+57 〃	+.05 〃	June 10	0.71
										20	0.67
										30	0.68
										July 10	0.74
										20	0.82
Before neap tides.	4	00	−57 〃	+.05 〃	Before spring tides.	4	00	+63 〃	−.01 〃	30	0.92
	3	18	−60 〃	+.03 〃		3	18	+61 〃	+.02 〃	Aug. 9	1.01
	3	12	−62 〃	.00 〃		3	12	+59 〃	+.04 〃	19	1.10
	3	06	−64 〃	−.02 〃		3	06	+56 〃	+.07 〃	29	1.18
	3	00	−66 〃	−.05 〃		3	00	+52 〃	+.09 〃	Sept. 8	1.23
	2	18	−67 〃	−.08 〃		2	18	+49 〃	+.11 〃	18	1.26
	2	12	−67 〃	−.10 〃		2	12	+44 〃	+.13 〃	28	1.26
	2	06	−66 〃	−.13 〃		2	06	+41 〃	+.15 〃	Oct. 8	1.24
	2	00	−65 〃	−.16 〃		2	00	+37 〃	+.17 〃	18	1.20
	1	18	−62 〃	−.18 〃		1	18	+32 〃	+.18 〃	28	1.14
	1	12	−58 〃	−.21 〃		1	12	+28 〃	+.19 〃	Nov. 7	1.06
	1	06	−52 〃	−.23 〃		1	06	+23 〃	+.20 〃	17	0.97
	1	00	−44 〃	−.25 〃		1	00	+19 〃	+.21 〃	27	0.89
	0	18	−35 〃	−.27 〃		0	18	+14 〃	+.22 〃	Dec. 7	0.83
	0	12	−25 〃	−.28 〃		0	12	+10 〃	+.23 〃	17	0.79
	0	06	−13 〃	−.29 〃		0	06	+5 〃	+.23 〃	27	0.80
	0	00	0 〃	−.29 〃		0	00	0 〃	+.23 〃	Jan. 6	0.85

*The factor p applies to the "increase in semirange of tide," and not to the "increase in lunitidal intervals." It is due to the declinations of the sun and moon and to the solar parallax.

[a] In these the phase inequality is obtained from observation. The inequalities due to parallax and declination of sun and moon are in accordance with Bernoulli's equilibrium theory. They are as tabulated by Lubbock in the Philosophical Transactions for 1836, pp. 58, 59, 257–262. For Devonport these, too, are based upon observations; of course the table for correction for moon's declination would naturally, in this case, involve that due to the sun.

[b] These tables are based upon Tables 24, 25, and 31.

Factor expressing the effect of the moon's parallax upon the mean range of tide.

Time.		Factor q.	Time.		Factor q.	Time.		Factor q.	Time.		Factor q.
	d.			d.			d.			d.	
After perigean tides.	0	1·17	Before apogean tides.	7	0·99	After apogean tides.	0	0·86	Before perigean tides.	7	0·98
	1	1·16		6	0·96		1	0·86		6	1·02
	2	1·15		5	0·93		2	0·87		5	1·06
	3	1·13		4	0·90		3	0·88		4	1·09
	4	1·09		3	0·88		4	0·90		3	1·13
	5	1·06		2	0·87		5	0·93		2	1·15
	6	1·02		1	0·86		6	0·96		1	1·16
	7	0·98		0	0·86		7	0·99		0	1·17

In the column headed "Increase in lunitidal intervals" the negative values are often spoken of as the *priming* and the positive ones as the *lagging* of the tide.

The *vulgar establishment*, being the interval at "full and change," may be obtained from the mean lunitidal interval by entering the first table as many hours before spring tides as are contained in the age of the phase inequality.

In making use of these tables for prediction purposes, the mean range (Mn) should be first multiplied by the factor q expressing the parallax effect; this corrected range should then be used in ascertaining the variation due to phase in the lunitidal interval and in obtaining the semirange of tide.

If several "tables of phase effects" like the above be constructed with as many assumed values of S_2/M_2, or of $(Sg - Np)/2Mn$, the accuracy of the results will be somewhat increased. This table is based upon Table 24, wherein S_2/M_2 was taken as 0·46531.

Of course this table of parallax effect can be replaced by a more accurate one involving the perigean and apogean ranges (Pn, An); but to realize this increased accuracy these ranges would have to be known from observation, and the age of the parallax inequality should also be observed and not assumed to be that of the phase inequality.

A general table giving the diurnal inequality at any time, and involving certain nonharmonic constants, would hardly be feasible, because of the numerous arguments involved.* Nevertheless, constants like the tropic high-and low-water inequalities in height (HWQ, LWQ), the corresponding intervals, the great and small tropic ranges (Gc, Sc) are serviceable in describing the character of the tide.

In Chapter III, Part III, a scheme is given for the determination of the following nonharmonic quantities:

HWI	Mn
LWI	Gc
Tropic HHWI	Sc
Tropic LHWI	Gt
Tropic LLWI	Sl
Tropic HLWI	Sg
Age of phase inequality	Np
Age of parallax inequality	Pn
Age of diurnal inequality	An
	HWQ
	LWQ
	LW (on staff)
	Tropic LLW (on staff)
	Mean LLW (on staff)

In the determination of these, the portion of the work relating exclusively to harmonic constants may, of course, be omitted.

In the reductions for finding the spring range, and especially that for finding the neap (Part III, §§ 29, 30), it is important to know the exact age of the inequality and to then use a group no

* See §§ 66, 67, Part III.

more extensive than is necessary, say four tides in case of a long series and eight, for a short series. For this reason the age should be found by the inequality method, § 55 (especially where there are shallow water components), instead of by noting when the interval has its mean value.

In order to obtain good results and to avoid the labor of making certain corrections, series six or twelve months in length should be used. If the harmonic constants have been determined from hourly ordinates or from high and low waters in accordance with Part II, the ages derived from them may be used to advantage in taking out the above quantities. In all cases the longer the series the narrower may be the group, and so the more definite the quantities found.

The nonharmonic constants given in the Tide Tables by the United States Coast and Geodetic Survey are

$$\begin{gathered}\text{HWI, LWI, tropic HHWI, tropic LLWI,}\\ \text{Mn, Sg, Np, Gc, HWQ, LWQ, } \mathrm{D_1HWI},\\ 2\mathrm{D_1}\text{, mean sea level above tropic LLW.}\end{gathered} \tag{147}$$

$2D_1$ denotes the tropic range of the diurnal wave and D_1HWI its lunitidal interval. The quantities ranking next in importance to those given are probably the ages of the phase and diurnal inequalities.

These can be compared and other nonharmonic constants supplied by aid of the following exact or approximate theoretical relations between the various ranges, intervals, etc., obtained, directly or indirectly, from Part III:

$$2\,\mathrm{Mn} = \mathrm{Sg} + \mathrm{Np} + \tfrac{1}{4}\frac{(\mathrm{Sg}-\mathrm{Np})^2}{\mathrm{Sg}+\mathrm{Np}}. \tag{148}$$

$$2\,\mathrm{Mn} = \mathrm{Gt} + \mathrm{Sl}. \tag{149}$$

$$\mathrm{Gc} - \mathrm{Sc} = \mathrm{HWQ} + \mathrm{LWQ}. \tag{150}$$

$$\mathrm{Gt} = \tfrac{3}{4}\,\mathrm{Gc} + \tfrac{1}{4}\,\mathrm{Mn}, \text{ or } \mathrm{Mn} + \tfrac{1}{3}\,(\mathrm{HWQ} + \mathrm{LWQ}); \tag{151}$$

the former to be used where the greater inequality equals or exceeds, say, $\frac{1}{4}$ Mn; the latter, when both inequalities are small.

The depression of average lower low water below mean low water is

$$\frac{\mathrm{LWQ}}{3} + \frac{1}{\mathrm{LWQ}}\left[\frac{\mathrm{Gc}-\mathrm{Mn}}{5}\right]^2, \text{ when LWQ} > \text{HWQ}; \tag{152}$$

or

$$\mathrm{Gt} - \mathrm{Mn} - \frac{\mathrm{HWQ}}{3} - \frac{1}{\mathrm{HWQ}}\left[\frac{\mathrm{Gc}-\mathrm{Mn}}{5}\right]^2, \text{ when HWQ} > \text{LWQ}. \tag{153}$$

$$\tfrac{1}{2}\,\mathrm{Mn} = \text{depression of mean low water below mean sea level.} \tag{154}$$

$$\tfrac{1}{2}\,\mathrm{Sg} = \text{depression of low-water springs below mean sea level.} \tag{155}$$

$$2\,\mathrm{D_1} = \sqrt{\mathrm{HWQ}^2 + \mathrm{LWQ}^2}. \tag{156}$$

$$\mathrm{HWI} - \mathrm{LWI} = \text{duration of rise,} \tag{157}$$

$$\mathrm{LWI} - \mathrm{HWI} = \text{duration of fall,} \tag{158}$$

adding $12^h\,25^m$ when necessary to make the result positive.

$$\text{Tropic HHWI} + \text{tropic LHWI} + \text{tropic LLWI} + \text{tropic HLWI} = 2\,(\mathrm{HWI} + \mathrm{LWI}); \tag{159}$$

and, less accurately,

$$\text{Tropic LHWI} = 2\,\mathrm{HWI} - \text{tropic HHWI}, \tag{160}$$

$$\text{Tropic HLWI} = 2\,\mathrm{LWI} - \text{tropic LLWI}. \tag{161}$$

$$\mathrm{Mc} = \tfrac{1}{2}\,(\mathrm{Gc} + \mathrm{Sc}). \tag{162}$$

$$Gc = Mc + \tfrac{1}{2}(HWQ + LWQ). \tag{163}$$

$$\text{Tropic LLW below MSL} = \tfrac{1}{2}\,Mc + \tfrac{1}{2}\,LWQ, \text{ nearly.} \tag{164}$$

$$\text{Indian harmonic tide plane below MSL} = 0{\cdot}49\,(Sg + 2\,D_1). \tag{165}$$

59. *Prediction direct from observation.*

Suppose that a person start with a blank book containing as many pages as there are days in a year and labeled accordingly. Suppose the page ruled into 12 (or 24) columns, one for each possible degree of the moon's declination, viz.. 18° to 29°. Let the page be divided horizontally into 15 equal strips, one for each day of the moon's age, reckoned from full moon as well as from new.

In each of the rectangles thus formed let the observed heights and lunitidal intervals (obtained from the observed times and properly marked) be recorded. If observations be made throughout the node-equinox period, or about nineteen years, and thus tabulated, predictions can be made by referring to this record in the following manner:

Turn to the day of the year for which predictions are required. Select such tabular values as correspond most nearly to the age and decliuation of the moon for the given day. If no such tabular value can be found upon the page for the day in question, go a few days forward or backward until a tabular value is found which has very nearly the required arguments. The lunitidal intervals there tabulated give, when applied to the moon's transits for the day in question, the times of the tides. The heights are the tabular values unaltered. If in making the tabulation the effect of parallax upon the height of the tide is allowed for, it may be roughly introduced, when predictions are required, by multiplying the range of the tide by the cube of the value of the parallax for the day divided by the cube of its mean value, or by making use of the second table of § 58.

The moon's transits, declination, age, and parallax, when used, are supposed to be taken from the Nautical Almanac.

A very convenient method for obtaining fairly good predictions at a station having either great or small diurnal inequality, is the following:

Take a year's observed high and low waters, or preferably a year's accurate predictions, and copy down alongside each date the time of the moon's superior transit. To predict for any given day of any year, use that part of the year tabulated which is about the same season of the year as is the given date. Find a day having, as nearly as may be, the same hour of transit as has the given day. The times and heights of the tides for that day will be approximately the values required.

If only very rough predictions are required, all inequalities may be omitted. In this case it is sufficient to know the high-water lunitidal interval and the mean range of tide. This has been explained in § 51.

In regions where the diurnal inequality is small the page should be divided with reference to the various values of the moon's parallax instead of declination.

60. *Phase reductions.*

Along the coast of Europe and the Atlantic coast of America, the region of the West Indies excepted, the tide is of the semidiurnal type. Consequently, the greatest inequality is the phase or semimenstrual, due to the sun. This is more especially true of times than of heights.

If a person has secured a month of observations upon high and low water for any place along these coasts, he can, with very little computation, obtain reasonably good predictions. He has only to tabulate the tides, along with the times of the moon's transits, take the lunitidal intervals (using the transit immediately preceding) and distribute the intervals and heights according to the hour of transit, going from 0 to 12. This distribution constitutes a "second" or "phase reduction," an example of which is given below. Or, he may tabulate the times and heights without using intervals, according to the hour of the moon's transit. In the latter case the time of the tide can be ascertained without even adding the interval to the time of transit, because the times of the tide are supposed to be tabulated with reference to the hour of transit as an argument.

If the observations are less than a year in extent, their times, as well as the times of the transits, should, strictly speaking, be changed into apparent time before making a phase reduction; or the average value of the equation of time (Table 30) for the period of observations may

be substracted from the "hour of transit argument" of the reduction results, thus obtaining the same prediction table as before. The predictions made therefrom are, of course, in apparent time and must be changed into mean time by Table 30.

On the Atlantic coast of the United States, the phase inequality being small, the kind of time used is quite immaterial, mean time being almost as satisfactory as apparent. The prediction table got out in the kind of mean time used in the observations can be adapted to another kind of mean time by applying a constant (equal to the difference between the two kinds of time) to the tabular values.

An example of the method of making a "first" and a "phase" (or "second") reduction, is given for Tybee Island Light, Savannah River Eutrance, Georgia, for May 1–29, 1891. As a matter of convenience, the Greenwich time of the moon's transit is taken directly from the Nautical Almanac, merely changing astronomical to civil time. The observations were made in eighty-first meridian time, which is $5^h\ 24^m$ west, while the local meridian is $5^h\ 23^m{\cdot}37$ west. In tabulating the observations two lines are given to each day. All hours less than 12 are in the morning; all greater are in the afternoon, and when diminished by 12 give the usual reckoning; for instance, 15^h is 3 p. m.

Tybee Island Light, Georgia.

First reduction.

Date.	Moon's transits.	Time of— HW		Time of— LW		Lunitidal interval. HW		Lunitidal interval. LW		Height of— HW	Height of— LW	Remarks.
1891.	*h.* *m.*	*h.*	*m.*	*h.*	*m.*	*h.*	*m.*	*h.*	*m.*	*feet.*	*feet.*	
	(17 35)											Lat. = 32° 01′ 20″ N.
May 1	6 04	0	39	7	02	(7	04)	0	58	9·3	2·5	Long. = 80° 50′ 37″ = $5^h\ 23^m{\cdot}37$ W.
	(18 33)	13	10	19	20	7	06	(0	47)	8·3	2·7	
2	7 01	1	45	8	03	(7	12)	1	02	9·1	2·2	
	(19 28)	14	20	20	27	7	19	(0	59)	9·0	2·7	
3	7 55	2	45	9	05	(7	17)	1	10	9·3	2·0	
	(20 21)	15	07	21	40	7	12	(1	19)	9·5	2·6	
4	8 47	3	52	10	04	(7	31)	1	17	9·5	1·7	Observations in eighty-first meridian time
	(21 12)	16	05	22	45	7	18	(1	33)	10·0	2·5	= $5^h\ 24^m$
5	9 37	4	53	11	05	(7	41)	1	28	9·7	1·6	
	(22 02)	17	12	23	30	7	35	(1	28)	10·4	2·3	
6	10 28	5	55	....		(7	53)	....		9·8	......	
	(22 53)	18	20	12	04	7	52	1	36	10·8	1·5	Greenwich transits
7	11 18	6	36	0	48	(7	43)	(1	55)	9·9	2·0	
	(23 44)	18	50	12	56	7	32	1	38	11·1	1·3	
8	....	7	27	1	34	(7	43)	(1	50)	9·9	1·8	
	12 11	19	35	13	36	7	24	1	25	11·4	1·2	
9	(0 38)	8	05	2	09	(7	27)	(1	31)	9·8	1·7	To convert Greenwich transits to local
	13 05	20	31	14	19	7	26	1	14	11·0	1·4	transits, observation time, add $L - S +$
10	(1 33)	9	04	3	00	(7	31)	(1	27)	9·4	1·9	$0{\cdot}035\ (L - E) = 10^m{\cdot}7$, consequently
	14 01	21	16	14	59	7	15	0	58	10·5	1·8	to correct intervals subtract $10^m{\cdot}7$
11	(2 29)	9	35	3	42	(7	06)	(1	13)	8·8	2·3	
	14 57	21	57	15	38	7	00	0	41	10·1	2·2	
12	(3 25)	10	30	4	19	(7	05)	(0	54)	8·6	2·7	
	15 52	22	35	16	25	6	43	0	33	9·8	2·6	
13	(4 19)	11	18	4	55	(6	59)	(0	36)	8·4	3·1	
	16 45	23	22	17	18	6	37	0	33	9·5	3·2	
14	(5 10)	....		6	00	....		(0	50)	......	3·4	
	17 34	12	06	18	22	(6	56)	0	48	8·2	3·6	
15	(5 58)	0	32	6	47	6	58	(0	49)	9·1	3·7	
	18 21	13	09	19	05	(7	11)	0	44	8·1	3·9	
16	(6 43)	1	33	7	34	7	12	(0	51)	9·0	3·8	
	19 04	14	05	20	15	(7	22)	1	11	8·4	4·2	
17	(7 25)	2	15	8	22	7	11	(0	57)	8·8	3·8	
	19 46	15	13	21	17	(7	48)	1	31	8·7	4·1	
18	(8 06)	3	11	9	12	7	25	(1	06)	8·8	3·5	
	20 26	15	58	22	12	(7	52)	1	46	8·9	3·6	
19	(8 47)	4	08	10	04	7	42	(1	17)	8·7	2·9	
	21 07	16	30	22	44	(7	43)	1	37	9·1	3·0	
20	(9 27)	4	40	10	54	7	33	(1	27)	8·6	2·6	HWI LWI HW LW
	21 48	17	19	23	35	(7	52)	1	47	9.4	2·8	56 56 56 56
21	(10 10)	5	31	11	41	7	43	(1	31)	8·8	2·5	*h.* *m.* *h.* *m.* *feet* *feet*
	22 32	18	00	....		(7	50)	....		9·7	......	Sums 383 1742 34 1908 530·0 146·0
22	(10 55)	6	27	0	30	7	55	1	58	8·9	2·7	Means 7 21·5 1 10·5 9·46 2·61
	23 18	18	50	12	25	(7	55)	(1	30)	10·0	2·4	

Tybee Island Light, Georgia—Continued.

First reduction—Continued.

Date.	Moon's transits.	Time of— HW	Time of— LW	Lunitidal interval. HW	Lunitidal interval. LW	Height of— HW	Height of— LW	Remarks.
1891.	*h. m.*	*h. m.*	*h. m.*	*h. m.*	*h. m.*	*feet.*	*feet.*	
May 23	(11 43)	7 00	1 10	7 42	1 52	9·0	2·6	Mn = 9·46 — 2·61 = 6·85 feet
	....	19 15	13 05	(7 32)	(1 22)	10·4	2·2	
24	0 09	7 37	1 52	7 28	1 43	9·1	2·5	Correcting intervals for transits
	(12 35)	20 02	13 50	(7 27)	(1 15)	10·5	2·2	
25	1 03	8 20	2 33	7 17	1 30	9·1	2·3	
	(13 32)	20 54	14 27	(7 22)	(0 55)	10·5	2·2	HWI = $7^h\ 11^m$
26	2 01	9 14	3 19	7 13	1 18	9·1	2·4	LWI = 1 00
	(14 30)	21 38	15 11	(7 08)	(0 41)	10·4	2·4	
27	3 00	9 58	4 00	6 58	1 00	9·0	2·6	
	(15 30)	22 30	15 46	(7 00)	(0 16)	10·3	2·6	
28	4 00	10 49	4 30	6 49	0 30	9·1	2·8	
	(16 28)	23 12	16 47	(6 44)	(0 19)	10·2	2·9	
29	4 57	11 40	5 43	6 43	0 46	9·2	3·1	
	(17 24)		18 00		(0 36)		3·2	

Tybee Island Light, Georgia.

Phase or second reduction for high water. May 1-29, 1891.

Moon's upper and lower transits.	Lunitidal interval.	Height on staff.	No. of observations.	Moon's upper and lower transits.	Lunitidal interval.	Height on staff.	No. of observations.	Moon's upper and lower transits.	Lunitidal interval.	Height on staff.	No. of observations.
h. m.	*h. m.*	*feet.*		*h. m.*	*h. m.*	*feet.*		*h. m.*	*h. m.*	*feet.*	
12 11	7 24	11·4		13 05	7 26	11·0		14 01	7 15	10·5	
(0 38)	7 27	9·8		(1 33)	7 31	9·4		(2 29)	7 06	8·8	
0 09	7 28	9·1		1 03	7 17	9·1		14 57	7 00	10·1	
(12 35)	7 27	10·5		(13 32)	7 22	10·5		2 01	7 13	9·1	
								(14 30)	7 08	10·4	
93	106	40·8	4	73	96	40·0	4	118	42	48·9	5
0 23	7 26	10·20		1 18	7 24	10·00		2 24	7 08	9·78	
(3 25)	7 05	8·6		(4 19)	6 59	8·4		(5 10)	6 56	8·2	
15 52	6 43	9·8		16 45	6 37	9·5		17 34	6 58	9·1	
3 00	6 58	9·0		4 00	6 49	9·1		5 58	7 11	8·1	
(15 30)	7 00	10·3		(16 28)	6 44	10·2		(17 35)	7 04	9·3	
				4 57	6 43	9·2					
107	226	37·7	4	149	232	46·4	5	137	09	34·7	4
3 27	6 56	9·42		4 30	6 46	9·28		5 34	7 02	8·68	
6 04	7 06	8·3		7 01	7 19	9·0		(20 21)	7 31	9·5	
(18 33)	7 12	9·1		(19 28)	7 17	9·3		8 47	7 18	10·0	
18 21	7 12	9·0		7 55	7 12	9·5		(8 06)	7 52	8·9	
(6 43)	7 22	8·4		19 04	7 11	8·8		20 26	7 42	8·7	
				(7 25)	7 48	8·7		(8 47)	7 43	9·1	
				19 46	7 25	8·8					
101	52	34·8	4	159	132	54·1	6	147	186	46·2	5
6 25	7 13	8·70		7 26	7 22	9·02		8 29	7 37	9·24	
(21 12)	7 41	9·7		(22 02)	7 53	9·8		11 18	7 32	11·1	
9 37	7 35	10·4		10 28	7 52	10·8		(23 44)	7 43	9·9	
21 07	7 33	8·6		(22 53)	7 43	9·9		23 18	7 42	9·0	
(9 27)	7 52	9·4		(10 10)	7 50	9·7		(11 43)	7 32	10·4	
21 48	7 43	8·8		22 32	7 55	8·9					
				(10 55)	7 55	10·0					
131	204	46·9	5	180	308	59·1	6	123	149	40·4	4
9 26	7 41	9·38		10 30	7 51	9·85		11 31	7 37	10·10	

Tybee Island Light, Georgia—Continued.

Phase or second reduction for low water. May 1-29, 1891.

Moon's upper and lower transits.	Lunitidal interval.	Height on staff.	No. of observations.	Moon's upper and lower transits.	Lunitidal interval.	Height on staff.	No. of observations.	Moon's upper and lower transits.	Lunitidal interval.	Height on staff.	No. of observations.
h. m.	*h. m.*	*feet.*		*h. m.*	*h. m.*	*feet.*		*h. m.*	*h. m.*	*feet.*	
12 11 (0 38) 0 09 (12 35)	1 25 1 31 1 43 1 15	1·2 1·7 2·5 2·2		13 05 (1 33) 1 03 (13 32)	1 14 1 27 1 30 0 55	1·4 1·9 2·3 2·2		14 01 (2 29) 14 57 2 01 (14 30)	0 58 1 13 0 41 1 18 0 41	1·8 2·3 2·2 2·4 2·4	
93 0 23	114 1 28	7·6 1·90	4	73 1 18	66 1 16	7·8 1·95	4	118 2 24	291 0 58	11·1 2·22	5
(3 25) 15 52 3 00 (15 30)	0 54 0 33 1 00 0 16	2·7 2·6 2·6 2·6		(4 19) 16 45 4 00 (16 28) 4 57	0 36 0 33 0 30 0 19 0 46	3·1 3·2 2·8 2·9 3·1		(5 10) 17 34 (5 58) (17 24)	0 50 0 48 0 49 0 36	3·4 3·6 3·7 3·2	
107 3 27	163 0 41	10·5 2·62	4	149 4 30	164 0 33	15·1 3·02	5	126 5 32	183 0 46	13·9 3·48	4
6 04 (18 33) 18 21 (6 43)	0 58 0 47 0 44 0 51	2·5 2·7 3·9 3·8		7 01 (19 28) 7 55 19 04 (7 25) 19 46	1 02 0 59 1 10 1 11 0 57 1 31	2·2 2·7 2·0 4·2 3·8 4·1		(20 21) 8 47 (8 06) 20 26 (8 47)	1 19 1 17 1 06 1 46 1 17	2·6 1·7 3·5 3·6 2·9	
101 6 25	200 0 50	12·9 3·22	4	159 7 26	50 1 08	19·0 3·17	6	147 8 29	105 1 21	14·3 2·86	5
(21 12) 9 37 21 07 (9 27) 21 48	1 33 1 28 1 37 1 27 1 47	2·5 1·6 3·0 2·6 2·8		(22 02) 10 28 (22 53) (10 10) 22 32 (10 55)	1 28 1 36 1 55 1 31 1 58 1 30	2·3 1·5 2·0 2·5 2·7 2·4		11 18 (23 44) 23 18 (11 43)	1 38 1 50 1 52 1 22	1·3 1·8 2·6 2·2	
131 9 26	172 1 34	12·5 2·50	5	180 10 30	238 1 40	13·4 2·23	6	123 11 31	162 1 40	7·9 1·98	4

Phase or second reduction for HW *and* LW. May 1-29, 1891.

RECAPITULATION.

High water.				Low water.			
Moon's upper and lower transits.	Lunitidal interval.	Height on staff.	No. of observations.	Moon's upper and lower transits.	Lunitidal interval.	Height on staff.	No. of observations.
h. m.	*h. m.*	*feet.*		*h. m.*	*h. m.*	*feet.*	
0 23	7 26	10·20	4	0 23	1 28	1·90	4
1 18	7 24	10·00	4	1 18	1 16	1·95	4
2 24	7 08	9·78	5	2 24	0 58	2·22	5
3 27	6 56	9·42	4	3 27	0 41	2·62	4
4 30	6 46	9·28	5	4 30	0 33	3·02	5
5 34	7 02	8·68	4	5 32	0 46	3·48	4
6 25	7 13	8·70	4	6 25	0 50	3·22	4
7 26	7 22	9·02	6	7 26	1 08	3·17	6
8 29	7 37	9·24	5	8 29	1 21	2·86	5
9 26	7 41	9·38	5	9 26	1 34	2·50	5
10 30	7 51	9·85	6	10 30	1 40	2·23	6
11 31	7 37	10·10	4	11 31	1 40	1·98	4
	243 7 20·2	113·65 9·47	56		115 1 09·6	31·15 2·60	56

The first reduction is made by subtracting each transit from the time of high or low water which directly follows it, thus filling out the columns headed "Lunitidal interval." If the observations had been taken in local time, and the local time of the moon's transit across the local meridian had been used, these lunitidal intervals would represent the lag of the tide behind the moon; but in this example each interval is to a certain extent fictitious, and requires a correction in order to reduce it to its true local value. As a general rule, however, there is no need of correcting each interval, for only average intervals are of use, so that the mean of all the high-water and of the low-water intervals can be corrected by applying a constant once for all, which is obtained as follows:

Let S = west longitude in time of the time meridian used.
L = " " " " station.
E = " " " " ephemeris used for transits.

Then the correction to reduce the observation time to local time is

$$S - L, \tag{166}$$

and the correction to the intervals, due to the motion of the moon in her orbit while passing from the meridian of the ephemeris to the meridian of the station, is

$$0{\cdot}035\,(E - L). \tag{167}$$

Combining both corrections, we have for the entire interval correction expressed in hours

$$S - L + 0{\cdot}035\,(E - L). \tag{168}$$

In this example $S = 5^{h}{\cdot}400$, $L = 5^{h}{\cdot}390$, and $E = 0^{h}{\cdot}000$; hence, the correction is $-10^{m}{\cdot}7$, as stated on the first reduction sheet.

The phase on second reduction is made by distributing the intervals and heights of the first reduction according to the hour of the moon's transit. All intervals for high water which were obtained by using transits occurring between $0^{h}\,00^{m}$ and $0^{h}\,59^{m}$, as also between $12^{h}\,00^{m}$ and $12^{h}\,59^{m}$, together with the corresponding heights, are thrown into one group; those intervals resulting from transits between $1^{h}\,00^{m}$ and $1^{h}\,59^{m}$, and between $13^{h}\,00^{m}$ and $13^{h}\,59^{m}$, are also collected together into one group; and similarly for each hour of the moon's transit. The same distribution is then made for the low-water intervals and heights. Each group is summed and the mean obtained by dividing by the number of observations. The twelve means, one for each hour of transit from 0 to 11, are then brought together into a tabular form, designated as a "Recapitulation," and the mean of all obtained by dividing the sum by twelve. Before using these final mean intervals as the establishment for high or low water, they must be corrected in the same manner as the first reduction results were, which in this case is done by subtracting 10·7 minutes from each.

61. *Rough prediction of tides from phase reduction.*

The values contained in the recapitulation of the phase reduction may be represented graphically by plotting them upon profile paper and joining the points thus given with a curved or broken line. (See Fig. 8.) These curves can be used for making approximate predictions of the tide at the station, but their irregularities should first be smoothed out by estimation of the eye, or preferably by means of the Fourier series, using only those terms of the series whose coefficients have the subscripts 1 and 2, and the even-numbered hours, in the form given in § 71, Part II. In order to make this method of prediction clear, an example is given of the predicted times and heights of high and low waters at Tybee Island Light, Georgia, on the first five days of January, 1898, using the uncorrected curves of Fig. 8, although the results must be rougher than those which might have been obtained by aid of § 55.

In this example the Greenwich times of the moon's transits are taken from the Ephemeris or Nautical Almanac, merely changing the time into civil reckoning. The curves are then read at the times of transit, furnishing the values given in the lines marked HWI (high-water interval) and LWI (low-water interval). These readings of the interval curves added to the time of transit give the approximate times of high and low water. The height curves are also read at the times of transit. No distinction is made between upper and lower transits in this method of prediction; and whenever the hours of transit exceed twelve they are reckoned as applying to the curves at a time twelve hours earlier, but the real value of the time of transit is set down in the computation.

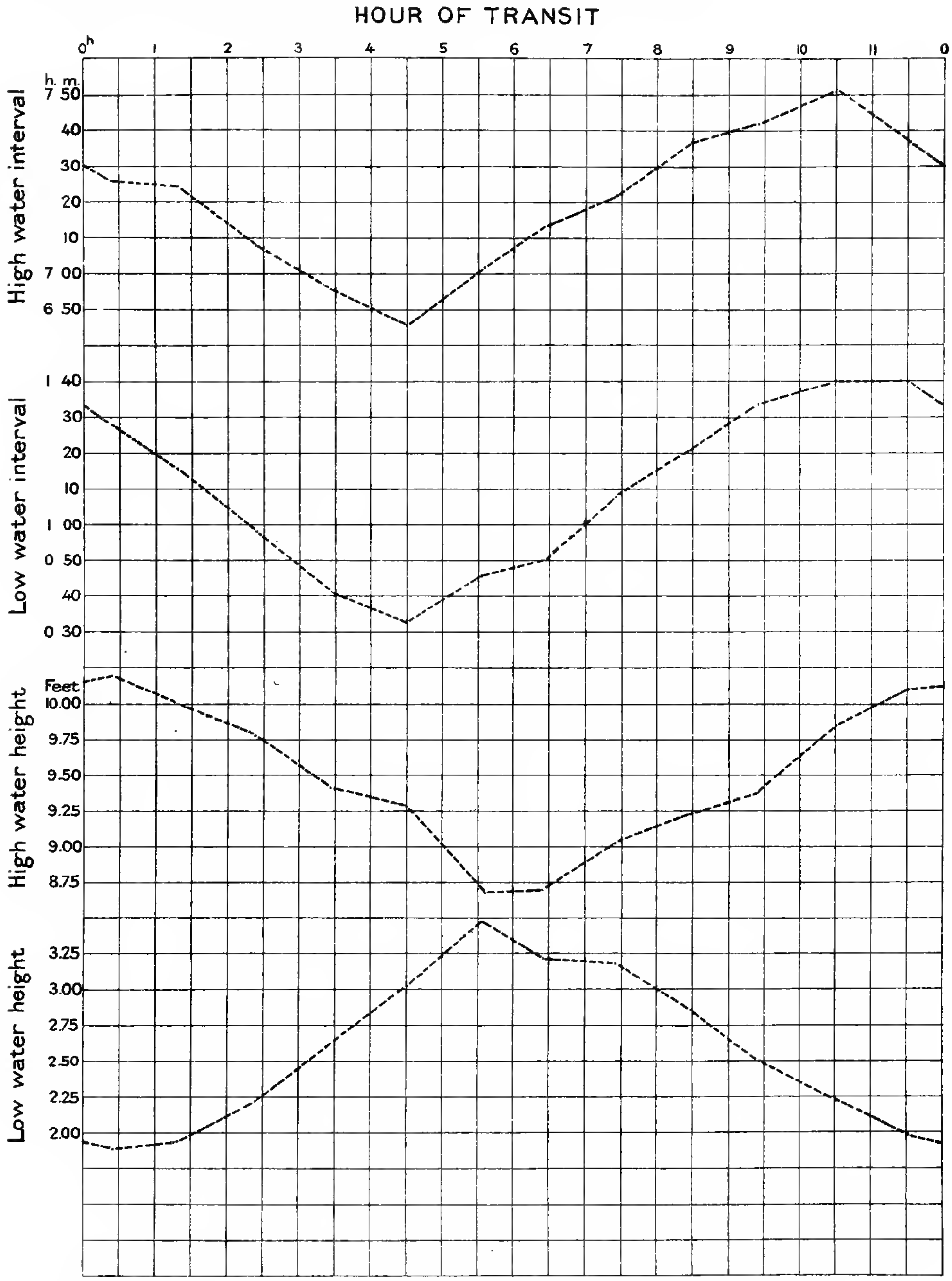


Fig. 8. Phase inequality, Tybee Id. Light, Ga.

Tybee Island Light, Georgia.

PREDICTION OF TIDES FROM PHASE REDUCTION.

	Dec., 1897.	January, 1898.				
	31	1	2	3	4	5
	h. m.	*h. m.*	*h. m.*	*h. m.*	*h. m.*	*h. m.*
HWI	7 14	7 20	7 29	7 38	7 43	7 50
☽'s upper transit	18 29	19 13	19 59	20 46	21 36	22 26
LWI	0 51	1 05	1 14	1 25	1 35	1 40
Time of HW	25 43	26 33	27 28	28 24	29 19	30 16
Time of LW	19 20	20 18	21 13	22 11	23 11	24 06
	ft.	*ft.*	*ft.*	*ft.*	*ft.*	*ft.*
Height of HW	8·7	8·9	9·1	9·3	9·4	9·8
Height of LW	3·2	3·2	3·0	2·7	2·5	2·2
	h. m.	*h. m.*	*h. m.*	*h. m.*	*h. m.*	*h. m.*
HWI	7 09	7 17	7 24	7 35	7 40	7 47
☽'s lower transit	6 07	6 51	7 36	8 22	9 11	10 01
LWI	0 48	0 58	1 09	1 20	1 30	1 38
Time of HW	13 16	14 08	15 00	15 57	16 51	17 48
Time of LW	6 55	7 49	8 45	9 42	10 41	11 39
	ft.	*ft.*	*ft.*	*ft.*	*ft.*	*ft.*
Height of HW	8·7	8·8	9·1	9·2	9·3	9·6
Height of LW	3·3	3·2	3·1	2·9	2·6	2·4

It will be observed that the computed time of tide frequently comes out greater than twenty-four hours; this is to be understood as indicating the number of hours which have elapsed from midnight, beginning the day on which the transit used occurs, until the given tide. By subtracting twenty-four hours and increasing the day by one such times may be expressed in the usual reckoning; for instance, the high water given as 25^h 43^m on December 31, 1897, corresponds to 1^h 43^m on January 1, 1898. The predicted times are expressed in eighty-first meridian time, the same as the observations were taken in. The predicted heights are as they would read on the tide staff upon which the observations were made. From the phase reduction it appears that mean low water read 2·6 feet on the staff; hence to reduce these predicted heights to the plane of mean low water subtract 2·6 feet from each. In case one desires to make predictions in any kind of time other than that in which the observations were taken and to refer the heights to any given plane of reference, the table of recapitulated values of the phase reduction should be so modified as to satisfy these conditions; and then the values should be plotted, and the curves used as before.

The phase inequality in height may be reduced to its mean value by dividing by the "factor for (Sg — Np)" § 58, and the remark there made concerning the effect of parallax upon the range applies here also.

62. *Types of tide.*

Observations show that cotidal lines off a shore or reef are nearly parallel to the same, resembling somewhat the contours of equal depths. This is what would naturally follow from the consideration of a free wave propagated over shallow areas. The same remark applies with about the same degree of precision to lines of equal lunitidal interval.

Example.—Lunitidal intervals for the (outer) coast of the United States:

Region.	Interval.		
	Semidaily tide HWI.	Diurnal wave Tropic D1 HWI.	Difference.
	h.	*h.*	*h.*
Eastport to N. and E. shores of Nantucket and Marthas Vineyard	11	8	+3
S. and W. shore of do. to Florida Strait	7½	8	—½
Cape St. Lucas to Cape Flattery	8½ to 12½ or 0	5 to 8½	+4
Cape Flattery to Kadiak Id.	0	8½	+4

These intervals must be increased for bays or tidal rivers. On the Atlantic coast the diurnal wave is small in comparison with the semidiurnal. On the Pacific coast the two may approach equality.

To ascertain the type of tide from the above values, draw a diurnal and a semidiurnal wave; then combine the two by adding the ordinates algebraically. (See Figs. 1, 7.)

The (tropic) diurnal high waters being three hours in advance of the semidiurnal high waters in the northern New England region, it follows that there is about the same inequality in the high as in the low waters, that the sequence of tide is from higher high to lower low, that the tropic higher high-water interval is less than HWI and should be marked *a*, that the tropic lower low-water interval should be greater than LWI and marked *b* if taken about six hours less than HWI: *a* indicates that the interval is to be applied to $\begin{smallmatrix}\text{an upper}\\ \text{a lower}\end{smallmatrix}$ transit, the moon's declination being $\begin{smallmatrix}\text{north;}\\ \text{south;}\end{smallmatrix}$ *b* indicates that the interval is to be applied to $\begin{smallmatrix}\text{a lower}\\ \text{an upper}\end{smallmatrix}$ transit, the moon's declination being $\begin{smallmatrix}\text{north.*}\\ \text{south.}\end{smallmatrix}$

For the remainder of the Atlantic coast, the high waters of both waves nearly coincide, thereby putting nearly all of the height inequality in the high waters. In fact, the low-water inequality is so small that the sequence of the tropic tides does not remain fixed throughout the year. (See § 21, Part III.) The tropic high-water interval is almost exactly equal to HWI and should be marked *a*.

On the Pacific coast the greater height inequality is, obviously, in the low waters, but there is a good amount in the high waters. The sequence is higher high to lower low. The tropic higher high-water interval is less than HWI, and should be marked *a* unless the small value (about zero) be used, in which case it should be marked *b*. The tropic lower low-water interval is greater than LWI, and should be marked *b* if taken about six hours less than the HWI marked *a*.

Particular localities.—The tides at Willets Point, N. Y. (see Figs. 9–19), have a comparatively long stand, somewhat resembling the tides at Havre. The alternate low waters (more definitely the "lower lows") approach at times to the condition of double tides or aggers, § 15.

The tide at Sandy Hook, N. J., is "regular;" that is, comparatively free from shallow water components. This curve is characteristic of the tides at ocean stations along the Atlantic coast of the United States. All along this coast the phase and diurnal inequalities in height are small in comparison with the range of tide. The changes in mean sea level which happen during the fortnight shown are not tidal, but meteorological.

At Philadelphia, Pa.,† the stand is short. The duration of fall is much longer than the duration of rise. In fact, there is a tendency toward a bore. This tendency is carried much further at Rambler Island and Volcano Island, as is shown by Fig. 19.

At Galveston, Tex., the tide is almost wholly diurnal. When the moon is north of the equator the tides follow one transit, and when south, the other. When she is upon the equator the tides nearly disappear.

At Mazatlan, Mexico, the tide has large phase and diurnal inequalities in height. The former causes the semidaily range of tide to disappear at times (near neap tides), and the lunitidal intervals to suddenly change in value. This feature is found in the tides at Port Adelaide, Australia, where it is locally termed the "dodging tide."

At Port Townsend, Wash., the diurnal wave is large whenever the declination of the moon is considerable. Its phase with respect to the semidiurnal is then such that nearly all of the diurnal height inequality occurs in the low waters, and so nearly all of the diurnal time inequality occurs in the high waters. Here the sequence of tides is uncertain even at the times of extreme declination.

At St. Michael, Alaska, the tide is essentially diurnal.

---

*Ferrel's diagram of the Boston tides, Tidal Researches, Fig. 2, has the interval of the diurnal wave wrong by a half lunar day.

†The shallow water component M_4 is here unusually large. M_4 causes most of the inequality between the duration of rise and fall.

At Honolulu, Hawaiian Islands, the diurnal height inequality is nearly all in the high waters, instead of the low.

At Tahiti the tide is largely solar, so that the solitidal intervals are generally more nearly constant than the lunitidal intervals. (For a plotting of the solitidal intervals, see Fig. 13 of Ferrel's Tidal Researches.)

At Havre, France, the tides have a long stand and approach the double-headed form. The phase inequality is large in comparison with this inequality on our Atlantic coast.

Fig. 19 is obtained from observations published in a Report on the Bore of the Tsien-tang Kiang, by Commander Moore, R. N. Haining is at a point where the estuary suddenly narrows; Rambler Island and Volcano Island~~s~~ are situated seaward about 28 and 75 statute miles, respectively.

CHAPTER V.

TIDAL WORK AND KNOWLEDGE BEFORE THE TIME OF NEWTON.

63. The maritime people of antiquity whose history has come down to us lived on or near the Mediterranean Sea, where the tide is generally small, and so they probably paid little attention to this periodic rising and falling of the waters. The numerous islands and straits, however, give rise to tidal races, and these in turn may be accompanied by whirlpools. The mythological Scylla and Charybdis, described by Homer,* were at a later period localized in the Straits of Messina, Scylla being a rock on the Italian shore, and Charybdis a whirlpool (now Galofaro) near the Sicilian shore.†

He places them near Trinacria (or Thrinacia), the three-cornered island of the sun, afterwards localized as Sicily.‡ The velocity of this race may be 6 miles per hour, while the range of the tide is but 1 foot. Homer says, "Under this § divine Charybdis sucks in black water. For thrice in a day she sends it out, and thrice she sucks it in terribly." Strabo, probably without sufficient grounds, is inclined to attribute to Homer some knowledge of ordinary tides, and says, "The assertion of thrice, instead of twice, is either an error of the author or a blunder of the scribe." ‖ Strabo subsequently makes another explanation, viz., that Circe says thrice instead of twice in order to exaggerate the perils which await Ulysses, and so to deter him from departing. Strabo adds, "However, this latter is a hyperbole which everyone makes use of; thus we say thrice-happy and thrice-miserable."¶ Gossellin remarks, "In the Euripus, which divides the Isle of Negropont from Bœotia, the waters are observed to flow in opposite directions several times a day. It was from this that Homer probably drew his ideas." ** According to the explanation adopted, "thrice" may be either an intensitive or an indefinite term.

The much more formidable tidal currents in the fjords of Norway and among the islands around Scotland have undoubtedly played an important part in the mythology of the ancient inhabitants of these countries.††

While the myths of Scylla and Charybdis undoubtedly owe their origin to tidal movements, and while the large tides of the Red Sea may possibly help to explain the passage of the Israelites,‡‡ it seems probable that the deluge was due to some earthquake disturbance whereby a portion of lower Mesopotamia may have been submerged.§§

---

*Odyssey, Bk. XII, lines 73 et seq.

†Strabo, Geography (Hamilton and Falconer's translation, Bohn's library): Bk. I, Ch. II, § 16; Bk. VI, Ch. II, § 3.

Mediterranean Pilot, Vol. I (1894), pp. 406–408.

Berghaus, Physikalischer Atlas (1892), No. 24.

At this point the Mediterranean Sea is divided into two parts, eastern and western. Each part has a tide of its own, and so the water at the two ends of the strait is not upon the same level. See § 40.

‡Geog., Bk. VI, Ch. II, § 1.

§I. e., wild fig-tree. Odyssey, Bk. XII, lines 104–106.

‖Geog., Bk. I, Ch. I, § 7.

¶Geog., Bk. I, Ch. II, § 36.

**Geog., Bk. I, Ch. I, § 7, footnote; Cf. ibid., Bk. I, Ch. II, § 30. See under Aristotle.

††For some information along this line, see Enc. Brit., article "Whirlpool." See also this manual under Hakluyt and Varenius.

‡‡Exodus, Chs. XIV, XV. Scaliger, Exercitatio LII, and Varenius, Geographia Generalis, Vol. I, Ch. 14, do not take this view. Cf. Lalande, Astronomie, Vol. III, p. 649.

It seems that in this vicinity Napoleon came near repeating Pharaoh's experience. Harper's Magazine, Vol. IV (1852), pp. 310, 311.

§§B. K. Emerson, "Geological myths," Science, Vol. IV (1896), pp. 328 et seq.

64. *Herodotus* (484–428 B. C.) says, referring to a certain bay or arm of the Red Sea, "And in it an ebb and flow takes place daily." *

This is the only mention he makes of tides proper, and he says nothing concerning their origin. He thus describes the annual overflow of the Nile:

Respecting the nature of this river, I was unable to gain any information, either from the priests or anyone else. I was very desirous, however, of learning from them why the Nile, beginning at the summer solstice, fills and overflows for a hundred days; and when it has nearly completed this number of days, falls short in its stream, and retires; so that it continues low all the winter, until the return of the summer solstice. †

Herodotus then gives several opinions which had been advanced by others to explain this characteristic of the Nile, and follows these by an explanation of his own, which amounts to attributing the low stages during the winter season to evaporation caused by the sun then being over upper Libya.

Although *Plato* (429–348 B. C.), in common with most of the Greek philosophers, believed the earth an animal, he does not attribute the rising and falling of waters to its breathing, but rather to oscillations of the fluid within the earth.

Fournier, in his Hydrographie (1643), does not regard this latter hypothesis as altogether untenable.

Aristotle (384–322 B. C.) seems to have been aware of the existence of tides in certain places; but he probably paid little attention to them, notwithstanding the allegement that his death was indirectly due to the study of their cause. ‡

The following extract is taken from the fourth chapter of his letter entitled "On the universe," addressed to Alexander, and which, although usually regarded as spurious, may nevertheless embody the ideas of Aristotle. Other than this, his writings refer to the tides but twice. He states that great tides are found in northern Europe, and that they are greater in a large sea than in a small one. §

The periodicity of the tide and its connection with the moon's motion are alluded to, but only as if by hearsay. Submersions of the coast and waves due to earthquakes would, naturally enough, have been better known to him.

Things analogous to these ‖ are found in the sea also; for, oftentimes chasms in the water are formed by the receding waves, and the incoming waves advance, sometimes again retreating and sometimes only rushing straight forward as seems to have happened to Helice and Bura. ¶ Oftentimes also fiery eruptions exist in the sea, fountains gush out, mouths of rivers are opened, trees spring up, deluges and whirlpools arise, corresponding to those which often accompany the wind; some in the middle of the deep sea, others in straits and bays. It is even said that many ebbings and risings of the sea always come around with the moon and upon certain fixed times. In a word, since the elements are mutually intermixed with one another, there appear, likewise in the air, the earth and the sea, analogous affinities which either may create and destroy certain properties [of each substance] or may preserve them in an unaltered state.

In speaking of northern Portugal, Strabo says:

The eastern part is mountainous and rugged, while the country beyond, as far as the sea, consists entirely of plains, with the exception of a few inconsiderable mountains. On this account Posidonious remarks that Aristotle was not correct in supposing that the ebb and flow of the tide was occasioned by the sea-coast of Iberia and Maurusia. For Aristotle asserted that the tides of the sea were caused by the extremities of the land being mountainous and rugged, and therefore both receiving the wave violently and also casting it back. Whereas Posidonius truly remarks that they are for the most part low and sandy. **

For two other views attributed to Aristotle, see under Plutarch and under Galileo.

65. When the army of *Alexander* approached the mouth of the Indus in their southward

---

* Herodotus (Cary's translation, Bohn's library), Bk. II, Ch. 11.

† Ibid., Bk. II, Ch. 19. The maximum height of the river occurs near the autumnal equinox.

‡ Lalande, Astronomie, Vol. III, p. 650, and Vol. IV, pp. 3–5. See also under Galileo. For accounts of the tides of the Euripus, see Lalande, Astronomie, Vol. IV, pp. 148–151; F. J. P. Babin, Phil. Trans., 1671, Abr. Vol. I, p. 592 A more recent account is given in Nature, Vol. 21 (1879), p. 186. According to this, the currents are due in part to the seiches in the Gulf of Talanta. In fact, near the moon's quadratures they control the currents, causing many reversals each day.

§ See under Varenius and Pliny.

‖ I. e., whirlwinds, etc.

¶ Towns of Greece submerged by an earthquake sea wave.

** Geography, Bk. III, Ch. III, § 3.

journey (325 B. C.), they were amazed at the tides, having never noticed any similar phenomenon in the Mediterranean Sea. Curtius (fl. c. 50 A. D.) narrates their experience on this occasion in his History of the Life and Reign of Alexander.*

On the third day the insinuations of the sea were perceptible in the river, blending their unequal waves by a gentle influx.†

To a second island, seated in the middle of the river, the navigators were then borne somewhat more slowly, because the stream was counteracted by the tide. They moor their vessels, and separate in parties to forage, without a presentiment of the disaster which overtakes mariners locally uninstructed.

About the third hour,‡ the ocean, according to a regular alternation, began to flow in furiously, driving back the river. The river—at first, arrested; then, impressed with a new force—rushed upward with more impetuosity than torrents descend a precipitous channel. The mass on board, unacquainted with the nature of the tide, saw only prodigies and symbols of the wrath of the gods. Ever and anon, the sea swelled; and, on plains recently dry, descended a diffused flood. The vessels lifted from their stations, and the whole fleet dispersed,—those who had debarked, in terror and astonishment at the calamity, ran from all quarters toward the ships. But tumultuous hurry is slow. These, with boat hooks, are hauling up their gallies: these, while fixing their seats, prevent the oars from being paired: some, hastening to sail, without waiting for the complement of mariners, impel languid hulls, unmanageable, crippled in the wings of navigation: other transports could not hold those who inconsiderately pressed into them: deficient, or redundant, numbers equally obstructed the impatient. Here was clamored, "Wait:"—here, "Row off." Dissonant voices, circulating inconsistent orders, prevented the multitude from acting by their own observation, or from hearing the general command. Nor availed the pilots; whose directions were either undistinguished in the tumult, or disobeyed by terrified and promiscuous crews.

Vessels dash together; and oars are by turns snatched away, to impel other gallies. A spectator would not imagine a fleet carrying the same army, but hostile navies commencing a battle. Prows strike against sterns: on the invading vessels, others drive aft. The fury of altercation carried the mariners to blows.

Now the tide had inundated all the fields skirting the river, only tops of knolls extant like little islands: to these, from the evacuated ships, the majority swam in consternation.

The dispersed fleet was, partly, riding in deep water, where the land was depressed into dells; and, partly, resting on shoals, where the flood had covered elevated ground. Suddenly breaks on the Macedonians a new alarm, more vivid than the former. The sea began to ebb; the deluge with a violent drain, to retreat into the frith, disclosing tracts just before deeply buried. Unbuoyed, the ships pitched, some upon their prows, some upon their sides. The fields were strewed with baggage, arms, loose planks, and fragments of oars. The soldiers, neither daring to descend to the ground, nor reconciling themselves to stay in the transports, awaited what calamities could follow heavier than the present. They scarcely believed what they suffered, and witnessed—shipwrecks on dry land, the sea in a river. Nor yet ended their unhappiness; for, ignorant that the speedy return of the tide would set their ships afloat, they predicted to themselves famine and death. Terrifying monsters, too, left by the waves, were vagrantly gliding around.§

Now night approached; and the desperate circumstances touched the king with concern: but no anxieties could overwhelm his invincible courage. All night, he superintended the watches: he sent forward horsemen to the mouth of the river, to bring intelligence when the access of the tide commenced. Meanwhile, he ordered the shattered ships to be refitted, the overset to be propped up; and the mariners to be prepared, and attentive, against the flux of the tide.

The night consumed in vigilance and exhortations, the horsemen are descried, flying back in full career, followed by the tide. By a gradual diffusion, the inundation began to raise the ships; presently, flooding all the fields, it set the fleet in motion. Along the banks, resounded from the soldiers and mariners shouts of boundless joy, celebrating an unhoped deliverance. "Whence reissued suddenly so great a sea? Whither the day before had it retreated? What were the nature of the element, elsewhere refusing and here acknowledging periodical laws?" with wonder they inquired.

From what had happened, the king conjectured the appointed time of the flux to be just after sunrise. To anticipate the tide, he, at midnight, descended the river with a few vessels; and, passing its mouth, advanced four hundred stadia into the sea. A favorite object accomplished, he sacrificed to Neptune and the local deities, and returned to the fleet.

66. *Seleucus*, a mathematician of Babylonia (365–283 B. C.), admitted the rotary movement of the earth, but thought the moon moved in the opposite direction. By this means, he thought, waves happening between the earth's center and the moon become accumulated and so form the tide; or that the tide results from winds set up by these opposing motions.‖ According to Posidonius,¶ Seleucus connected the irregular tides of the Persian Gulf with the moon and its various declinations. He observed tides from Phœnicia to the Atlantic coast of Spain.

Pytheas, of Massilia (fl. c. 325 B. C.), having navigated the ocean from the Strait of Gibraltar to the British Isles, and possibly to Iceland,** was one of the first to note the connection between

* Bk. IX, Ch. IX. Translated by Pratt.

† About 60 or 65 miles from the sea.

‡ About 9 o'clock a. m.

§ "Probably, for the most part, aquatic serpents."

‖ See under Plutarch.

¶ See under Strabo.

** Cf. Strabo, Geog., Bk. IV, Ch. V, § 5.

the tides and moon; and was, so far as known, the first person to point out the connection between the half-monthly variation in the tide and the phases of the moon. He is said to be the first to measure the heights of the tide. Plutarch (q. v.) in referring to Pytheas, seems to have the erroneous notion that high water occurs at full moon and low water at new moon. One of Aristotle's remarks, however, would seem to indicate that somebody had brought him (Aristotle) accounts of tides which had been referred to the moon. Pytheas wrote a work on matters pertaining to the ocean which has not come down to us.

Eratosthenes (276— B. C.), according to Strabo (q. v.), believed the tidal currents of the Mediterranean due to a difference in level at neighboring points.

Posidonius, the Stoic philosopher (c. 130–50 B. C.), continued the history of Polybius. Only fragments of his work exist. According to Strabo (q. v.) he was the author of a Treatise on the Ocean, and an authority on tidal knowledge. He is said to have tried to calculate the influence of the moon upon the tides.

67. *Strabo* (c. 54 B. C.—c. 24 A. D.).

Strabo believes that the uniformity and the considerable size of the ocean tides go to prove that all land is surrounded by the ocean. He says:

Those who have returned from an attempt to circumnavigate the earth, do not say they have been prevented from continuing their voyage by any opposing continent, for the sea remained perfectly open, but through want of resolution, and the scarcity of provision. This theory too accords better with the ebb and flow of the ocean, for the phenomenon, both in the increase and diminution, is every where identical, or at all events has but little difference, as if produced by the agitation of one sea, and resulting from one cause.

We must not credit Hipparchus, who combats this opinion, denying that the ocean is every where similarly affected; or that even if it were, it would not follow that the Atlantic flowed in a circle, and thus continually returned into itself. Seleucus, the Babylonian, is his authority for this assertion. For a further investigation of the ocean and its tides we refer to Posidonius and Athenodorus, who have fully discussed this subject: we will now only remark that this view agrees better with the uniformity of the phenomenon; and that the greater the amount of moisture surrounding the earth, the easier would the heavenly bodies be supplied with vapours from thence.*

Strabo's ideas concerning the figure and position of the earth, the force of gravity, and the nature of sea level may be seen from the following quotations:

We shall also assume that the earth is spheroidal, that its surface is likewise spheroidal, and above all, that bodies have a tendency towards its centre, which latter point is clear to the perception of the most average understanding. However we may show summarily that the earth is spheroidal, from the consideration that all things however distant tend to its centre, and that every body is attracted towards its centre of gravity; this is more distinctly proved from observations of the sea and sky, for here the evidence of the senses, and common observation, is alone requisite. The convexity of the sea is a further proof of this to those who have sailed; for they cannot perceive lights at a distance when placed at the same level as their eyes, but if raised on high, they at once become perceptible to vision, though at the same time further removed. So, when the eye is raised, it sees what before was utterly imperceptible.

Homer speaks of this when he says, "Lifted up on the vast wave he quickly beheld afar."

Sailors, as they approach their destination, behold the shore continually raising itself to their view; and objects which had at first seemed low, begin to elevate themselves. Our gnomons, also, are, among other things, evidence of the revolution of the heavenly bodies; and common sense at once shows us, that if the depth of the earth were infinite, such a revolution could not take place.†

However, so nice a fellow is Eratosthenes, that though he professes himself a mathematician, he rejects entirely the dictum of Archimedes, who, in his work "On Bodies in Suspension," says that all liquids when left at rest assume a spherical form, having a center of gravity similar to that of the earth. A dictum which is acknowledged by all who have the slightest pretensions to mathematical sagacity. He says that the Mediterranean, which, according to his own description, is one entire sea, has not the same level even at points quite close to each other; and offers us the authority of engineers for this piece of folly, notwithstanding the affirmation of mathematicians that engineering is itself only one division of the mathematics. He tells us that Demetrius intended to cut through the Isthmus of Corinth, to open a passage for his fleet, but was prevented by his engineers, who, having taken measurements, reported that the level of the sea at the Gulf of Corinth was higher than at Cenchrea, so that if he cut through the isthmus, not only the coasts near Ægina, but even Ægina itself, with the neighbouring islands, would be laid completely under water, while the passage would prove of little value. According to Eratosthenes, it is this which occasions the current in straits, especially the current in the Strait of Sicily, where effects similar to the flow and ebb of the tide are remarked. The current there changes twice in the course of a day and night, like as in that period the tides of the sea flow and ebb twice. In the Tyrrhenian sea the current which is called

*Strabo, Geography (Hamilton and Falconer's translation, Bohn's Library), Bk. I, Ch. I, §§ 8, 9.

†Geog., Bk. I, Ch. I, § 20.

descendent, and which runs towards the sea of Sicily, as if it followed an inclined plane, corresponds to the flow of the tide in the ocean. We may remark, that this current corresponds to the flow both in the time of its commencement and cessation. For it commences at the rising and setting of the moon, and recedes when that satellite attains its meridian, whether above [in the zenith] or below the earth [in the nadir]. In the same way occurs the opposite or ascending current, as it is called. It corresponds to the ebb of the ocean, and commences as soon as the moon has reached either zenith or nadir, and ceases the moment she reaches the point of her rising or setting. [So far Eratosthenes.]

The nature of the ebb and flow has been sufficiently treated of by Posidonius and Athenodorus. Concerning the flux and reflux of the currents, which also may be explained by physics, it will suffice our present purpose to observe, that in the various straits these do not resemble each other, but each strait has its own peculiar current. Were they to resemble each other, the current at the Strait of Sicily would not change merely twice during the day, (as Eratosthenes himself tells us it does,) and at Chalcis seven times; nor again that of Constantinople, which does not change at all, but runs always in one direction from the Euxine to the Propontis, and, as Hipparchus tells us, sometimes ceases altogether. However, if they did all depend on one cause, it would not be that which Eratosthenes has assigned, namely, that the various seas have different levels. The kind of inequality he supposes would not even be found in rivers only for the cataracts; and where these cataracts occur, they occasion no ebbing, but have one continued downward flow, which is caused by the inclination both of the flow and the surface; and therefore though they have no flux or reflux they do not remain still, on account of a principle of flowing which is inherent in them; at the same time they cannot be on the same level, but one must be higher and one lower than another. But who ever imagined the surface of the ocean to be on a slope, especially those who follow a system which supposes the four bodies we call elementary, to be spherical. For water is not like the earth, which being of a solid nature is capable of permanent depressions and risings, but by its force of gravity spreads equally over the earth, and assumes that kind of level which Archimedes has assigned it.*

Here are a few of the facts established by natural philosophers:

The earth and heavens are spheroidal.

The tendency of all bodies having weight, is to a centre.

Further, the earth being spheroidal, and having the same centre as the heavens, is motionless, as well as the axis which passes through both it and the heavens. The heavens turn round both the earth and its axis, from east to west. The fixed stars turn round with it, at the same rate as the whole. These fixed stars follow in their course parallel circles; the principal of which are, the equator, the two tropics, and the arctic circles. While the planets, the sun, and the moon, describe certain oblique circles comprehended within the zodiac.†

Elsewhere Strabo states that if we go west sufficiently far we shall reach India, unless some continent intervenes; and he suggests that there may be one or more such bodies of land on the parallel of Spain.‡ He ascribes the great heat felt in the torrid zone to the perpendicularity of the sun's rays, and not to the sun's proximity. He says that all parts of the earth are equally remote from the sun, since, in comparison with this body, the earth is but a point.§

The following quotations show that Strabo clearly discriminated between "tidal waves" and ordinary or true tides. He considers "tidal waves" to be due to disturbances in the bottom of the sea.

But the risings of rivers are not violent and sudden, nor do the tides continue any length of time, nor occur irregularly; nor yet along the coasts of our sea do they cause inundations, nor any where else. • Consequently we must seek for an explanation of the cause‖ either in the stratum composing the bed of the sea, or in that which is overflowed; we prefer to look for it in the former, since by reason of its humidity it is more liable to shiftings and sudden changes of position, and we shall find that in these matters the wind is the great agent after all. But, I repeat it, the immediate cause of these phenomena, is not in the fact of one part of the bed of the ocean being higher or lower than another, but in the upheaving or depression of the strata on which the waters rest.¶

It is likewise evidently a fiction, that there ever occurred an overwhelming flood-tide, for the ocean, in the influences of this kind which it experiences, receives a certain settled and periodical increase and decrease.**

He does not, however, state what he regards as the physical cause of the tides, although he has already hinted at vapors surrounding the heavenly bodies. He merely, after Athenodorus *likens* the phenomenon to the breathing of a living creature, as do the poets of to-day.††

For after the manner of living creatures, which go on inhaling and exhaling their breath continually, so the sea in a like way keeps up a constant motion in and out of itself.

---

*Geog., Bk. I, Ch. III, §§ 11, 12.
†Geog., Bk. II, Ch. V, § 2.
‡Geog., Bk. I, Ch. IV, § 6.
§Geog., Bk. XV, Ch. I, § 24.
‖Of inundations.
¶Geog., Bk. I, Ch. III, § 5.
**Geog., Bk. VII, Ch. II, § 1.
††Geog, Bk. I, Ch. III, § 8; Bk. III, Ch. V, § 7.

He describes the tides around Spain and Portugal,* the Persian Gulf,† England,‡ Italy,§ and Denmark.‖

The following account of the tides at Cadiz describes the phase inequality and properly connects it with the age of the moon, but the description of a supposed annual increase (in range) has been doubtless based upon insufficient observations:

Now he ¶ asserts that the motion of the sea corresponds with the revolution of the heavenly bodies, and experiences a diurnal, monthly, and annual change, in strict accordance with the changes of the moon. For [he continues] when the moon is elevated one sign of the zodiac above the horizon, the sea begins sensibly to swell and cover the shores, until she has attained her meridian; but when that satellite begins to decline, the sea again retires by degrees, until the moon wants merely one sign of the zodiac from setting; it then remains stationary until the moon has set, and also descended one sign of the zodiac below the horizon, when it again rises until she has attained her meridian below the earth; it then retires again until the moon is within one sign of the zodiac of her rising above the horizon, when it remains stationary until the moon has risen one sign of the zodiac above the earth, and then begins to rise as before. Such he describes to be the diurnal revolution.** In respect to the monthly revolution, [he says] that the spring-tides occur at the time of the new moon, when they decrease until the first quarter; they then increase until full moon, when they again decrease until the last quarter, after which they increase till the new moon; [he adds] that these increases ought to be understood both of their duration and speed. In regard to the annual revolution, he says that he learned from the statements of the Gaditanians, that both the ebb and flow tides were at their extremes at the summer solstice: and that hence he conjectured that they decreased until the [autumnal] equinox; then increased till the winter solstice; then decreased again until the vernal equinox; and [finally] increased until the summer solstice.††

The following quotation refers to a condition of flood-tide resembling a bore:

For in the navigation of the rivers, ‡‡ the sailors run considerable danger both in ascending and descending, owing to the violence with which the flood-tide encounters the current of the stream as it flows down. The ebb-tides are likewise the cause of much damage in these estuaries, for resulting as they do from the same cause as the flood-tides, they are frequently so rapid as to leave the vessel on dry land; and herds in passing over to the islands that are in these estuaries are sometimes drowned [in the passage] and sometimes surprised in the islands, and endeavouring to cross back again to the continent, are unable, and perish in the attempt. §§

Strabo is the first writer who gives, although incredulously and upon the authority of others, some account of the tropic tides and the diurnal inequality:‖‖

Posidonius tells us that Seleucus, a native of the country next the Erythræan Sea, states that the regularity and irregularity of the ebb and flow of the sea follow the different positions of the moon in the zodiac; that when she is in the equinoctial signs the tides are regular, but that when she is in the signs next the tropics, the tides are irregular both in their height and force; and that for the remaining signs the irregularity is greater or less, according as they are more or less removed from the signs before mentioned. Posidonius adds, that during the summer solstice and whilst the moon was full, he himself passed many days in the temple of Hercules at Gades, but could not observe any thing of these annual irregularities. ¶¶

In reference to the annual inequality in water level, it may be of interest to here quote from Strabo some passages concerning the periodic stages of the Nile and the rivers of India:

Of this kind are the rising of the Nile, and the alluvial deposition at its mouth. There is nothing in the whole country to which travellers in Egypt so immediately direct their inquiries, as the character of the Nile; nor do the inhabitants possess any thing else equally wonderful and curious, of which to inform foreigners; for in fact, to give them a description of the river, is to lay open to their view every main characteristic of the country. It is the question put before every other by those who have never seen Egypt themselves.***

---

* Geog., Bk. III, Ch. II, §§ 4, 5, 7; Ch. III, § 1; Ch. V, §§ 7, 8.

† Geog., Bk. III, Ch. V, § 9; Bk. XVI, Ch. III, § 6.

‡ Geog., Bk. IV, Ch. V, § 3.

§ Geog., Bk. V, Ch. I, §§ 5, 7; Bk. VI, Ch. II, §§ 3, 11.

‖ Geog., Bk. VII, Ch. II, § 1.

¶ Posidonius.

** HWI $= 1^h\,45^m$, LWI $= 7^h\,58^m$, Mn $= 8{\cdot}7$ feet, at Cadiz.

†† Geog., Bk. III, Ch. V. § 8.

‡‡ Rivers between the Strait of Gibraltar and Cape St. Vincent.

§§ Geog., Bk., III, Ch. II, § 4.

‖‖ For the Persian Gulf at Bushire, lat. 29° 00′, long. 50° 52′, $M_2 = 0^{ft}{\cdot}90$, $S_2 = 0^{ft}{\cdot}31$, $K_1 = 0^{ft}{\cdot}86$, $O_1 = 0^{ft}{\cdot}60$, $M_2° = 210°{\cdot}5$, $S_2° = 262°$, $K_1° = 285°$, $O_1° = 253°$, according to Report of Indian Survey, 1893–4, p. XLII; ∴ Mn $= 2^{ft}{\cdot}10$, Gc $= 3^{ft}{\cdot}80$, HWQ $= 2^{ft}{\cdot}83$, LWQ $= 0^{ft}{\cdot}72$.

¶¶ Geog., Bk. III, Ch. V, § 9.

*** Geog., Bk. I, Ch. II, § 29.

Nearchus says, that the old question respecting the rise of the Nile is answered by the case of the Indian rivers, namely, that it is the effect of summer rains.*

For the Euphrates, at the commencement of summer, overflows. It begins to fill in the spring, when the snow in Armenia melts.†

The ancients understood more by conjecture than otherwise, but persons in later times learnt by experience as eye-witnesses, that the Nile owes its rise to summer rains, which fall in great abundance in Upper Ethiopia, particularly in the most distant mountains. On the rains ceasing, the fulness of the river gradually subsides.‡

Elephantina is an island in the Nile, at the distance of half a stadium in front of Syene; in this island is a city with a temple of Cnuphis, and a nilometer like that at Memphis. The nilometer is a well upon the banks of the Nile, constructed of close-fitting stones, on which are marked the greatest, least, and mean risings of the Nile; for the water in the well and in the river rises and subsides simultaneously. Upon the wall of the well are lines, which indicate the complete rise of the river, and other degrees of its rising. Those who examine these marks communicate the result to the public for their information. For it is known long before, by these marks, and by the time elapsed from the commencement, what the future rise of the river will be, and notice is given of it. This information is of service to the husbandmen with reference to the distribution of the water; for the purpose also of attending to the embankments, canals, and other things of this kind. It is of use also to the governors, who fix the revenue; for the greater the rise of the river, the greater it is expected will be the revenue.§

This passage is of interest in reference to tide gauges and tidal prediction.

68. *Plutarch* (fl. 50–100 A. D.) gives the theories of several philosophers concerning the cause of the tides, but his remarks on Pytheas show his unfamiliarity with the phenomenon. He says:

Aristotle and Heraclides say, they proceed from the sun, which moves and whirls about the winds; and these falling with a violence upon the Atlantic, it is pressed and swells by them, by which means the sea flows; and their impression ceasing, the sea retracts, hence they ebb. Pytheas the Massilian, that the fulness of the moon gives the the flow, the wane the ebb. Plato attributes it all to a certain oscillation of the sea, which by means of a mouth or orifice causes the alternate ebb and flow; and by this means the seas do rise and flow contrarily. Timæus believes that those rivers which fall from the mountains of the Celtic Gaul into the Atlantic produce a tide. For upon their entering upon that sea, they violently press upon it, and so cause the flow; but they disemboguing themselves, there is a cessation of the impetuousness, by which means the ebb is produced.‖ Seleucus the mathematician attributes a motion to the earth; and thus he pronounceth that the moon in its circumlation meets and repels the earth in its motion; between these two, the earth and the moon, there is a vehement wind raised and intercepted, which rushes upon the Atlantic Ocean, and gives us a probable argument that it is the cause the sea is troubled and moved.¶

While Plutarch states that most philosophers have regarded the earth as an animal, he says nothing about ascribing the tides to its respiration or to its alternate drinking in and spouting out a certain portion of the water. Such notions were entertained by the Stoics Solinus, Apollonius of Tyana, and others. (See under Pomponius Mela and under Kepler, §§ 70, 77.)

It cannot be said that the Greeks, as a rule, had occasion to become familiar with tidal phenomena on any impressive scale. But the Romans, toward and after the time of Cæsar, had frequently to contend with those enormous tides which visit the coasts of Portugal, France, and Great Britain.

69. *Pliny the Elder* (23–79 A. D.).

Pliny describes, in the following extract taken from his Natural History,** the principal phenomena of the tides. It will be seen that he is aware of a nearly constant lunitidal interval for a given place; of the phase inequality; of the retard, or age, of this inequality; of the fact that higher tides occur near the equinoxes than near the solstices; and of the fact that the tides on the outer coast rise higher than those along the shores of the Mediterranean.

Much has been said about the nature of waters; but the most wonderful circumstance is the alternate flowing and ebbing of the tides, which exists, indeed, under various forms, but is caused by the sun and the moon. The tide flows twice and ebbs twice between each two risings of the moon, always in the space of twenty-four hours.

---

* Geog., Bk. XV, Ch. I, § 25.

† Geog., Bk. XVI, Ch. I, § 9.

‡ Geog., Bk. XVII, Ch. I, § 5. In the Philosophical Transactions for 1666 is a review of a French book by de la Chambre, in which he claims that the overflow of the Nile is not due to the rain, but to the niter contained in its muddy banks. This being heated by the sun ferments and mingling with the waters swells the river, causing it to overflow its banks. In the same volume there is given a review of Isaac Vossius' "De Nili et Aliorum Fluminum Origine."

§ Geog., Bk. XVII, Ch. I, § 48. Cf. Lockyer, The Dawn of Astronomy, Ch. XXIII; "The Egyptian Year and the Nile."

‖ This idea was revived in the seventeenth century by Scipio Claramontius and refuted by Riccioli.

¶ "Of Those Sentiments Concerning Nature with which Philosophers were Delighted" (Morales, Goodwin's translation, Vol. III) Bk. III, Ch. XVII.

** Historia Naturalis (Bostock and Riley's translation), Book II, chapters 99 (97)–102 (99). A. D. 77.

First, the moon rising with the stars swells out the tide, and after some time, having gained the summit of the heavens, she declines from the meridian and sets, and the tide subsides. Again, after she has set, and moves in the heavens under the earth, as she approaches the meridian on the opposite side, the tide flows in; after which it recedes until she again rises to us. But the tide of the next day is never at the same time with that of the preceding; as if the planet was in attendance, greedily drinking up the sea, and continually rising in a different place from what she did the day before. The intervals are, however, equal, being always of six hours; not indeed in respect of any particular day or night or place, but equinoctial hours,* and therefore they are unequal as estimated by the length of common hours, since a greater number of them fall on some certain days or nights, and they are never equal everywhere except at the equinox. This is a great, most clear, and even divine proof of the dullness of those, who deny that the stars go below the earth and rise up again, and that nature presents the same face in the same states of their rising and setting; for the course of the stars is equally obvious in the one case as in the other, producing the same effect as when it is manifest to the sight.

There is a difference in the tides, depending on the moon, of a complicated nature, and, first, as to the period of seven days. For the tides are of moderate height from the new moon to the first quarter; from this time they increase, and are the highest at the full: they then decrease. On the seventh day they are equal to what they were at the first quarter, and they again increase from the time that she is at first quarter on the other side. At her conjunction with the sun they are equally high as at the full. When the moon is in the northern hemisphere, and recedes further from the earth, the tides are lower than when, going toward the south, she exercises her influence at a less distance. After an interval of eight years, and the hundredth revolution of the moon, the periods and the heights of the tides return into the same order as at first, this planet always acting upon them; and all these effects are likewise increased by the annual changes of the sun, the tides rising up higher at the equinoxes, and more so at the autumnal than at the vernal; while they are lower about the winter solstice, and still more so at the summer solstice; not indeed precisely at the points of time which I have mentioned, but a few days after; for example, not exactly at the full nor at the new moon, but after them; and not immediately when the moon becomes visible or invisible, or has advanced to the middle of her course, but generally about two hours later than the equinoctial hours; the effect of what is going on in the heavens being felt after a short interval; as we observe with respect to lightning, thunder, and thunderbolts.

But the tides of the ocean cover greater spaces and produce greater inundations than the tides of the other seas; whether it be that the whole of the universe taken together is more full of life than its individual parts, or that the large open space feels more sensibly the power of the planet, as it moves freely about, than when restrained within narrow bounds. On which account neither lakes nor rivers are moved in the same manner. Pytheas of Massilia informs us, that in Britain the tide rises 80 cubits. Inland seas are inclosed as in a harbor, but, in some parts of them, there is a more free space which obeys the influence. Among many other examples, the force of the tide will carry us in three days from Italy to Utica, when the sea is tranquil and there is no impulse from the sails. But these motions are more felt about the shores than in the deep parts of the seas, as in the body the extremities of the veins feel the pulse, which is the vital spirit, more than the other parts. And in most estuaries, on account of the unequal rising of the stars in each tract, the tides differ from each other, but this respects the period, not the nature of them; as is the case in the Syrtes.

There are, however, some tides which are of a peculiar nature, as in the Tauromenian Euripus, where the ebb and flow is more frequent than in other places, and in Euboea, where it takes place seven times during the day and the night. The tides intermit three times during each month, being the 7th, 8th, and 9th day of the moon. At Gades, which is very near the temple of Hercules, there is a spring inclosed like a well, which sometimes rises and falls with the ocean, and, at other times, in both respects contrary to it. In the same place there is another well, which always agrees with the ocean. On the shores of the Bætis, there is a town where the wells become lower when the tide rises, and fill again when it ebbs; while at other times they remain stationary. The same thing occurs in one well in the town of Hispalis, while there is nothing peculiar in the other wells. The Euxine always flows into the Propontis, the water never flowing back into the Euxine.

All seas are purified at the full moon; some also at stated periods. At Messina and Mylae a refuse matter, like dung, is cast up on the shore, whence originated the story of the oxen of the sun having had their stable at that place. To what has been said above (not to omit anything with which I am acquainted) Aristotle adds, that no animal dies except when the tide is ebbing. The observation has been often made on the ocean of Gaul; but it has only been found true with respect to man.

Hence we may certainly conjecture, that the moon is not unjustly regarded as the star of our life. This it is that replenishes the earth; when she approaches it, she fills all bodies, while, when she recedes, she empties them.† From this cause it is that shell-fish grow with her increase, and that those animals which are without blood more particularly experience her influence; also, that the blood of man is increased or diminished in proportion to the quantity of her light; also, that the leaves and vegetables generally, as I shall describe in the proper place, feel her influence, her power penetrating all things.

70. *Information about tides common to the Romans.*

That the rising and falling of the tide was common knowledge among the Romans may be readily seen by consulting Latin-English lexicons under such words as "æstus" and "tumesco."

---

* I. e., mean solar hours.

† The sympathy of the moon for moist bodies, etc., was a prevalent notion at the time of Varenius.

Virgil* (70–19 B. C.) and Horace† (65–8 B. C.) make some mention of the phenomenon; but Cicero‡ (106–43 B. C.), Lucan§ (39– – A. D.), and Claudianus|| (fl. c. 400 A. D.) state or suggest that the times of the tide are governed by the moon's motion. Cæsar¶ (100–44 B. C.) and Macrobius** (fl. 400 A. D.) state how the spring tides are connected with the full moon. Seneca†† (3–65 A. D.) even states that the equinoctial tides when the moon and sun are in conjunction generally exceed in size all others. This is much more nearly true than Pliny's statement that tides rise higher at the equinoxes, for it is the *spring tides* which partake of this semiannual increase.‡‡

Pompouius Mela (fl. c., 50 A. D.) says that it is not known whether the tides are produced by the earth's breathing, by deep caverns, or by the moon.§§

71. *Early tide table for London Bridge.*—In the Philosophical Transactions for 1837, p. 103, Lubbock says:

I am much indebted to Mr. Yates for notice of a very ancient tide table which exists in a MS. in the British Museum. It is in the Codex Cottonianus, Julius DVII., which appears to have been written in the 13th century, and to have belonged to St. Albans Abbey. It contains calendar and other astronomical or geographical matters, some of which are the productions of John Wallingford, who died Abbot of St. Albans A. D. 1213. At p. 45b. is a table on one leaf, showing the time of high water at London Bridge, "flod at london briggo," thus:

Ætas Lunæ.	*h.*	*m.*
1	3	48
2	4	36
3	5	24
4	6	12
......	......	......
28	1	24
29	2	12
30	3	0

N. B. The numbers increase by a constant difference of forty-eight minutes. The first column gives the moon's age in days.

* Qua vi maria alta tumescant,
Objicibus ruptis, rursusque in se ipsa residant.—Georg., II, 479-480.

† Quæ mare compescant causæ. . . . —Epist., I, XII, 16.

‡ Quid de fretis, aut de marinis æstibus plura dicam? quorum accessus et recessus lunæ motu gubernantur.—De Divinatione II, 34.

§ An sidere mota secundo
Tethyos unda vagæ lunaribus æstuet horis:
Flammiger an Titan. . . . . —Pharsalia, I, 413–415.

|| Certis ubi legibus advena Nereus
Æstuat, et pronas pupes nunc amne secundo,
Nunc redeunte, vehit; nudataque littora fluctu
Deserit, Oceani lunaribus æmula damnis.—De VI. Cons. Honorii, 496–499.

¶ Eadem nocte accidit, ut esset luna plena; qui dies maritimos æstus maximos in Oceano efficere consuevit.—B. G., IV, XXIX.

** Oceanus quoque in incremento suo hunc numerum tenet; nam primo nascentis lunæ die fit copiosor solito; minitur paulisper secundo; minoremque videt eum tertius, quam secundus: et ita decrescendo ad diem septimum pervenit. Rursus octavus dies manet septimo par; et nonus fit similis sexto . . . tertio quoque duodecimus; et tertius decimus fit similis secundo, quartus decimus primo. Tertia vero hebdomas eadem facit quæ prima; quarta eadem, quæ secundo.—Somnium Scipionis, I, 6.

†† Ut solet æstus æquinoxialis, sub ipsum lunæ solisque coitum, omnibus aliis major undare.—Naturales Quæstiones, III, 28.

‡‡ See "Table of phase effects," § 35; also under Bacon, Varenius, Wallis, and Childrey. Lalande, Astronomie, Vol. IV, pp. 83–113, examines the question of equinoctial spring tides, but many of his conclusions are erroneous. It would be interesting to know whether or not early notions concerning equinoctial tides were not obtained from real or supposed equinoctial storms or floods rather than from the tides proper. But Strabo states that unusually large tides occur at the solstices.

§§ Neque adhuc satis cogitum est, anhelitune suo id muudus efficiat, retractamque cum spiritu regerat undam undique, si, (ut doctioribus placet) unum animal est: an sint depressi aliqui specus, quo reciprocata maria residant, atque unde se rursus exuberantia attollant: an luna causas tantis meatibus præbeat.—De Situ Orbis, III, L.

Most of the above Latin writers are quoted by Lalande, Astronomie, Vol. IV. See also Riccioli, Almagestum Novum, and Boscovich, "Dissertatio de maris æstu" (Rome, 1747). For some account of the tides in the Mediterranean, see Lalande, Astronomie, Vol. IV, pp. 119 et seq; also Alessandro Cialdi, "Cenni sul Moto Oudoso del Mare e sulle Correnti di Esso" (Rome, 1856), pp. 80 et seq.

Hence it would appear that high water at London on full and change was at that epoch 3^h 48^m, or more than an hour later than at present. The time of high water at London on full and change is given in Mr. Riddle's *Navigation* and in other works 2^h 45^m: Flamsteed made it 3^h.

At 0^h *ætas lunæ* this table gives 3^h as it also does at 30^d ($=\frac{24 \times 60}{48}$ = the assumed length of a synodical month in days). The present value at 0^d is about 2^h. It is to be noticed, however, that no regard is paid to the phase inequality in these tabular values, and that the age of the moon roughly determines the time of transit; and to this a constant lunitidal interval of 3^h has been applied. It appears from Flamsteed's remarks* that up to his time the rule involved in this old table was generally followed although some had calculated the time of transit more carefully.

Julius Cæsar Scaliger (1484–1558). *Exercitatio* LII,† entitled "De maris motu," is a general exposition of the tides, the views therein suggested, taken in connection with the time at which it was written, give to it considerable historic value.

Without professing to know the cause of the tides, Scaliger remarks that because the flow and ebb recur at definite and stated times, the moving cause must be definite in character.

His approach to the gravitational theory of tides appears in the following quotation:

For since [the tide] is observed to follow the course of the moon, they have judged it created by the moon. [You say] but the moon does not touch the waters. This has been a difficulty with some of the Peripatetics: likewise the magnet ought to give them difficulty. Because there is motion in the iron, although not in contact with the stone, wherefore may not the sea follow the body of the noblest orb? But it seems manifest. Surely there is a stillness of the sea at the times of the quadratures which is called a calm by the people. At the times of full moon the seas are rougher: so that they seem to restrain themselves with the desire of the moon. . . .

The tide then, in the usual sense of the word, is a duplex motion, and truly duplex: for in itself is a return. One part is in conformity with the motion of the *primum mobile*, the other is recurrent and contrary thereto: but both occur on definite times. For there are two motions just as in the contraction and expansion of the heart.

He speaks of the tides in the Arctic Ocean, around Great Britain, in the South Sea, the Adriatic Sea, the Red Sea, the Indus River, the Garonne River, the Euripus, and elsewhere; and tries to assign causes for their peculiarities.

He infers a declinational inequality in the tides because the moon changes her declination and so rises not always in the same place. He believes the long stretch of the Western Continent to explain the alternation of flow and ebb.

As with most of the early writers he finds difficulty in accounting for the ebb:

Wherefore does the sea ebb? Not only because of a dislike for the shores and a casting back, but also because it follows its loves, viz., the moon. It differs from the movement of the ocean.

Extracts from Hakluyt's Collection of the Early Voyages, Travels, and Discoveries of the English Nation.‡

72. A whirlpool on the Coast of Norway, called Malestrande.

Giraldus Cambrensis (who florished in the yeere 1210, vnder king Iohn) in his booke of the miracles of Ireland, hath certaine words altogether alike with these, videlicet:

Not farre from these Islands (namely the Hebrides, Island &c.) towards the North there is a certaine woonderful whirlpoole of the sea, whereinto all the waues of the sea from farre haue their course and recourse, as it were without stoppe: which, there conueying themselves into the secret receptacles of nature, are swallowed vp, as it were, into a bottomlesse pit, and if it chance that any shippe doe passe this way, it is pulled, and drawen with such a violence of the waves, that eftsoones without remedy, the force of the whirlepoole deuoureth the same.

The Philosophers describe foure indraughts of this Ocean sea, in the foure opposite quarters of the world, from whence many doe coniecture that as well the flowing of the sea, as the blasts of the winde, haue their first originall.§

. . . .

Instructions and notes very necessary and needfull to be obserued in the purposed voyage for discouery of Cathay Eastwards, by Arthur Pet, and Charles Iackman: given by M. William Burrough. 1580.

* Phil. Trans., 1682–3 [p. 12]; Abr. Vol. II, p. 555.

† Exotericarum Exercitationem ad Cardanum (Frankfort 1592; first published in 1557).

‡ London, 1809–12. A new edition; the original, London, 1599.

§ Vol. I. pp. 134–135.

And when you come vpon any coast where you find floods and ebs, doe you diligently note the time of the highest and lowest water in euery place, and the slake or still water of full sea, and lowe water, and also which way the flood doeth runne, how the tides doe set, how much water it hieth, and what force the tide hath to driue a ship in one houre, or in the whole tide, as neere as you can iudge it, and what difference in time you finde betwene the running of the flood, and the ebb. And if you finde upon any coast the currant to runne alwayes one way, doe you also note the same duely, how it setteth in euery place, and obserue what force it hath to driue a ship in one houre &c.*

This shows that the writer was aware of the fact that the duration of flood and ebb currents are not generally equal.

73. The voyage and trauell of M. Cæsar Fredericke, Marchant of Venice, into the East India, and beyond the Indies . . . . . . translated out of Italian by M. Thomas Hickocke. [1563.]

From Martauan I departed to goe to the chiefest Citie in the kingdome of Pegu, which is also called after the name of the kingdome, which voyage is made by sea in three or foure daies; they may goe also by lande, but it is better for him that hath marchandize to goe by sea and lesser charge. And in this voyage you shall haue a Macareo, which is one of the most marueilous things in the world that Nature hath wrought, and I neuer saw any thing so hard to be beleeued as this, to wit, the great increasing & diminishing of the water there at one push or instant, and the horrible earthquake and great noyse that the said Macareo maketh where it commeth. We departed from Martauan in barkes, which are like to our Pylot boates, with the increase of the water, and they goe as swift as an arrow out of a bow, so long as the tide runneth with them, and when the water is at the highest, then they drawe themselues out of the Chanell towardes some banke, and there they come to anker, and when the water is diminished, then they rest on dry land: and when the barkes rest dry, they are as high from the bottome of the Chanell, as any house top is high from the ground. They let their barkes lie so high for this respect, that if there should any shippe rest or ride in the Chanell, with such force commeth in the water, that it would ouerthrowe shippe or barke: yet for all this, that the barkes be so farre out of the Chanell, and though the water hath lost her greatest strength and furie before it come so high, yet they make fast their prowe to the streme, and oftentimes it maketh them very fearefull, and if the anker did not holde her prow vp by strength, shee would be ouerthrowen and lost with men and goods. When the water beginneth to increase, it maketh such a noyse and so great that you would thinke it an earthquake, and presently at the first it maketh three waues. So that the first washeth ouer the barke, from steumme to sterne, the second is not so furious as the first, and the thirde rayseth the Anker, and then for the space of sixe houres while the water encreaseth, they rowe with such swiftnesse that you would thinke they did fly: in these tydes there must be lost no iot of time, for if you arriue not at the stagions before the tyde be spent, you must turne backe from whence you came. For there is no staying at any place, but at these stagions, and there is more danger at one of these places then at another, as they be higher and lower one then another. When as you returne from Pegu to Martauan, they goe but halfe the tide at a time, because they will lay their barkes vp aloft on the bankes, for the reason aforesayd. I could neuer gather any reason of the noyse that this water maketh in the increase of the tide, and in deminishing of the water. There is another Macareo in Cambaya, but that is nothing in comparison of this.†

74. Another testimonie of the voyage of Sebastian Cabot to the West and Northwest, taken out of the sixt Chapter of the third Decade of Peter Martyr of Angleria.

As hee traueiled by the coastes of this great land,‡ (which he named Baccalaos) he saith that hee found the like course of the waters toward the West, but the same to runne more softly and gently then the swift waters which the Spaniards found in their Nauigations Southwards. Wherefore it is not onely more like it to be true, but ought also of necessitie to be concluded that betweene both the lands hitherto vnknowen, there should be certaine great open places whereby the waters should thus continually passe from the East vnto the West: which waters I suppose to be driuen about the globe of the earth by the uncessant mouing and impulsions of the heauens, and not to bee swallowed vp and cast vp againe by the breathing of Demogorgon, as some haue imagined, because they see the seas by increase and decrease to ebbe and flowe.§

. . . .

To prove by authoritie a passage to be on the Northside of America, to goe to Cathaia, and the East India.

Also it appeareth to be an Island, insomuch as the Sea runneth by nature circularly from the East to the West, following the diurnal motion of Primum Mobile, which carieth with it all inferiour bodies moueable, aswel celestiall as elemental: which motion of the waters is most euidently seene in the Sea, which lieth on the Southside of Afrike.‖ . . . . .

---

* Vol. I, p. 492.

† Vol. II, p. 362.

‡ Possibly as far south as North Carolina, but probably not far south of Delaware. See U. S. C. & G. Survey Report, 1890, pp. 475–476. The name probably has reference to the fishing grounds off Newfoundland.

§ Hakluyt, Vol. III, p. 30.

‖ Hakluyt, Vol. III, p. 36. "The sea hath three motions. 1. Motum ab oriente in occidentem. 2. Motum fluxus & refluxus. 3 Motum circularem. Ad cœli motum elementa omnia (excepta terra) mouentur." This last sentence expresses the opinion held by Columbus and the other early navigators.

Futhermore, the current in the great Ocean, could not haue beene maintained to runne continually one way, from the beginning of the world unto this day, had there not beene some thorow passage by tho fret aforesayd,* and so by circular motion bee brought againe to maintayne it selfe: For the Tides and courses of the sea are maintayned by their interchangeable motions: as fresh riuers are by springs, by ebbing and flowing, by rarefacation aud condensation. †

So that it resteth not possible (so farre as my simple reason can comprehend) that this perpetuall current cau by any means be maintained, but onely by continuall reaccesse of the same water, which passeth thorow the fret, and is brought about thither againe, by such circular motion as aforesayd. ‡

. . . And first it may be called in controuersie, whether any current continually be forced by the motion of Primum mobile, round about tbe world, or no: For learned men doe diuersly handle that question. The naturall course of all water is downeward, wherefore of congruence they fall that way where they finde the earth moste lowe and deepe: in respect whereof, it was erst sayd, the seas doe strike from the Northern landes Southerly.§ Violently the seas are tossed and troubled diuers wayes with the windes, encreased and diminished by the course of the Moone, hoised vp & downe through the sundry operations of the Sunne and the starres: finally, some be of opinion, that the seas be carried in part violently about the world, after the dayly motion of the highest moueable heauen, in like manner as the elements of ayre and fire, with the rest of the heauenly spheres, are from the East unto the West. And this they doe call their Easterne current, or leuant streame.‖

. . . .

75. The sixt book of the first Decade, to Lodouike Cardinal of Aragonie. Written by Peter Martyr of Angleria Milenoes, Counsayler to the Kyng of Spaine. The third voyage of Colonus the Admirall [1498].

No great space from this Ilande, [Puta] euer towarde the West, the Admiral [Colonus] saith he found so outragious a fal of water, running with such a violence from the East to the West, that it was nothing inferiour to a mightie streame falling from high mountaynes. Hee also confessed, that since the first day that euer hee knewe what the sea meant, hee was neuer in such feare. Proceeding yet somewhat further in this daungerous voyage, he founde certaine goulfes of eight myles, as it had bin the entraunce of some great hauen, into the which the sayde violent streames did fall. These goulfes or streyghtes hee called Os Draconis, that is, the Dragones mouth: and the Iland directly ouer against the same, hee called Margarita. Out of these strayghtes, issued no lesse force of freshe water, whiche encountering with the salt, dyd striue to passe foorth, so that beetwene both the waters, was no small conflict: But entering into the goulfe, at the length hee founde the water thereof very fresh and good to drinke.¶

76. *Francis Bacon* (1561–1626).

The chief merit of Bacon's essay entitled "De fluxu et refluxu maris," apart from its suggestions of inquiry into the tides of various countries, is the insistence upon the progressive character of the tide causing it to move from place to place; that is, the waters do not boil up, as it were, and produce everywhere simultaneous tides. He thinks, however, that the semimenstrual and semiannual inequalities may occur everywhere simultaneously, or at least upon the same day. He says:

That this** should be done so quickly, namely, twice a day; as if the earth, according to that foolish conceit of Apollonius, were taking respiration, and breathing out water every six hours and then taking it in again; is a very great difficulty.

He notes that this simultaneous rising is not proven from reports concerning certain wells, nor even from the (then supposed) fact that tides are simultaneous at Florida and the coasts of Europe. He remarks that this may result naturally enough from the tide coming from the Indian Ocean around southern Africa.

Besides the sexhorary motion, the semimonthly, and the semiannual, "whereby the tides receive a great and remarkable increase at the equinoxes," he mentions a monthly motion which he may have indistinctly connected with the moon's parallax.

He believes the earth to stand fixed while all of the universe outside, including the air and waters of the sea, has a tendency westward,—the heavenly bodies moving much more rapidly than the air or the waters of the sea. He believes that the two continents so obstruct the tidal waters,

---

* About the north of Labrador.

† "The flowing is occasioned by reason that the heate of the moone boyleth and maketh the water thinne by way of rarefaction."

‡ Hakluyt, Vol. III, p. 36, 37.

§ Cf. Saint-Pierre.

‖ Hakluyt, Vol. III, p. 52.

¶ Vol. V, p. 196.

** I. e., rise and fall of the tide.

which would otherwise progress uniformly but slowly from east to west around the earth, that their progressive motion is converted into a motion whose period is a half lunar day; also, that, because of this westward motion, those gulfs or bays which open eastward should have larger tides than similar bodies of water opening westward.

Concerning the coincidences in the periods of the tides and of the heavenly bodies he says:

Yet it will not immediately follow (and we would have men observe this) that things which correspond in the course and periods of time, or even in the manner of carriage, are in their nature subordinate, and the cause one of the other. For I do not go so far as to assert that the motions of the moon or sun are set down as the causes of the inferior motions which are analogous to them, or that the sun and moon (as is commonly said) have dominion over those motions of the sea (though such thoughts easily find entrance into men's minds by reason of their veneration for heavenly bodies); indeed in that very half-monthly motion (if rightly observed) it would be a very strange and novel kind of obedience, for the tides at the new and full moon to be affected in the same way, while the moon is affected in opposite ways; and many other things might be adduced which would destroy these fancies about dominations, and lead us rather to conclude that these correspondences arise out of the universal passions of matter, and the primary combinations of things, not as if one were governed by the other, but that both emanate from the same origins and fellow causes. Nevertheless (however it be) what I have said remains true, that nature delights in correspondences, and scarce admits anything unique or solitary.*

77. *William Gilbert* (1540–1603). Gilbert, in his New Philosophy,† asserts that the moon and earth have a mutual attraction for each other analogous to magnetic attraction; that the tides are produced by this force of the moon and not by its rays or its light. He cannot see how the ebbing of the tide follows from the direct attraction unless the interior of the earth contain humors which retreat into the earth when the tide forces cease, and so cause the surface of the sea to descend. He finds difficulty also in understanding why earth and moon do not fall together.

John Kepler (1571–1630). As early as 1598, Kepler took exception to the views of Galileo concerning the cause of the tides. In the introduction to the Cosmographical Mystery he says:

For he who attributes the motion of the seas to the motion of the earth, clearly assumes a violent motion; but he who says that the seas adhere to the moon, makes in part a natural assumption.‡

In his Foundations of Astrology (1602) he asserts, in Thesis 15, that, as proven by experience, all things swell up with the waxing moon and subside when she is waning. This foreknowledge is useful in housekeeping, farming, medicine, and navigation. But he says:

Physicists have not yet fully ascertained the reason for this sympathy.§

In Thesis 47 he likens the circulation of the waters of the earth to the circulation of living animals, and suggests that the observations of many years must needs be collated in order to investigate any long-period circulation which may exist. He says:

Hereupon Caesius attributes something to the nineteen-year cycle of the moon; from whom, indeed, all faith can not be taken away. For mariners say the greatest tides return upon the same days of the year after the period of 19 years; and the moon, loaded with vapors, seems suitable for this purpose, because she possesses either an excess or a deficit of moisture.‖

This is probably the first hint at a 19 year inequality in the tides. Pliny supposed an inequality of 100 lunar months to exist.

In the introduction to the Motions of the Planet Mars, Kepler lays down the following axioms concerning gravitation and the cause of the tides (1609). He is the first to assert that the attractive forces exercised by earth and moon upon each other are proportional to their respective masses.

Therefore the true doctrine of gravity depends upon these axioms:

Any corporeal substance, seeing that it is corporeal, is by nature destined to rest in any place, in which it is placed alone outside the sphere of virtue of a cognate body.

Gravity is a mutual corporeal affection among cognate bodies to union or conjunction (in which class of things is also the magnetic faculty) such that the earth attracts a stone by as much more as the stone travels toward the earth.

---

* The Works of Francis Bacon (Edition Spedding et al.), Vol. V, p. 448.

† De Mundo Nostro Sublunari, Philosophia Nova, (1651). See also De Magnete, Magneticisque Corporibus, et de et de Magno Magnete Tellure" (1600); translation by Mottelay (1893).

‡ Opera Omnia (Edition Ch. Frisch, 1865–71) Vol. I, p. 64.

§ Ibid., p. 422.

‖ Ibid., p. 430.

Heavy bodies (especially if we place the earth in the center of the universe) are not carried to the center of the universe, as to the center of the universe, but as to the center of a round cognate body such as the earth. Therefore wherever the earth is located or wherever it is transported by its animal faculty, heavy bodies are always drawn toward it.

If the earth be not round, heavy bodies will not be borne from everywhere straight to the middle point of the earth, but will be borne to diverse points from diverse sides.

If two stones be located in any part of the universe, mutual neighbors outside the sphere of influence of the third cognate body, these stones in the similitude of two magnetic bodies will come together in an intermediate place, the first approaching the other by so much space, as the other is a heavy mass in comparison.

If the moon and earth were not retained by animal force or some other equal [force], each in its orbit, the earth would ascend toward the moon a fifty-fourth part of the space, and the moon would descend toward the earth about 53 parts of the space, where they would be united: provided, however, that the substance of both be of one and the same density.

If the earth should cease to attract its waters, all marine waters would be elevated and would flow into the body of the moon.

The sphere of the attracting virtue, which is in the moon, extends to the earth and incites the waters under the Torrid Zone, because of its meeting [with them] wheresoever she happens to be in the zenith of the place; [they are incited] insensibly in enclosed seas, but sensibly where there are very broad beds of the ocean, and abundant liberty of reciprocation for the waters; by which cause the shores in the zones and neighboring climes are made bare and even as far as the Torrid Zone; the neighboring oceans cause the waters of gulfs to be more reduced. Therefore in a broader bed of the ocean, the waters being in motion, it may happen that in its more narrow gulfs, provided not too closely confined, the waters, when the moon is present, may even seem to flee from her: in fact, they subside when the abundance of water is diminished outside.

The moon speedily traversing the heavens, although the waters cannot follow as quickly, causes a westward flow in the Torrid Zone, until it impinges against opposing shores and is deflected from them; the assembly or army of waters is indeed dissolved by the departure of the moon, which [water] is in its march toward the Torrid Zone, because deserted by the attraction which had called it forth, and with its vigor taken away, as in water vases, goes back and leaps against its shores and conceals them: and this impetus begets through the absence of the moon another impetus, until the returning moon receives and moderates the curbs of this impetus and at the same time with her motion turns them about. Thus shores equally bare are all filled at the same hours, but the more distant shores are filled later, some in diverse manners because of diverse approaches of the ocean.*

In the fourth book of his Harmonics† (1619) Kepler likens the tide to the breathing of terrestrial animals and especially to the breathing of fishes; but it does not follow from this that he renounced his earlier views.

Of those who attributed the tides to some attraction of the moon analogous to magnetic attraction, may be mentioned, Scaliger, Gilbert, the College of Jesuits at Coimbra, Antonio de Dominis, and Stevin.

78. *Galileo Galilei* (1564–1642).

The fourth dialogue of Galileo's System of the World‡ is devoted to the discussion of the tides. His object is to show that they are due to the non-uniform motion (in space) of the particles of the sea and solid earth, and that their presence goes to prove the earth's axial rotation and its motion around the sun.§ Since the earth turns from west to east and also moves eastward in its orbit, the actual motion of a place in the nighttime must be greater than the actual motion of the same place in the daytime; hence the diurnal acceleration and retardation of the otherwise uniform motion in space of any given sea, and hence the tidal cause whose period is a solar day. Galileo does not realize that the tides call for the lunar instead of the solar day. In 1616 Kepler points out to Galileo that the lunar day should be taken.‖

He notices that there are several "varieties" of tidal movements depending upon the localities. Considerable tides may be accompanied by weak currents or by strong currents, and small tides may be accompanied by strong currents, as happens around several islands of the Mediterranean. Such "varieties" he thinks would be present if a vessel (like the Mediterranean Sea) be subjected to a non-uniform movement, but not otherwise. He notices that (analogous to the pendulum) the undulations of waters in short vessels are more frequent than in long vessels; also that an increase of depth increases their frequency.¶

---

* Opera Omnia, Vol. III, p. 151.

† Opera Omnia, Vol. V, p. 255.

‡ Galileo's earlier paper on tides, written in 1616 and entitled "Discorso sopra il flusso e reflusso del mare" (Opere, Vol. II), contains the same theory as that found in his System of the World.

§ Kepler points out that this is no proof of the earth's motion, Opera Omnia, Vol. VI, p. 180.

‖ Kepler, Opera Omnia, Vol. II, pp. 116, 117.

¶ Cf. Forel's seiche period, § 31.

In reference to certain other hypotheses the dialogue reads: *

And there dwelleth not many miles from hence a famous Peripatetick, that alledgeth a cause for the same newly fished out of a certain Text of *Aristotle*, not well understood by his Expositors, from which Text he collecteth, that the true cause of these motions doth only proceed from the different profundities of Seas: for that the waters of greatest depth being greater in abundance, and therefore more grave, drive back the Waters of lesse depth, which being afterwards raised, desire to descend, and from this continual colluctation or contest proceeds the ebbing and flowing. Again those that referre the same to the Moon are many, saying that she hath particular Dominion over the Water; and at last a certain Prelate † hath published a little Treatise, wherein he saith that the moon wandering too and fro in the Heavens attracteth and draweth towards it a Masse of Water, which goeth continually following it, so that it is full Sea alwayes in that part which lyeth under the Moon; and because, that though she be under the Horizon, yet nevertheless the Tide returneth, he saith that no more can be said for the salving of that particular, save onely, that the Moon doth not onely naturally retain this faculty in her self; but in this case hath power to confer it upon that degree of the Zodiack that is opposite unto it. Others, as I believe you know, do say that the Moon is able with her temperate heat to rarefie the Water, which being rarefied, doth thereupon flow.

Later on in the dialogue still other theories are referred to.

Galileo's indistinct notion of the tidal period and his contempt for the idea that the moon is the principal cause of the tides may be gathered from the following extracts:

The period of six hours therefore is no more proper or natural than those of other intervals of times, though indeed its the most observed, as agreeing with our Mediterrane, which was the onely Sea that for many Ages was navigated: though neither is that period observed in all its parts; for that in some more angust places, such as are the *Hellespont*, and the *Ægean* Sea, the periods are much shorter, and also very divers amongst themselves; for which diversities, and their causes incomprehensible to *Aristotle*, some say, that after he had a long time observed it upon some cliffes of *Negropont*, being brought to desperation, he threw himself into the adjoyning *Euripus*, and voluntarily drowned himself.

. . . .

Now follow the two other Periods, Mouthely, and Annual, which do not bring with them new and different Accidents, other than those already considered in the diurnal Period; but they operate on the same Accidents, by rendring them greater and lesser in several parts of the Lunar Moneth, and in several times of the Solar Year; as if that the Moon and Sun did each conceive it self apart in operating and producing of those Effects; a thing that totally clasheth with my understanding, which seeing how that this [movement] of Seas is a local and sensible motion, made in an immense mass of Water, it cannot be brought to subscribe to Lights, to temperate Heats, to predominacies by occult Qualities, and to such like vain Imaginations, that are so far from being, or being possible to be Causes of the Tide; that on the contrary, the Tide is the cause of them, that is, of bringing them into the brains more apt for loquacity and ostentation, than for the speculation and discovering of the more abstruse secrets of Nature; which kind of people, before they can be brought to prononnce that wise, ingenious, and modest sentence, *I know it not*, suffer to escape from their mouths and pens all manner of extravagancies.

Galileo thinks the monthly inequality due to a supposed retardation and acceleration in the earth's orbital motion, caused by the moon being alternately outside and inside of the earth's orbit, and his reason for this is that distant bodies have longer periodic times of revolution about the sun than near ones. He believes that this irregularity of motion may be too small to have been observed by astronomers and yet sufficiently great for producing the tides.

But how each Planet governeth itself in its particular revolution, and how precisely the structure of its Orb is framed; which is that which is vulgarly called the *Theory* of the *Planets*, we cannot as yet undoubtedly resolve.

. . . .

But amongst all the famous men that have philosophated upon this admirable effect of Nature, I more wonder at *Kepler* than any of the rest, who being of a free and piercing wit, and having the motion ascribed to the Earth, before him, hath for all that given his ear and assent to the Moons predominancy over the Water, and to occult properties, and such like trifles.

Because of its author, the system of Galileo attained recognition from numerous sources. *Gassendi* (1592–1655), in his De Æstu Maris follows Galileo in the main; and it appears by his later writings that Kepler may in part have renounced his own more rational views. Riccioli explains Galileo's system at considerable length; the attempts of Balianus and Wallis in this direction are given below. *Fournier* at times, in his Hydrographie, resumes Galileo's theory as modified by Gassendi, but he was not altogether pleased with it.

In justice to Galileo it should be added that he at one time contemplated a treatise on the theory of tides, but that his religious persecutors rendered it well-nigh impossible for him to continue his scientific work.

---

* Thomas Salusbury's translation, found in his Mathematical Collections and Transactions (London, 1661).

† Antonio de Dominis, Archbishop of Spalatro.

79. *John Baptist Riccioli* (1598–1671). In the second, fourth, and ninth books of his Almagestum Novum,* Riccioli treats of the tides.

Galileo's system or theory is explained and refuted at some length. He quotes the theses laid down by Kepler, and mentions a great number of other theories both ancient and modern, thus giving much historic value to his work.

Riccioli divides the ordinary motions of the sea into three classes: Motion in latitude, in longitude, and in height. The last he regards as the tide—the *æstus* or the *fluxus ac refluxus.* The motions from east to west are real or supposed ocean currents, but the north-and-south motions are, at least in part, tidal. For example, he speaks of their period being twenty-four hours around the Molucca and Philippine Islands.

He gives some account of the tides of Europe including the Mediterranean Sea, and makes some mention of the tides in America. For example, he states that at the island of Martinique and in the Caribbean Sea the sea rises scarcely 1 foot.

He gives a table showing for various localities the times of high water at new and full moon; also a table, after Fournier, showing the times of tide for each day of the month. The values upon successive days generally become later by 48 minutes and recur after an interval of 15 days.

John Baptist Balianus.† This writer undertakes to supplement the most obvious defect of Galileo's theory by supposing the moon to describe the orbit about the sun, and the earth to accompany the moon revolving about the latter every month. This would give alternate accelerations and retardations in motions in space of any given point of the earth, and the period would be a lunar instead of a solar day.

Later, Dom Jacques Alexandre, Benedictine, in a paper on tides which took the prize of the Academy of Bordeaux in 1726 adopts the hypothesis of Balianus. M. de Mairan refutes this in the Mémoires de l'Académie, 1727.

Jeremiah Horrox (c. 1619–1640). Horrox, the astronomer, observed tides for three months in 1640 shortly before his death, which occurred when he was only 21 or 22 years of age.

Horrocks appears to have been the first person who undertook the prosecution of a continuous course of observations of the tides, for the express purpose of obtaining a series of facts which might form the ground work of a philosophical investigation of the subject. ‡

It seems, however, that the heights of remarkably high tides upon the Thames have been recorded ever since the Norman conquest,§ and that Candale made in 1575 several observations on the tides at the mouth of the Garonne.‖

80. *René Descartes* (1596–1650).

In the fourth part of his Principles of Philosophy, Descartes attempts (c. 1644) to explain the tides in accordance with his theory of the movements of the heavenly bodies set forth in the third part. According to the vortex theory, the sun, being at the center of the solar system and surrounded by a boundless fluid, does, by its axial rotation, set in motion the fluid layer adjacent to it; this layer imparts its motion to the adjoining layer; this in turn to the next, and so on to the distant parts of space. The planets placed in their respective fluid layers are in this way carried about the sun. Similarly, each planet, because of its axial rotation, is the center of a secondary vortex system. In this are involved not only the satellites attending the planet, but likewise the air and water by which it is surrounded. By means of the intervening fluid, he supposes the moon to exert a pressure upon the atmosphere and, in some way, produce the tides, low water when she is upon the meridian and high water when 90 degrees distant. He supposes that spring tides are due to the moon's approaching the earth in a syzygy, and that neap tides are due to her receding from the earth when in quadrature.

The vortex theory of the universe appealed to the popular imagination, because there seemed to be no occult or unfamiliar properties of matter involved. It exerted a considerable influence for about one hundred years. The last honor paid to it by the Academy of Sciences of Paris was in the year 1740 when the essay of Cavalleri received a prize along with the essays of Bernoulli, Maclaurin, and Euler.

* First printed in 1651.

† Riccioli, Almagest. Novum, Bk. 9.

‡ Grant's History of Physical Astronomy, p. 428.

§ Childrey, Phil. Trans. (1670); Abr. Vol. I, p. 516.

‖ Lalande, Astronomie, Vol. IV, p. 35.

In Propositions LII and LIII, Bk. II, Newton shows the inability of the vortex theory to account for the planetary motions. For, in order that the planets be so carried, their periodic times must be as the squares of their distances from the sun, and, moreover, their densities must be the same as that of the surrounding fluid.

As already noted (§ 66), Seleucus held a view somewhat similar to that of Descartes. Among writers upon tides who adopted this explanation are Varenius and Jacques Cassini.

81. *Bernhardus Varenius* or *Bernhard Varen* (1622-1670).

The *Geographia Generalis* of Varenius appeared in 1650; in 1733 the work was translated into English by Dugdale,* and it is from this that the following quotations upon the subject of the tides are taken.

Although apparently not satisfied with Descartes's explanation, he accepts it, with some amendments, as the most plausible one known to him.

He states that, as shown by observation, water has but one natural motion, viz. from a higher to a lower place; also that—

> *The general Motion of the Sea is twofold; the one is constant, and from East to West: the other is composed of two contrary Motions, and called the Flux and Reflux of the Sea, by which, at certain Hours, it flows towards the shores, and at others back again.*

He instances numerous straits where the east-to-west motion is very strong.

> *The Cause of this general Motion of the Sea from East to West is uncertain.*
>
> THE *Aristotelians* (tho' neither they, nor their Master, nor any *European* Philosopher, had the least Notion of these Things, before the *Portuguese* sailed thro' the ocean in the *Torrid Zone*) suppose, that it is caused by the Prime Motion of the Heavens, which is common not only to all the Stars, but even, in part, to the Air and Ocean; and by which they, and all things, are carried from East to West. Some *Copernicans* (as *Kepler, etc.*), altho' they acknowledge the Moon, to be the prime cause of this Motion, yet they make the Motion of the Earth not a little contribute to it, by reason that the Water, being not joined to the earth, but contiguous only, cannot keep up with it's quick Motion towards the East; but is retarded and left toward the West; and so the Sea is not moved from one Part of the Earth to another, but the Earth leaves the Parts of the Sea one after another.
>
> OTHERS, who are satisfied with neither of these Causes, have recourse to the Moon; which they will have to be the Governess of all Fluids, and therefore to draw the Ocean round with her from East to West. If you ask, how she performs this? They answer, it is, by an occult Quality, a certain Influence, a Sympathy, her Vicinity to the Earth, and such like. It is very probable indeed the Moon, some way or other, causes this Motion, because it is observed to be much more violent at the New and Full Moon, than about the Quadratures, when it is, for the most Part, but small.
>
> THE ingenious *des Cartes* mechanically explains how the Moon may cause this Motion, both in the Water, and the Air.

He then attempts to explain this motion by Descartes's theory.

After attempting to derive the phenomenon of the tides from considering the westward flowing current he says:

> HENCE we may determine, that the Flux and Reflux of the Sea is no way distinct from that general Motion, which we explained in the former Proposition, whereby the Ocean is perpetually moved from East to West; for it is only a certain Mode or Property of that Motion.
>
> . . . .
>
> *To explain the Cause of the Swelling and Swaging of the Sea, vulgarly called it's Flux and Reflux.*
>
> THERE is no Phænomenon in Nature that has so much exercised and puzzled the Wits of Philosophers and learned Men as this. Some have thought the Earth and Sea to be a living Creature, which, by it's Respiration, causeth this ebbing and flowing. Others imagined that it proceeds, and is provoked, from a great Whirl-pool near *Norway*, which, for Six Hours, absorbs the Water, and afterwards, disgorges it in the same space of Time. *Scaliger*, and others, supposed that it is caused by the opposite Shores, especially of *America*, whereby the general Motion of the Sea is obstructed and reverberated. But most Philosophers, who have observed the Harmouy that these Tides have with the Moon, have given their Opinion, that they are entirely owing to the Influence of that Luminary. But the Question is, what is this Influence? To which they only answer, that it is an occult Quality, or Sympathy, whereby the Moon attracts all moist Bodies. But these are only Words, and signify no more than that the Moon does it by some means or other, but they do not know how: Which is the Thing we want.

He then returns to Descartes for an explanation of the phenomenon and offers some amendments.

* "A Compleat System of General Geography: . . . . Originally written in Latin by Bernhard Varenius, M. D., since improved and illustrated by Sir Isaac Newton and Dr. Juri London, 1734, Vol. I, Ch. 14.

Having noted that greater tides happen at the syzygies than at the quadratures because the moon is nearer to the earth in the first instance, he says:

YET in some Places there are higher Tides at the Full Moon than at the New, which I cannot account for, unless they be the Effects of it's greater Light at that time. Nor can it be otherwise explained, why at the Full Moon Vegetables and Animals are impregnated with a greater quantity of Sea Moisture, than at the New, tho' even then the Tides are every whit as high. It is very wonderful what one *Twist*, a *Dutchman*, relates in his Description of *India*. He says, that in the Kingdom of *Guzarat* (where he lived many Years) their Oysters, and Crabs, and other Shell Fish, are not so fat and juicy at the Full Moon as at the New, contrary to their Nature in all other Places. Nor is it less admirable, that on the Coast of the same Kingdom, near the Mouths of the River *Indus*, the Sea swells, and is troubled, at the New Moon, when not far from hence, *viz.* in the Sea of *Calicut*, the greatest Rise of the Waters is at the Full. But it is requisite that we should have repeated Enquiries and Observations about these Matters, before we pretend to solve their Phaenomena.

. . . .

DES Cartes pretends to account for this Phaenomenon by his Hypothesis, but I cannot apprehend his Meaning by his Words, nor how it can be deduced from it. It is probable, that the Sun and the general Winds may contribute much to raise these Tides, when, in the equinoxes, the Sun is vertical to the Ocean in the middle of the *Torrid Zone*, and therefore may cause both the Wind and Water to rage, and the former to agitate the latter. The contrary of which may happen about the Solstices. Or we may say, that these extraordinary Tides then happen by the same Reason, and proceed from the same Cause that frequent Rains and Inundations proceed from in these Seasons.

. . . .

THE Flux is caused by the Pressure of the Moon, or the celestial Matter, between it and the Sea, and continues no longer than the Cause forces it: but in the Ebb, the Sea only flows from a higher to a lower Place, which is the natural Motion of the Water.

The tides are generally highest in those places over which the moon is vertical—

BECAUSE those Places are more pressed, and the swelling of the Sea is greater, over which the Moon squeezeth the celestial Matter, whereby greater Tides are produced: but where the incumbent Matter is less squeezed, and other Causes conspire, the Alteration will be less.

. . . .

SINCE the Moon, in the Meridian, is nearer any Place than when she is in the Horizon, (because the Hypotenuse of any right-angled Triangle is longer than the Perpendicular) it follows (by Proposition 16, of this Chapter) that then it ought to be High Water in that Place (where she is full South). And when she is full North, or in the lower Part of the Meridian Circle, it ought to be also High Water in the same Place, because, tho' she be not there, yet the opposite Part of the Vortex of the Earth is straitned, and hath the same Effect, as if the Body of the Moon itself were present.

. . . .

Then follow descriptions of tides in numerous localities, also a somewhat improved table for finding the time of the moon's transit from the age of the moon. As usual, no account is taken of the sun's effect because the moon alone is supposed to be responsible for the tides.

The Gyrations of the Sea, which we call Vortexes, or Whirlpools, are of three Kinds.

SOME Whirlpools only turn the Water in a Round; others at Times absorb, and emit or vomit it up; and some again suck it in, but do not cast it out. And doubtless there is a fourth Kind somewhere in the Channel of the Sea, which may throw out Water but takes none in. I do not remember any such to be recorded by Authors; only upon the dry Land there are several observed. The *Dutch* Mariners call these Whirlpools *Maelstroom*.

THERE are but very few of these, at least, that have been taken Notice of.

BETWEEN *Negropont* and *Greece* there is a famous Whirlpool; called the *Euripus*, much talked of because of the fabulous Story of *Aristotle's* dying there. *Scaliger* endeavours to explain it thus. It is not much amiss (says he) to suppose the Water, received into the Caverns, in the Cliffs of the Rocks below, issueth from thence; for by the continual running in of the Water the little rocky Bays are filled, and being full, they emit what they received, thro' winding and subterraneous Passages; whose Capacity is such, that they pour out the Water for so many Hours, whereby the Tides are now obstructed or repelled, and a little after forwarded or helped. But any one may perceive the insufficiency of this Cause.

THE *Maelstroom* on the Coast of *Norway*, is the swiftest and largest known Vortex; for it is said to be thirteen *Dutch* Miles in Circuit; in the middle of which there is a Rock, which the People thereabouts call the *Mouske*. This Whirlpool, for six Hours, sucks in whatever approaches it, or comes nigh it; not only Water, but Whales, loaded Ships, and other Things; and in as many Hours disgorges them all again, with a hideous Noise, Violence, and whirling round of the Water. The Cause is latent.

BETWEEN *Normandy* in *France*, and *England*, there is a Whirlpit, toward which Ships are drawn with an incredible Celerity; but when they come near the middle of the Swallow, they are, with the same Force, thrown out again.

Isaac Vossius (1618–1689), in his book entitled De Motu Marium et Ventorum, contends that the tides are produced by the heat of the sun, and that their apparent connection with the moon is only a casual synchronism.*

*Reviewed by Wallis, Phil. Trans. (1666) [Vol. I, p. 286]; Abr., Vol. I, p. 105.

82. *Dr. John Wallis* (1616–1703).

Largely in accord with Galileo's explanation of the tides in his System of the World, is the hypothesis of Dr. Wallis, found in Philosophical Transactions for the year 1666.

Galileo's theory makes the tides depend upon the non-uniform motion (in space) of the different parts of the earth; that is, the places having night go eastward faster than those having day. Wallis takes into account the fact that the earth's center does every month describe an orbit about the common center of gravity of earth and moon, and not about the moon itself as Balianus assumes. This causes the places having the moon below the horizon to move faster than those having the moon above. This gives an acceleration and retardation having a period of a lunar day—a period which Galileo did not obtain. Neither writer, however, makes it clear why there should be two high waters and two low waters daily instead of one.

[To show that the tides cannot be caused except by the existence of attraction, we may proceed as follows:

Suppose the axial rotation of the earth were zero; then (at least for a long time) a given hemisphere of the earth would face a certain fixed star as the earth is carried around the center of gravity of itself and the moon. Every particle of the earth would then have equal and parallel motions and so, of course, equal centrifugal forces. Hence there could be no differential or tidal forces from this cause alone.

But given sufficient time, and some kind of mutual attraction between earth and moon which are supposed to be revolving, without axial rotation, about their common center of gravity; the two bodies will eventually face each other and revolve upon their axes once a month as if parts of one rigid body, and the spherical surface of the water will then have become spheroidal, chiefly, we may now suppose, on account of the centrifugal force. But the amount of this centrifugal deformation is obviously zero when the axial rotation is zero, and it can become sensible only when the axial rotation becomes, or tends to become, monthly. But in the case of nature, the earth has an axial rotation whose period is constant and no approach to the month. Hence this centrifugal force can have no effect upon the tide.*]

He tries to account for the spring and neap tides by the fact that when the moon is full the velocity (in space) of the earth's center about the sun is a minimum; when new, a maximum; and when in quadrature, the velocity has its mean value.

In quotations given below it will be noticed how nearly Wallis comes to the solution of the problem of universal gravitation.†

The sea's ebbing and flowing has so great a connexion with the moon's motion, that in a manner all philosophers have attributed much of its cause to the moon, which either by some occult quality, or particular influence which it has on moist bodies, or by some magnetic virtue, drawing the water towards it, which should therefore make the water highest where the moon is vertical, or by its gravity and pressure downwards upon the terraqueous globe, which should make it lowest, where the moon is vertical, or by whatever other means, has so great an influence on, or at least connexion with, the sea's flux and reflux, that it would seem very unreasonable to separate the consideration of the moon's motion from that of the sea: the periods of tides, to say nothing of the greatness of them near the new and full moon, so constantly waiting on the moon's motion, that it may be well presumed, that either the one is governed by the other, or at least both by some common cause.

I consider therefore, that in the tides, or the flux and reflux of the sea, besides extraordinary extravagances, or irregularities, whence great inundations or strangely high tides follow, (which yet perhaps may prove not to be so merely accidental as they have been thought to be, but might from the regular laws of motion, if well considered, be both well accounted for, and even foretold;) these three notorious observations are made of the reciprocation of tides.

1. The diurnal reciprocation; whereby twice in somewhat more than 24 hours, we have a flood and an ebb;

---

* The idea that the tides are due in part to a centrifugal disturbing force supposed to be set up because the earth is carried around the center of gravity of it and the moon or sun, has appeared from time to time since the days of Wallis. E. g., Hube, "Vollständiger und fasslicher Unterricht in der Naturlehre," Pt. III (Leipzig, 1794), pp. 2 0 et seq.; Alexander Wilcocks, "An Essay on the Tides" (Phila., 1855); P. E. Chase, Jour. Frank. Inst., Vol. 47 (1864), pp. 137, 208; Newcomb, Popular Astronomy (1878), p. 91. This part of the explanation does not appear in Newcomb and Holden's Astronomy. It is, however, still given in some text-books and popular lectures. Bernoulli, in the third and fourth chapters of his essay on tides, especially mentions the inability of this centrifugal force to alter the figure of the earth.

In this connection, see Thomson and Tait, § 813; also Darwin, Nature, Vol. 43 (1891), p. 609.

† "Hypothesis on the Flux and Reflux of the Sea," Phil. Trans. (1666) [Vol. I, pp. 263–281]; Abr., Vol. I, pp. 89–101. In quoting, Hutton's abridgment will generally be followed.

or a high-water and low-water. 2. The menstrual; whereby in one synodical period of the moon, suppose from full-moon to full-moon, the time of those diurnal vicissitudes moves round through the whole compass of the *Νυχθήμερον*, or natural day of 24 hours; as for instance, if at the full moon the full sea be at such or such a place just at noon, it shall be the next day at the same place somewhat before one of the clock; the day following, between one and two; and so onward, till at the new moon it shall be at midnight; the other tide, which in the full moon was at midnight, now at the new moon coming to be at noon; and so forward, till at the next full moon the full sea shall at the same place come to be at noon again: Again, that of the spring tides and neap tides; about the full moon and new moon the tides are at the highest, at the quadratures the tides are at the lowest; and at the times intermediate, proportionably. 3. The annual,* whereby it is observed, that at some part of the year, the spring tides are yet much higher than the spring tides at others, which times are usually taken to be at the spring and autumn or the two equinoxes; but I have reason to believe, as well from my own observations for many years, as of others who have alike observed it, that we should rather assign the beginnings of February and November, than the two equinoxes.

In his explanation, as already intimated, he attributes the tides to the rotation of the earth upon its axis, the revolution of moon and earth about their common center of gravity, and the revolution of this common center about the sun,—gravity serving merely as a tie to connect the bodies, their motions causing the tides.

From this quotation it is seen that the semi-annual variation in the phase inequality (in height) was known to Wallis although he was for some years mistaken in thinking the times of its maxima could fall far from the equinoxes.† He attempts to explain this non-coincidence with the equinoxes by the inequality in the length of the solar day, or rather by the cause of the inequality In further discussion of the subject, he distinctly suggests an annual variation in the phase inequality (in height) due to the sun being in apogee or perigee.‡ To clear up these questions he very properly suggests the observing of low waters as well as of high, and the selecting of stations near the open sea.§

To return to the question of gravitation. In reply to an objection to this portion of his theory, he says:||

To the first objection of those you mention, That it appears not how two bodies that have no tie can have one common centre of gravity; that is, for so I understand the intendment of the objection, can act or be acted in the same manner as if they were connected: I shall only answer, that it is harder to show how they have than that they have it. That the loadstone and iron have somewhat equivalent to a tie, though we see it not, yet by the effects we know. And it would be easy to show that two loadstones at once applied in different positions to the same needle, at some convenient distance, will draw it, not to point directly to either of them, but to some point between both; which point is, as to those two, the common centre of attraction; and it is the same as if some one loadstone were in that point. Yet have these two loadstones no connection or tie, though a common centre of virtue, according to which they jointly act. And as to the present case, how the earth and moon are connected, I will not now undertake to show, nor is it necessary to my purpose; but that there is somewhat that does connect them, as much as what connects the loadstone and the iron which it draws, is past doubt to those who allow them to be carried about by the sun, as one aggregate or body, whose parts keep a respective position to one another: Like as Jupiter with his four satellites, and Saturn with his one. Some tie there is that makes those satellites attend their lorde, and move in a body; though we do not see that tie, nor hear the words of command. And so here.

This is a close approach to Newton's discovery, but the *law* of the supposed attraction is not stated. Halley, however, in 1684 concluded that the centripetal force in planetary orbits must be inversely as the square of the distance.¶ Kepler had asserted, in 1609, that the force is proportional to the masses. Newton discovered and established the law of universal gravitation in 1682, but the details of its extension to physical and astronomical questions extended over a few succeeding years,—until the publication of the Principia in 1687.

---

* See under Strabo, Pliny, Seneca, and Bacon.

† See Table 31, observing the variation in S_2 due to T_2 and solar K_2; or the table given in § 58, factor for (Sg—Np). Captain Sturmy speaks of these as the "annual spring tides" and states that he has observed that they happen in March and September. Phil. Trans. (1668) [Vol. III, p. 813]; Abr. Vol. I, p. 290. See note under § 70.

‡ Phil. Trans. (1666) [Vol. I, p. 283]; Abr., Vol. I, p. 102; see also Phil. Trans. (1670); Abr., Vol. I, pp. 521, 522.

§ Phil. Trans. (1666) [Vol. I, pp. 283–285]; Abr., Vol. I, pp. 112, 113.

|| "An Appendix, written by Way of Letter to the Publisher, being an Answer to some Objections made by several Persons to the preceding Discourse." Phil. Trans. (1666) [Vol. I, pp. 281–286]; Abr., Vol. I, pp. 101–107.

¶ Phil. Trans. (1676); Abr., Vol. II, p. 327, note; Grant's History of Physical Astronomy, pp. 27–30.

See Newton's Principia, Bk. I, Prop. IV, Cor. 6, Scholium.

Also Humboldt's Kosmos, Vol. III, pp. 18–21 (Cosmos, Vol. III, pp. 18–22, Otté's translation).

Ibid., Vol. II, pp. 348, 349 (Vol. II, pp. 689–691, Otté's trans.).

Also Lalande's Astronomie, §§ 3379–3382, and Brewster's Life of Sir Isaac Newton, Ch. XI.

Later replies by Wallis are found in the Transactions for 1668 and 1670.

83. *Sir Robert Moray** (–1673). Moray doubts the supposition that the increase of tides from neaps to spring exactly follows the law of sines. He points out that the irregularities in the time required by the moon in going from new to full, or *vice versa*, will preclude any exact law in the matter.

He proposes what was probably the first box gauge; i. e., a gauge with a float.†

Among other things, he recommends observing the height of the tide and the velocity of the current every 15 minutes; the exact heights of high and low water; the direction and velocity of the wind; the state of the weather, including barometric readings. He would have the series continued for some months, or rather, years.

Samuel Colepresse. From observations made at and near Plymouth in 1667, Colepresse arrives at the following conclusions:‡

The diurnal tides, from about the latter end of March till the latter end of September, are about a foot higher in the evening than in the morning, that is, in every tide that happens after noon and before midnight. On the contrary, the morning tides, from Michaelmas till Lady-day in March again, are constantly higher by about a foot than those that happen in the evening. And this proportion holds in both, in the intermediate times of increase and decrease. The highest monthly spring tide is always the third tide after the new or full moon, if a cross wind do not oppose the water, as the north-east or north-west usually does. The highest springs make the lowest ebbs. The water neither flows nor ebbs alike in respect of equal degrees; but its velocity increases with the tide, till just at mid-water or half flood, at which time the velocity is strongest, and so decreases proportionably till high water or full sea. As appears by the following scheme, collected from observations made at several times and places; which, though taken at Plymouth Haven, where even the water usually rises about sixteen feet, yet it may indifferently serve for other places, where it may rise as many fathoms, or not so high, by a proportional addition or subtraction.

		Height.				Height.	
Time of flowing.	1 hr.	1 feet	6 inch.	Time of ebbing.	1 hr.	1 feet	6 inch.
	2	2	6		2	2	6
	3	4	0		3	4	0
	4	4	0		4	4	0
	5	2	6		5	2	6
	6	1	6		6	1	6

It will be seen that Colepresse was familiar with the diurnal inequality in the height of high water and knew that if for one half of the year higher high water fall in the evening, say, for the other half it would fall in the morning.

[If we use the letters $\frac{a}{b}$ to mark such lunitidal intervals as give a higher high or lower low water when applied to $\begin{matrix}\text{an upper north or lower south}\\ \text{a lower north or an upper south}\end{matrix}$ transit of the moon, then the truth or falsity of statements like the above may be ascertained by the following rule which is an obvious consequence of the equilibrium theory of tides:

When the interval is marked a the great tide is nearer to $\begin{matrix}\text{noon} + \text{interval}\\ \text{midnight} + \text{interval}\end{matrix}$ o'clock in the $\begin{matrix}\text{summer}\\ \text{winter}\end{matrix}$ half-year than to $\begin{matrix}\text{midnight} + \text{interval}\\ \text{noon} + \text{interval}\end{matrix}$ o'clock.

When the interval is marked b the reverse is true.

By great tide is here meant the higher high or the lower low water.]

His table shows the rise and fall for each lunar hour reckoned from the times of low water and of high water, the tide having a range of 16 feet.

Henry Philips§ draws a circle whose circumference is divided into 12 equal parts representing the 12 hours of transit of the moon. The diameter upon which these points are projected has written upon it the values of the interval which correspond to the several hours of transit. These are got, not from observations alone, but by assuming that the variation in the interval follows the law of sines so that, the diameter's length representing the extreme observed variation, equal

* "Considerations and Inquiries concerning Tides," Phil. Trans. (1666) [Vol. I, pp. 298–301]; Abr., Vol. I, pp. 113–115.

† Lalande, Astronomie, Vol. IV, p. 36, describes a somewhat elaborate box gauge proposed in about 1675.

‡ "Tides observed at Plymouth," Phil. Trans. (1668) [Vol. II, pp. 632–634]; Abr., Vol. I, p. 227.

§ "Time of the Tides observed at London," Phil. Trans. (1668) [Vol. III, pp. 656–659]; Abr. Vol. I, pp. 239, 240.

divisions upon it represent equal times.* He is wrong in assuming that the longest interval corresponds to the zero or twelfth hour of transit and the shortest to the sixth. This was pointed out by Flamsteed, the first astronomer royal, about 15 years later. He says that Philips "was certainly the first that brought the inequality to a rule."† Philips says that by changing the values written upon the diameter, the same construction is applicable elsewhere.

Capt. Samuel Sturmy. From the extract given below describing the tides in Hong-Road, near Bristol, it will be seen that Sturmy as well as Colepresse was familiar with the phenomenon of high-water diurnal inequality, and knew that if for one half of the year the higher high water fall in the evening, say, for the other half, it would fall in the morning; he knew, moreover, that this inequality is independent of the springs and neaps.‡

Concerning our diurnal tides, we observe, that from about the latter end of March till the latter end of September, they are about 1 foot 3 inches higher in the evening than in the morning; that is, when high water happens after the sun is past the meridian, or in the tides between noon and midnight: But from Michaelmas till Lady-day we find the contrary, the day tides being in that season higher by 15 inches than the night tides, or the tides between midnight and noon. And this proportion holds in both, after the gradual increase of the tides from neap to the highest spring, and the like decrease of their height till neap again. As for the highest menstrual spring-tide, it is always the third after the full moon or change-day, if it be not kept back by north-east winds.

Sturmy gives a table, similar to that given by Colepresse, showing the rise and fall in Hong-Road, near Bristol. He then describes the "boar" in the Severn.

In the Severn, 20 miles above Bristol, near Newnham, 160 miles from the river's mouth (Lundy,) the head of the flood, in spring-tides rises in height like a wall near nine feet high, and so runs for many miles together, covering at once all the shoals which were dry before; at which time all vessels that lie in the way of these head tides, or boars, as they are popularly called, are commonly overset, or carried upon the banks; and the head of the tide being past, such vessels are left dry again. It flows there but 2 hours and 18 feet in height, and it ebbs ten hours. The reason of the said boar is doubtless the straightening and shoaling of the river in that place, it being there but half a mile broad; as it is but 20 perches over three miles higher, running tapering to Gloucester.

84. *Joseph Childrey.* Childrey does not agree with Wallis in the mistaken notion that the highest tides happen about Allhallowtide and Candlemas. He says that English seamen, as a rule, believe them to occur near the equinoxes. He thinks that Wallis's November high tides must have been due to freshets. After giving numerous instances of remarkably high tides when the moon was in perigee, from the year 1250 to 1669, he says:§

Further, what inclines me to believe that the perigæosis of the moon is of some concernment in this matter, is, because it is a maxim among our Kentish seamen, that they never have two running spring tides (as they call them) together, but that the next spring tide, after a high running spring, is proportionably weak and slack; which, if true, is very correspondent to my opinion, because, if the moon be in perigæo at this spring tide, she will be in apogæo at the next.

But I conceive the best touchstone to prove the soundness of my opinion is, to have it observed, whether those neap-tides be not apparently higher that happen at the moon's being in perigæo either at the first or last quarter: because it is a received and demonstrable truth in astronomy, that the moon being in perigæo at either quarter, comes then nearer the earth than when it is in perigæo at the change or full.

Assuming the tides to be due to the rotary motions of earth and moon (i. e., Wallis's theory), Childrey evidently believes that the parallax inequality occurs with the neap tides as well as with the spring; and, moreover, that such inequality is affected by the moon's phase. This virtually infers an inequality in the parallax effect, whose period is the synodic period of the anomalistic month and the half synodical month, or about one-half of 412 days; this is the inequality due to the evection. Its effect, however, is quite the reverse of what he surmises it to be,‖ owing to the wrong astronomical notion which he entertains, and which was generally received before it was supplanted by the hypothesis of Horrox.¶ That is, the parallax inequality in height is really increased in the syzygies and diminished in the quadratures.**

* See Part III, §§ 2, 47, and Table 24.

† Phil. Trans. (1682 or 1683) [Vol. XIII, pp. 10–13]; Abr. Vol. 2, pp. 555–557.

‡ "Tides observed in Hong-Road, four Miles from Bristol." Phil. Trans. (1668) [Vol. III, pp, 813–817]; Abr., Vol. I, pp. 290, 291.

§ "Animadversions on Dr. Wallis's Hypothesis about the Flux and Reflux of the Sea." Phil. Trans. (1670) [Vol. V, pp. 2061–2068]; Abr. Vol. I, pp. 516–520.

‖ See Tables 1 and 34.

¶ See Whewell's History of Inductive Sciences, Vol. I, p. 457. Also Flamsteed's letter, Phil. Trans. (1675) [Vol. X, pp. 368–370]; Abr., Vol. II, pp. 220, 221. Also Godfray, Lunar Theory (1885), p. 113.

** See Tables 2 and 34.

John Flamsteed (1646–1719). Flamsteed published (in the Philosophical Transactions) a tide table giving the times of high water at London Bridge for the year 1683, and continued its publication for several succeeding years.

He corrects Mr. Phillips's table from observations which he caused to be made.

In his description of the 1683 table he says: *

Hitherto our tide-tables have only showed the time of that one high-water which next follows the moon's southing; but in this new table I have given the times of both. . . .

This table may be reduced and made to serve for any other port of his majesty's dominions and neighbouring countries, by only subtracting or adding so much time to the high-waters noted in it, as the high-water observed in the said place shall be found to precede or follow the time of the high-water the same day. For by such accounts as I have met with and received of the tides in remote places, I find there is every where, about England, the same difference between the spring and neap-tides, that is here observed in the river Thames.

I could easily have made and given you a table for this reduction, if I dare have relied on the account our mariners give of the tides in other ports; but I find their opinions different, except where they have copied from each other in their calendars; by reason of the afore-mentioned difference between the times of the moon's southings and the true high-waters; for which reason I forbear it, till further experience shall have informed us better.

About a year later Flamsteed published a table of tidal differences to be applied to the London tides. † This is probably the earliest known table of tidal differences.

85. *Dr. Edmund Halley* (1657–1742).

Halley notes the connection of the moon's declination and the tides at Tonquin, which Francis Davenport's observations had shown to be diurnal in their character. But the law which he proposes for ascertaining the height of high or low water for various longitudes of the moon, is obviously incorrect. He says: ‡

The effect of the moon on the waters in the production of the tides in the port of Tonquin is the more surprising, as it seems different in all its circumstances from the general rule, whereby the motion of the sea is regulated in all other parts of the world that I have yet heard of. For first, each flux is of about 12 hours duration, and its correspondent reflux as long; so that there is but one high water in 24 hours. Then there are in each month two intermissions of the tides, about 14 days asunder, when there is no sensible flood or rising of the waters to be observed, but the sea is in a manner stagnant. Thirdly, that the increase of the water has its 14 days period between the aforesaid intermissions; and at 7 days end makes the highest tides; from which time the water again gradually abates, and the flood is weaker till it comes to a stagnation, both increase and decrease observing the same rule in being exceedingly slow in their beginning and end, and swift in the middle. Lastly, and which is most strange, the rising moon in the one half of each month makes high water, and the setting moon in the other half.

These particulars considered, together with the tables showing the days of the water's stagnation in each month, gave me a light into the secret of this strange appearance, so as to be able to bring the hitherto unaccountable irregularity of these tides to a certain rule. And first it appears that the intermissions of the tides happen nearly on those days that the moon enters the signs of Aries and Libra, or passes the equinoctial, which divides the moon's course nearly into two equal parts, as well as the sun's; and from hence it follows, that the tropical moons in ♋ and ♑, are those which occasion the greatest flux and reflux.§ It also appears that the moon in northern signs brings in the flood, whilst she is above the horizon, so as to make high water at her setting, and on the contrary, that whilst she is in southern signs, it flows all the time the moon is below the horizon, and so makes high water at her rising. But it is to be observed, that though the moon pass swiftly from south to north when she is in or near ♈, and from north to south when in or near ♎, yet the motion of the sea, which is the cause of this tide, is scarcely discernible for 3 or 4 days, when the moon passes the said equinoctial points; whence it appears, that though the declination of the moon be that whereby these tides are regulated, yet the increase and decrease of the water is by no means proportionate to that of her declination, that changing swiftly, where the increase of the water is observed to be most slow. It seems therefore, and I propose it as a probable conjecture, that the increase of the waters should be always proportionate to the versed sines of the doubled distances of the moon from the equinoctial points.

In the same discussion he suggests the existence of an inequality in the "spring range" (i. e., great tropic range) dependent upon the obliquity of the lunar orbit to the plane of the earth's equator.‖ He says:

There is yet another thing well worth inquiry, viz. seeing that this motion of the sea is more or less, as the moon is farther from or nearer to the equinoctial, it is not unlikely that some years may have much higher spring tides

* "A correct Tide Table, showing the true Times of the High-waters at London Bridge, to every Day in the Year 1683," Vol. XIII (1682–83) [pp. 10-15]; Abr., Vol. II, pp. 555–557.

† Phil. Trans. (1683–84) [Vol. XIV, pp. 458–462]; Abr., Vol. III, p. 3.

‡ "A Theory of the Tides at the Bar of Tonquin." Phil. Trans. (1684) [Vol. XIV, pp. 685–688]; Abr., Vol. III, pp. 67–69.

§ See under Strabo; also § 7.

‖ See Part III, §§ 48, 49; also Tables 10, 13, 14, and 32.

than others, according to the various obliquity of the moon's orbit to the equinoctial; for when the ascending node is in ♈, as it was anno 1671, and will be anno 1690, the moon in ♋ and ♑ deviates from the equator full 28½°, and but 18½° when the same node is in ♎, as it was anno 1680.

In the year 1697 Halley calls attention to the excellence of Newton's Principia in explaining the cause and phenomena of the tides.* He notes that the moon's disturbing force would cause the spherical surface of the ocean to become spheroidal; that sun and moon have similar effects; that spring tides correspond to new and full moon, and neap tides to the quarters; that equinoctial spring tides are, *cæteris paribus*, the highest, but that the nearness of the sun in the winter displaces them somewhat, making them in February and October; that there should generally be a diurnal inequality; that tidal inequalities should have an age; and that even diurnal tides, like those at Tonquin, may be accounted for.

86. The preceding pages show the diversity of views concerning the cause of tides and tidal currents entertained before the law of gravitation was established. Among those described or alluded to are: The discharging of rivers into the sea (Timæus); winds, set up by the sun or moon, striking the water (Aristotle, Heraclides, Seleucus); bodily oscillations of large bodies of water within the earth (Plato); the surface of the sea being on a slope (Eratosthenes); vapors surrounding the moon (Strabo); submarine caverns; the breathing of an earth animal (Apollonius); water increasing with the waxing moon (Pliny); sympathy whereby the moon attracts moist bodies; rarefaction of the water caused by the moon or sun; occult qualities of the moon; westward diurnal motion of the *primum mobile;* the vortex theory (Descartes); a whirlpool off the coast of Norway; the absolute motion of a fixed point on the earth's surface not being uniform at all times, thereby setting up a variable centrifugal force (Galileo, Wallis); and the heat of the sun.

Some of the other notions which have been advanced to explain the phenomenon of the tide are: Unequal depths causing diverse densities in the water; submarine heat, fermentations, and vapors; a libration of the earth; and the force of rays of light from the sun and moon.

More particulars along this line are given by Riccioli, Lalande, Peschel, Ruge, and Günther.

---

* "The true Theory of the Tides, extracted from Mr. Isaac Newton's treatise, entitled Philosophiæ Naturalis Principia Mathematica; being a Discourse presented with that Book to the late King James." Phil. Trans. (1697) [Vol. XIX pp. 445 et seq.]; Abr., Vol. IV, pp. 142–149.

CHAPTER VI.

NEWTON TO LAPLACE.

Sir Isaac Newton (1642–1727).

87. Before referring to Newton's work upon tides it may be well to state some of the consequences of the law of gravitation when applied to certain astronomical questions which have a direct bearing upon the subject. In Proposition LXVI, Book I, of the *Principia*, the disturbing force of a third body is considered. Applying his results to the case of the moon as disturbed by the sun we obtain the following:

The moon's motion is by the action of the sun retarded while in the first and third quadrants, but accelerated while in the second and fourth (Cor. 2).

Cœteris paribus, the moon moves more swiftly in the syzygies than in the quadratures (Cor. 3).

Cœteris paribus, the moon is nearer to the earth in the syzygies than in the quadratures (Cor. 5).

The line of apsides advances in the long run although its motion is at times retrograde. It advances most rapidly when the line of apsides is in syzygy and most slowly when in quadrature (Cor. 5, 7).

The eccentricity of the moon's orbit will be the greatest when the apsides are in the syzygies, and least when in the quadratures (Cor. 9).

The nodes being either stationary or having a retrograde motion, will for any revolution of the moon be carried backwards (Cor. 11).

The disturbing force of the sun is a little greater at conjunction than at opposition (Cor. 12).

The disturbing force of the sun is, very nearly, inversely as the cube of its distance from the earth's center (Cor. 14).

The action of the sun upon the redundant matter in the equatorial regions of the earth will cause the equinoxes to be carried backwards (Cor. 20).

In Proposition LXXI, Book I, it is shown that the attraction of a spherical shell upon an external point is the same as if all matter of the shell were collected at its center.

From the construction given in Proposition XXV, Book III, it follows (considering the great distance of the sun) that at the quadratures the disturbing force of the sun upon the moon is directed towards the earth, but at the syzygies the disturbing force is twice as great and is directed from the earth.

In Proposition XXVIII, Book III, it is found that the moon's distance from the earth in the syzygies is to its distance in the quadratures (setting aside the consideration of the eccentricity) as 69 to 70, very nearly.

In Proposition XXXII, Book III, the mean motion of the lunar nodes is $19^\circ\ 18'\ 1''\ 23'''$ per sidereal year; and in Proposition XXXIX the precession of the equinoxes due to both moon and sun is very nearly $50''$.

Newton regards the tide in three different ways:

First. Book I, Prop. LXVI, Cors. 18, 19, a kinetic theory or hypothesis. The motion of a particle of water revolving with the earth about a fixed axis is disturbed by an extraneous body, just as the moon revolving about the earth is disturbed by the sun causing the inequality of variation. This necessitates low water and maximum eastward velocity (with respect to the fixed surface of the earth) at the time of upper or lower transit; also high water and maximum westward velocity at moonrise or moonset. For, supposing an analogy to the moon's variation, the particle becomes a little nearer to the earth's center, and moves a little more rapidly (in space) as the body nears meridian. He makes no attempt to apply this theory to observed tides.*

* Prof. J. Challis, in "A mathematical theory of tides," Phil. Mag., Vol. 39 (1870), pp. 31, 32, does not believe that this hypothesis of Newton's necessitates low water at the times of transits, but rather high water. Challis erroneously assumes that the time of the tide depends upon the vertical forces instead of the horizontal. Cf. § 41.

Second. Cor. 20. His next hypothesis is that (disregarding friction, etc.) high water occurs when the disturbing *vertical* force is zero, instead of when its value becomes a maximum, as in the uncorrected equilibrium theory. This occurs about three hours before or after the transit of the tidal body. He assumes that friction may retard the times of the tides somewhat and that "the motion of ascent or descent impressed by these (astronomical) forces may by the *vis insita* of the water continue a little longer or be stopped a little sooner by impediments in its channel."

Third. Book III, Prop. XXXVI, an equilibrium theory or hypothesis, in which the density of the earth is that of water.

88. In Proposition XXIV, Book III, Newton treats of the principal phenomena pertaining to tides. He says:

"By Cors. 19 and 20, Prop. LXVI, Book I, it appears that the waters of the sea ought twice to rise and twice to fall every day, as well lunar as solar; and that the greatest height of the waters in the open and deep seas ought to follow the appulse of the luminaries to the meridian of the place by a less interval than 6 hours; as happens in all that eastern tract of the *Atlantic* and *Æthiopic* seas between *France* and the *Cape of Good Hope;* and on the coasts of *Chili* and *Peru* in the *South Sea;* * in all which shores the flood falls out about the second, third, or fourth hour, unless where the motion propagated from the deep ocean is by the shallowness of the channels, through which it passes to some particular places, retarded to the fifth, sixth, or seventh hour, and even later. The hours I reckon from the appulse of each luminary to the meridian of the place, as well under as above the horizon; and by the hours of the lunar day I understand the 24th parts of that time which the moon, by its apparent diurnal motion, employs to come about again to the meridian of the place which it left the day before. The force of the sun or moon in raising the sea is greatest in the appulse of the luminary to the meridian of the place; but the force impressed upon the sea at that time continues a little while after the impression, and is afterwards increased by a new though less force still acting upon it. This makes the sea rise higher and higher, till this new force becoming too weak to raise it any more, the sea rises to its greatest height. And this will come to pass, perhaps, in one or two hours, but more frequently near the shores in about three hours, or even more, where the sea is shallow.

The two luminaries excite two motions, which will not appear distinctly, but between them will arise one mixed motion compounded out of both. In the conjunction or opposition of the luminaries their forces will be conjoined, and bring on the greatest flood and ebb. In the quadratures the sun will raise the waters which the moon depresses, and depress the waters which the moon raises, and from the difference of their forces the smallest of all tides will follow. And because (as experience tells us) the force of the moon is greater than that of the sun, the greatest height of the waters will happen about the third lunar hour. Out of the syzygies and quadratures, the greatest tide, which by the single force of the moon ought to fall out at the third lunar hour, and by the single force of the sun at the third solar hour, by the compounded forces of both must fall out in an intermediate time that approaches nearer to the third hour of the moon than to that of the sun. And, therefore, while the moon is passing from the syzygies to the quadratures, during which time the 3d hour of the sun precedes the 3d hour of the moon, the greatest height of the waters will also precede the 3d hour of the moon, and that, by the greatest interval, a little after the octants of the moon; and, by like intervals, the greatest tide will follow the 3d lunar hour, while the moon is passing from the quadratures to the syzygies. Thus it happens in the open sea; for in the mouths of rivers the greater tides come later to their height.

But the effects of the luminaries depend upon their distances from the earth; for when they are less distant, their effects are greater, and when more distant, their effects are less, and that in the triplicate proportion of their apparent diameter. Therefore it is that the sun, in the winter time, being then in its perigee, has a greater effect, and makes the tides in the syzygies something greater, and those in the quadratures something less than in the summer season; and every month the moon, while in the perigee, raises greater tides than at the distance of 15 days before or after, when it is in its apogee. Whence it comes to pass that two highest tides do not follow one the other in two immediately succeeding syzygies.

The effect of either luminary doth likewise depend upon its declination or distance from the equator; for if the luminary was placed at the pole, it would constantly attract all the parts of the waters without any intension or remission of its action, and could cause no reciprocation of motion. And, therefore, as the luminaries decline from the equator toward either pole, they will, by degrees, lose their force, and on this account will excite lesser tides in the solstitial than in the equinoctial syzygies. But in the solstitial quadratures they will raise greater tides than in the quadratures about the equinoxes; because the force of the moon, then situated in the equator, most exceeds the force of the sun. Therefore the greatest tides fall out in those syzygies, and the least in those quadratures, which happen about the times of both equinoxes: and the greatest tide in the syzygies is always succeeded by the least tide in the quadratures, as we find by experience. But, because the sun is less distant from the earth in winter than in summer, it comes to pass that the greatest and least tides more frequently appear before than after the vernal equinox, and more frequently after than before the autumnal.

Moreover, the effects of the luminaries depend upon the latitudes of places.†

* To ascertain the amount of truth in these statements, consult a cotidal chart. The cotidal hour diminished by the west longitude of the place (in hours) will give its lunitidal interval (in lunar hours).

† Motte's translation, 1st Am. ed.

He then assumes, without proof, that the surface of the sea takes the form of a spheroid whose axis points at the position of the moon three hours before the given time.' He shows that the greater high water should follow an upper north / a lower south transit for places in north latitude, and *vice versa* for places in south latitude. This, in a general way, explains the diurnal inequality. He adds:

And the greatest difference of the floods will fall out about the times of the solstices; especially if the ascending node of the moon is about the first of Aries. So it is found by experience that the morning tides in winter exceed those of the evening, and the evening tides in summer exceed those of the morning; at *Plymouth* by the height of one foot, but at *Bristol* by the height of 15 inches, according to the observations of *Colepress* and *Sturmy*.

This explains in a satisfactory manner the annual and nodal variation in the diurnal inequality.*

Continuing, Newton gives an explanation of the smallness of the diurnal inequality which seems to have passed unquestioned until the subject was investigated by Laplace: †

But the motions which we have been describing suffer some alteration from that force of reciprocation, which the waters, being once moved, retain a little while *by their vis insita*. Whence it comes to pass that the tides may continue for some time, though the actions of the luminaries should cease. This power of retaining the impressed motion lessens the difference of the alternate tides, and makes those tides which immediately succeed after the syzygies greater, and those which follow next after the quadratures less. And hence it is that the alternate tides at *Plymouth* and *Bristol* do not differ much more one from the other than by the height of a foot or 15 inches, and that the greatest tides of all at those ports are not the first but the third after the syzygies. And, besides, all the motions are retarded in their passage through shallow channels, so that the greatest tides of all, in some straits and mouths of rivers, are the fourth or even the fifth after the syzygies.

Farther, it may happen that the tide may be propagated from the ocean through different channels towards the same port, and may pass quicker through some channels than through others; in which case the same tide, divided into two or more succeeding one another, may compound new motions of different kinds. Let us suppose two equal tides flowing towards the same port from different places, the one preceding the other by six hours; and suppose the first tide to happen at the third hour of the appulse of the moon to the meridian of the port. If the moon at the time of the appulse to the meridian was in the equator, every six hours alternately there would arise equal floods, which, meeting with as many equal ebbs, would so balance one the other, that for that day, the water would stagnate and remain quiet. If the moon then declined from the equator, the tides in the ocean would be alternately greater and less, as was said; and from thence two greater and two lesser tides would be alternately propagated towards that port. But the two greater floods would make the greatest height of the waters to fall out in the middle time betwixt both; and the greater and lesser floods would make the waters to rise to a mean height in the middle time between them, and in the middle time between the two lesser floods the waters would rise to their least height. Thus in the space of 24 hours the waters would come, not twice, as commonly, but once only to their greatest, and once only to their least height; and their greatest height, if the moon declined toward the elevated pole, would happen at the 6th or 30th hour after the appulse of the moon to the meridian; and when the moon changed its declination, this flood would be changed into an ebb. An example of all which Dr. *Halley* has given us, from the observations of seamen in the port of *Batsham*, in the kingdom of *Tunquin*, in the latitude of 20° 50′ north. In that port, on the day which follows after the passage of the moon over the equator, the waters stagnate: when the moon declines to the north, they begin to flow and ebb, not twice, as in other ports, but once only every day: and the flood happens at the setting, and the greatest ebb at the rising of the moon. This tide increases with the declination of the moon till the 7th or 8th day; then for the 7 or 8 days following it decreases at the same rate as it had increased before, and ceases when the moon changes its declination, crossing over the equator to the south. After which the flood is immediately changed into an ebb; and thenceforth the ebb happens at the setting and the flood at the rising of the moon; till the moon, again passing the equator, changes its declination. There are two inlets to this port and the neighboring channels, one from the seas of *China*, between the continent and the island of *Leuconia;* the other from the *Indian* sea, between the continent and the island of *Borneo*. But whether there be really two tides propagated through the said channels, one from the *Indian* sea in the space of 12 hours, and one from the sea of *China* in the space of 6 hours, which therefore happening at the 3d and 9th lunar hours, by being compounded together, produce those motions; or whether there be any other circumstances in the state of those seas, I leave to be determined by observations on the neighbouring shores.

Thus I have explained the causes of the motions of the moon and of the sea. Now it is fit to subjoin something concerning the quantity of those motions.

In the next proposition,‡ it is shown that the disturbing force of the sun upon the moon at quadrature is the 1/638092·6 part of the force of gravity at the earth's surface, this being one half its value at syzygy.

---

* See Table 32.

† Cf. Wallis, Phil. Trans. (1666) [p. 275]; Abr. Vol. I, p. 96: "Though the next tide have not the same cause also, the impetus contracted will have influence upon the next tide."

‡ Bk. III, Prop. XXV.

89. Proposition XXXVI, Book III, resumes the question of the tides, and begins by passing from the sun's disturbing force upon the moon to the force tending to move the sea. This is the first attempt at a quantitive determination of the tidal forces and constitutes an important landmark in the development of the subject. He uses the sun rather than the moon because the ratio of the mass of the moon to that of the earth was then an unknown quantity.

But, descending to the surface of the earth, these forces are diminished in proportion of the distances from the centre of the earth, that is, in the proportion of $60\frac{1}{2}$ to 1; and therefore the former force on the earth's surface is to the force of gravity as 1 to 38 604 600;* and by this force the sea is depressed in such places as are 90 degrees distant from the sun. But by the other force, which is twice as great, the sea is raised not only in the places directly under the sun, but in those also which are directly opposed to it;† and the sum of these forces is to the force of gravity as 1 to 12 868 200.‡ And because the same force excites the same motion, whether it depresses the waters in those places which are 90 degrees distant from the sun, or raises them in the places which are directly under and directly opposed to the sun, the aforesaid sum will be the total force of the sun to disturb the sea, and will have the same effect as if the whole was employed in raising the sea in the places directly under and directly opposed to the sun, and did not act at all in the places which are 90 degrees removed from the sun.§

And this is the force of the sun to disturb the sea in any given place, where the sun is at the same time both vertical, and in its mean distance from the earth. In other positions of the sun, its force to raise the sea is as the versed sine of double its altitude above the horizon of the place directly,‖ and the cube of the distance from the earth reciprocally.

Cor. Since the centrifugal force of the parts of the earth, arising from the earth's diurnal motion, which is to the force of gravity as 1 to 289, raises the waters under the equator to a height exceeding that under the poles by 85 472 *Paris* feet,¶ as above, in Prop. XIX, the force of the sun, which we have now shewed to be to the force of gravity as 1 to 12 868 200, and therefore is to that centrifugal force as 289 to 12 868 200, or as 1 to 44 527, will be able to raise the waters in the places directly under and directly opposed to the sun to a height exceeding that in the places which are 90 degrees removed from the sun only by one *Paris* foot and $11\frac{1}{30}$ inches; for this measure is to the measure of 85 472 feet as 1 to 44 527.

The next proposition, entitled "To find the force of the moon to move the sea," and in which further quantitive results are obtained, is as follows:

The force of the moon to move the sea is to be deduced from its proportion to the force of the sun, and this proportion is to be collected from the proportion of the motions of the sea, which are the effects of those forces. Before the mouth of the river *Avon*, three miles below *Bristol*, the height of the ascent of the water in the vernal and autumnal syzygies of the luminaries (by the observations of *Samuel Sturmy*) amounts to about 45 feet, but in the quadratures to 25 only. The former of those heights arises from the sum of the aforesaid forces, the latter from their difference. If, therefore, S and L are supposed to represent respectively the forces of the sun and moon while they are in the equator, as well as in their mean distances from the earth, we shall have $L + S$ to $L - S$ as 45 to 25, or as 9 to 5.

At *Plymouth* (by the observations of *Samuel Colepress*) the tide in its mean height rises to about 16 feet, and in the spring and autumn the height thereof in the syzygies may exceed that in the quadratures by more than 7 or 8 feet. Suppose the greatest difference of those heights to be 9 feet, and $L + S$ will be to $L - S$ as $20\frac{1}{2}$ to $11\frac{1}{2}$, or as 41 to 23; a proportion that agrees well enough with the former. But because of the great tide at *Bristol*, we are rather to depend upon the observations of *Sturmy;* and, therefore, till we procure something that is more certain, we shall use the proportion of 9 to 5.

But because of the reciprocal motions of the waters, the greatest tides do not happen at the times of the syzygies of the luminaries, but, as we have said before, are the third in order after the syzygies; or (reckoning from the syzygies) follow next after the third appulse of the moon to the meridian of the place after the syzygies; or, rather (as *Sturmy* observes) are the third after the day of the new or full moon, or rather nearly after the twelfth hour from the new or full moon, and therefore fall nearly upon the forty-third hour after the new or full of the moon. But in this port they fall out about the seventh hour after the appulse of the moon to the meridian of the place; and therefore follow next after the appulse of the moon to the meridian, when the moon is distant from the sun, or from opposition with the sun by about 18 or 19 degrees *in consequentia.* So the summer and winter seasons come not to their height in the solstices themselves, but when the sun is advanced beyond the solstices by about a tenth part of its whole course, that is, by about 36 or 37 degrees. In like manner, the greatest tide is raised after the appulse of the moon to the meridian of the place, when the moon has passed by the sun, *or the opposition thereof,* by about the tenth part of the whole motion from *one greatest* tide to *the next following greatest* tide. Suppose that distance about $18\frac{1}{2}$ degrees; and the sun's force in this distance of the moon from the syzygies and quadratures will be of less moment to augment and diminish that part of the motion of the sea which proceeds from the motion of the moon than in the

*$38\ 604\ 600 = 638\ 092{\cdot}6 \times 60\frac{1}{2}$.

† Newton here substitutes the simple equilibrium hypothesis for the one previously entertained.

‡$12\ 868\ 200 = 38\ 604\ 600 \div 3$.

§ I. e., this force corresponds to the range of solar tide.

‖ The sun is supposed to move in the plane of the equator and the observer to be located upon the equator. See note to § 93.

¶ 1 Paris foot $= \frac{1}{6}$ toise $= 1{\cdot}06575$ feet $= 0{\cdot}325$ metre.

syzygies and quadratures themselves in the proportion of the radius to the co-sine of double this distance, or of an angle of 37 degrees; that is, in proportion of 10 000 000 to 7 986 355; and, therefore, in the preceding analogy, in place of S we must put 0·7986355S.

But farther; the force of the moon in the quadratures must be diminished, on account of its declination from the equator; for the moon in those quadratures, or rather in $18\frac{1}{2}$ degrees past the quadratures, declines from the equator by about 23° 13′; and the force of either luminary to move the sea is diminished as it declines from the equator nearly in the duplicate proportion of the co-sine of the declination; and therefore the force of the moon in those quadratures is only 0·8570327L; whence we have L + 0·7986355S to 0·8570327L — 0·7986355S as 9 to 5.*

Farther yet; the diameters of the orbit in which the moon should move, setting aside the consideration of eccentricity, are one to the other as 69 to 70; and therefore the moon's distance from the earth in the syzygies is to its distance in the quadratures, *cæteris paribus*, as 69 to 70; and its distances, when $18\frac{1}{2}$ degrees advanced beyond the syzygies, where the greatest tide was excited, and when $18\frac{1}{2}$ degrees passed by the quadratures, where the least tide was produced, are to its mean distance as 69·098747 and 69.897345 to $69\frac{1}{2}$. But the force of the moon to move the sea is in the reciprocal triplicate proportion of its distance; and therefore its forces, in the greatest and least of those distances, are to its force in its mean distance as 0·9830427 and 1·017522 to 1. From whence we have 1·017522L × 0·7986355S to 0 9830427 × 0·8570327L — 0·7986355S as 9 to 5; and S to L as 1 to 4·4815. Wherefore since the force of the sun is to the force of gravity as 1 to 12 868 200, the moon's force will be to the force of gravity as 1 to 2 871 400.

Cor. 1. Since the waters excited by the sun's force rise to the height of a foot and $11\frac{1}{30}$ inches, the moon's force will raise the same to the height of 8 feet and $7\frac{5}{22}$ inches; and the joint forces of both will raise the same to the height of $10\frac{1}{2}$ feet; and when the moon is in its perigee to the height of $12\frac{1}{2}$ feet, and more, especially when the wind sets the same way as the tide. And a force of that quantity is abundantly sufficient to excite all the motions of the sea, and agrees well with the proportion of those motions; for in such seas as lie free and open from east to west, as in the *Pacific* sea, and in those tracts of the *Atlantic* and *Ethiopic* seas which lie without the tropics, the waters commonly rise to 6, 9, 12, or 15 feet; but in the *Pacific* sea, which is of a greater depth, as well as of a larger extent, the tides are said to be greater than in the *Atlantic* and *Ethiopic* seas; for to have a full tide raised, an extent of sea from east to west is required of no less than 90 degrees. In the *Ethiopic* sea, the waters rise to a less height within the tropics than in the temperate zones, because of the narrowness of the sea between *Africa* and the southern parts of *America*. In the middle of the open sea the waters cannot rise without falling together, and at the same time, upon both the eastern and western shores, when, notwithstanding, in our narrow seas, they ought to fall on those shores by alternate turns; upon which account there is commonly but a small flood and ebb in such islands as lie far distant from the continent. On the contrary, in some ports, where to fill and empty the bays alternately the waters are with great violence forced in and out through shallow channels, the flood and ebb must be greater than ordinary; as at *Plymouth* and *Chepstow Bridge* in *England*, at the mountains of *St. Michael*, and the town of *Auranches*, in *Normandy*, and at *Cambaia* and *Pegu* in the *East Indies*. In these places the sea is hurried in and out with such violence, as sometimes to lay the shores under water, sometimes to leave them dry for many miles. Nor is this force of the influx and efflux to be broke till it has raised and depressed the waters to 30, 40, or 50 feet and above. And a like account is to be given of long and shallow channels or straits, such as the *Magellanic* straits, and those channels which environ *England*. The tide in such ports and straits, by the violence of the influx and efflux, is augmented above measure. But on such shores as lie toward the deep and open sea with a steep descent, where the waters may freely rise and fall without that precipitation of influx and efflux, the proportion of the tides agrees with the forces of the sun and moon.

Cor. 2. Since the moon's force to move the sea is to the force of gravity as 1 to 2 871 400, it is evident that this force is far less than to appear sensibly in statical or hydrostatical experiments, or even in those of pendulums. It is in the tides only that this force shews itself by any sensible effect.

Cor. 3. Because the force of the moon to move the sea is to the like force of the sun as 4·4815 to 1, and those forces (by Cor. 14, Prop. LXVI, Book 1) are as the densities of the bodies of the sun and moon and the cubes of their apparent diameters conjunctly, the density of the moon will be to the density of the sun as 4·4815 to 1 directly, and the cube of the moon's diameter to the cube of the sun's diameter inversely; that is (seeing the mean apparent diameters of the moon and sun are 31′ $16\frac{1}{2}$″ and 32′ 12″), as 4 891 to 1 000. But the density of the sun was to the density of the earth as 1 000 to 4 000; and therefore the density of the moon is to the density of the earth as 4 891 to 4 000, or as 11 to 9. Therefore the body of the moon is more dense and more earthly than the earth itself.

Cor. 4. And since the true diameter of the moon (from the observations of astronomers) is to the true diameter of the earth as 100 to 365, the mass of matter in the moon will be to the mass of matter in the earth as 1 to 39·788.

90. In "The system of the world" the tides are discussed somewhat as in the third book. The matter is not arranged in formal propositions, a popular treatment of the subject having been the author's intention at one time.

He here calls attention to the impossibility of the lunitidal interval being of uniform length

---

*Assuming that Newton is, in all cases, dealing with equinoctial spring and neap tides (instead of equinoctial syzygial tides), then equinoctial spring range: equinoctial neap range $= L + S : L\cos^2\delta - S = 9:5$, δ being the moon's declination when at equinoctial quadrature (generally about $23\frac{1}{2}$°). The above assumption seems justified in light of the subsequent parallax corrections; and so Newton's proportion is erroneous in so far as the age of the phase inequality is involved. Cf. Airy, Tides and Waves, Art. 17; Ferrel, Tidal Researches, Introduction, § 5.

across the Atlantic Ocean, because rise in one place necessitates fall in another; also to the fact that "the greatness of the tides depends upon the greatness of the sea."

He imagines a pair of vertical canals, one in the axis of the tidal spheroid and the other perpendicular thereto, both passing through the earth's center. Then, knowing that the whole weight of the water in either canal or leg is proportional to the square of its length, and that the attraction of the earth upon a particle at its surface is 12 868 200 times that of the sun upon the same particle, the square roots of 12 868 200 and 12 868 200 + 1, or 12 868 201, are proportional to the two semiaxes of the tidal spheroid. Their difference thus found is 9, or, more accurately, $9\frac{1}{5}$ Paris inches. This is about two-fifths of the result given in Book III, because the mutual attraction of the disturbed fluid particles is ignored.

He estimates that the moon's tidal force is $5\frac{1}{3}$ that of the sun, instead of 4·4815, as in Cor. 3, Prop. XXXVII, and so the ranges of tides are 9 inches for the sun and 4 feet for the moon. He thinks that the theoretical range of tide may be doubled or trebled because of the reciprocation (oscillatory motion) in the motion of the waters. The mass of the moon here obtained is 1/29, instead of 1/39·788, the result given in the third book.

Daniel Bernoulli (1700–1782).

91. In the year 1738 the Académie des Sciences at Paris proposed the problem of the tides as the subject of a prize essay. The prize was divided, in 1740, among Daniel Bernoulli, professor of anatomy and botany at Basel; Maclaurin, professor of mathematics at Edinburgh; Euler, professor of mathematics at St. Petersburg, and the Jesuit Antoine Cavalleri. The three first mentioned founded their theories upon the principle of universal gravitation, while Cavalleri adopted the Cartesian system of vortices (tourbillons), a system which most philosophers had already abandoned because of Newton's more rational theory. The essays of Bernoulli, Maclaurin, and Euler are to be found in Le Seur and Jacquier's edition of the Principia. They bear the respective titles, "Traité sur le flux et reflux de la mer," "De causa physica fluxus et refluxus maris," and "Inquisitio physica in causam fluxus ac refluxus maris."

Laplace has given a review of the essays of Bernoulli and Euler in the thirteenth book of his Mécanique Céleste, and Ferrel gives a similar review in the introduction to his Tidal Researches.

Each of the three writers begins his essay with some historical remarks on the subject.

92. Bernoulli was the first to develop the equilibrium theory or hypothesis sufficiently far to give it a practical value in the prediction of tides; hence this theory is often associated with his name. He proceeds upon the following suppositions: That for tidal purposes we may assume the undisturbed surface of the earth to be spherical, i. e., we may disregard the ellipticity of the earth's meridian; that the earth is composed of concentric layers, any one of which has a uniform density throughout; that either luminary causes the earth to assume the form of an ellipsoid of revolution whose axis points toward the center of the luminary; that the nucleus of the earth is practically rigid, and is surrounded by a homogeneous sea, shallow in comparison with the radius of the earth.

He determines the ellipticity of the tidal spheroid, and so the range of the tide (due to the sun), by imagining a vertical canal or well, directly under the body, cut to the earth's center, there communicating with a similar opening 90° distant from the first—an artifice employed by Newton in finding the effect of the centrifugal force at the earth's equator. He denotes the range of the solar tide by σ and the lunar tide by δ. Various theoretical values of the ranges are obtained upon assuming different laws for the density of the earth. Upon the assumption that the earth's density is uniformly that of water, Bernoulli finds σ about 23 inches, or Newton's result. He regards this range too small to be consistent with observed tides. That most of his conclusions based on various assumptions as to density are erroneous may be seen upon comparing them with the following obvious considerations: If we give to the earth a certain mass arranged in concentric shells, the height of the tide is independent of the law of change of density along a radius. If, on the other hand, we take for our estimate of the earth's mass, or mean density, a portion of surface material, water for instance, then the mass will, of course, be much affected by the assumed law of increase or decrease of density from the surface toward the center. Consequently any law which assumes the density to decrease from the surface toward the center will give larger tides than upon the assumption of uniform density, and *vice versa* for a law which supposes an increase in density below the surface. Bernoulli reaches conclusions quite opposite to these.

93. In the fifth chapter of his essay Bernoulli finds that the fall of tide from high water is proportional to the square of the sine of the moon's hour angle—the transits of the moon are assumed to occur at the times of high water.* [Of course a similar rule obtains for the height of the tide (surface of the sea) reckoned from low water. To establish these rules, assume an ellipse of small eccentricity, and upon its major and minor axes as diameters describe circles; the departure of the ellipse (along a radius) from either circle which it touches is proportional to the sine of the angle which the radius makes with the one drawn through the point of contact. This may be readily seen upon writing the polar equation of the ellipse referred to its center.]

After observing that the solar and lunar tides may be treated separately, because either produces but a small deformation in the figure of the earth, he gives an expression for the height of the combined tide. This height, when referred to mean sea level, may be written

$$\text{Height of tide} = \tfrac{1}{2}\,\varsigma - \sigma^2\varsigma + \tfrac{1}{2}\,\delta - \rho^2\delta \tag{169}$$

where σ is the sine of the hour angle of the sun and ρ the same for the moon. If m denote the sine of the angle between the sun and the moon, and n the cosine, then

$$\rho = \mathrm{m}\sqrt{1-\sigma^2} - \mathrm{n}\sigma. \tag{170}$$

From the square of this equation $\rho d\rho$ is readily obtained, σ being the variable. From the differential of the expression for the height, which must be zero at the times of high or low waters,

$$\rho d\rho = -\frac{\varsigma}{\delta}\sigma d\sigma. \tag{171}$$

Equating the two expressions for $\rho d\rho$, we obtain as the value of σ at a high or low water

$$\sigma = \pm\left(\tfrac{1}{2} \pm \frac{\mathrm{A}}{2\sqrt{4+\mathrm{A}^2}}\right)^{\frac{1}{2}}, \tag{172}$$

where

$$\mathrm{A} = \frac{1}{\mathrm{mn}}\left[\mathrm{m}^2 - \mathrm{n}^2 - \frac{\varsigma}{\delta}\right]. \tag{173}$$

The value of ρ is the same as that of σ when A is replaced by B, which quantity is obtained from the expression for A by interchanging ς and δ.

Regarding the angle between the sun and moon as variable,

$$\rho = \pm\left(\tfrac{1}{2} \pm \frac{\mathrm{B}}{2\sqrt{4+\mathrm{B}^2}}\right)^{\frac{1}{2}} \tag{174}$$

becomes a maximum when

$$\frac{d\mathrm{B}}{d\mathrm{m}} = 0. \tag{175}$$

This gives

$$\mathrm{m} = \sqrt{\frac{\varsigma+\delta}{2\delta}} \tag{176}$$

for the sine of the angle between sun and moon when the lunar tide is the most perturbed in time by the solar. The corresponding value of the cosine is

$$\mathrm{n} = \sqrt{\frac{\delta-\varsigma}{2\delta}}. \tag{177}$$

This gives to ρ the value

$$\left(\tfrac{1}{2} - \frac{\sqrt{\delta^2-\varsigma^2}}{2\delta}\right)^{\frac{1}{2}}; \tag{178}$$

* This is equivalent to saying that the fall is proportional to the versed sine of twice the moon's hour angle, since $2\sin^2\theta = 1-\cos 2\theta$; consequently a cosine curve represents the rise and fall of the tide. Cf. § 47.

or approximately,

$$\frac{\sigma}{2\delta}\tag{179}$$

when δ is much larger than σ.

Near the syzygies the lunar tide or tidal force is displaced (in longitude) by an angle whose sine is

$$\frac{\sigma}{\sigma+\delta}\times \text{m};\tag{180}$$

and the solar tide by an angle whose sine is

$$\frac{\delta}{\sigma+\delta}\times \text{m}.\tag{181}$$

Near the quadratures the sine of the angle by which the lunar tide is displaced is

$$\frac{\sigma}{\delta-\sigma}\times \text{n}.\tag{182}$$

After considering the available evidence, Bernoulli assumes the ratio of the solar to the lunar tide, or σ/δ, to be $\frac{2}{5}$. By aid of the expressions just obtained, he finds that at the syzygies (springs) the tides become later each day, not by 50 minutes, but by 35, and at the quadratures (neaps) by 85 minutes.* He then gives a table showing how much the time of tide departs from the time of the moon's transit because of the sun's action. This seems to be the first table of its kind since the attempts of Philips and Flamsteed already referred to, and which were not based upon the theory of gravitation. The distance between sun and moon is taken to each 10° from 0° to 180°. The intervals as well as the age or *retard* of the tide are each supposed to be zero; the value of σ/δ is assumed to be $\frac{2}{5}$. He computes the angle, whose sine is ρ, from the formula

$$\rho=\pm\left(\tfrac{1}{2}\pm\frac{\text{B}}{2\sqrt{4+\text{B}^2}}\right)^{\frac{1}{2}}.\tag{183}$$

He then converts this displacement into minutes of time by multiplying the number of degrees by, as it seems, exactly 4.

His second table shows the same quantities as the first, excepting that three distances of the moon—perigean, mean, and apogean—are provided for instead of the mean only; also that the age of the tide is assumed to be 20°, or one and one-half days, instead of zero. This amounts to writing the tabular values in the second table opposite an argument 20° greater than that in the first table. The perigean or apogean values of the departure of the time of tide from the time of transit may be approximately obtained from the values computed for mean distance by multiplying the latter by

$$\frac{\delta}{\text{range of lunar tide at perigee}},$$

or

$$\frac{\delta}{\text{range of lunar tide at apogee}}.$$

Since the range of tide due to either luminary varies inversely as the cube of its distance from the earth,† the perigean, mean, and apogean ranges should have values proportioned to

$$\frac{1}{(0\cdot945)^3},\ \frac{1}{1^3},\ \frac{1}{(1\cdot055)^3};$$

or roughly to

$$3,\ 2\tfrac{1}{2},\ 2,$$

the reciprocals of which are proportional to

$$\tfrac{5}{6},\ 1,\ \tfrac{5}{4}.$$

---

* Table 24 gives for the daily retardation of the tide at the times of spring and neap tides, $50-36\,\frac{\text{S}_2}{\text{M}_2}$ minutes and $50+96\,\frac{\text{S}_2}{\text{M}_2}$ minutes, respectively; or 36 and 88 minutes if $\text{S}_2/\text{M}_2=\frac{2}{5}$. On account of the moon's variation, the moon's lagging is not 50 minutes per day, but 51 in syzygy and 49 in quadrature.

† As he demonstrates in Ch. VII, § 7.

These are the factors used by Bernoulli in constructing his second table.

His third table gives the relative value of the height or range of tide for each 10° of distance between sun and moon. Having the values of ρ given in the first table, the corresponding values of σ become known by the formula

$$\rho = \mathrm{m}\sqrt{1-\sigma^2} - \mathrm{n}\sigma. \tag{184}$$

These values substituted in the expression

$$\tfrac{1}{2}\,6 - \sigma^2\,6 + \tfrac{1}{2}\,\delta - \rho^2\,\delta \tag{185}$$

would give the height of the tide. Bernoulli mentions this fact, but prefers to use a formula for the range of the resultant tide involving 6, δ, and the angle between sun and moon. His formula is

$$\mathrm{M} = 6 + \delta - 2\,\mathrm{m}^2\,6 + \frac{2\,\mathrm{m}^2\,\mathrm{n}^2\,6^2}{\delta} - \frac{2\,\mathrm{m}^2\,\mathrm{n}^4\,6^3}{\delta^2}.* \tag{186}$$

The equation between ρ and σ gives, when ρ is small,

$$\sigma = \mathrm{m} - \mathrm{n}\rho\,; \tag{187}$$

and the equation between ρ and B gives, approximately,

$$\rho = \frac{1}{\mathrm{B}}\,; \tag{188}$$

or, less accurately,

$$\rho = \frac{\mathrm{mn}\,6}{\delta}. \tag{189}$$

These values of ρ and σ substituted in the expression for the height of tide give $\frac{1}{2}$ M.

In his third table he again assumes $6/\delta = \frac{2}{5}$ and takes $6 + \delta$ as unit height.

The value of M is approximately equal to

$$\mathrm{n}^2\,\mathrm{A} + \mathrm{m}^2\,\mathrm{B}\dagger \tag{190}$$

where

$$\mathrm{A} = \delta + 6,\ \mathrm{B} = \delta - 6. \tag{191}$$

By finding the variation in the lunar tide for perigean and apogean distances of the moon, the table just referred to can be made the basis of a more general one which, with the arguments increased by 20°, constitutes Bernoulli's fourth table.

Resuming the values of the lunar tide in terms of the solar, viz.:

$$3\,6,\ 2\tfrac{1}{2}\,6,\ 2\,6,$$

according as the moon is at its perigean, mean, or apogean distance, we find for the several cases

$$\begin{aligned}
\mathrm{A}' &= 1{\cdot}2\,\delta + 6 = 1{\cdot}14\,(\delta + 6) = 1{\cdot}14\,\mathrm{A},\\
\mathrm{B}' &= 1{\cdot}2\,\delta - 6 = 1{\cdot}33\,(\delta - 6) = 1{\cdot}33\,\mathrm{B};\\
\mathrm{A}' &= \delta + 6 = 1{\cdot}00\,\delta + 6 = \mathrm{A},\\
\mathrm{B}' &= \delta - 6 = 1{\cdot}00\,\delta - 6 = \mathrm{B};\\
\mathrm{A}' &= 0{\cdot}8\,\delta + 6 = 0{\cdot}86\,(\delta + 6) = 0{\cdot}86\,\mathrm{A},\\
\mathrm{B}' &= 0{\cdot}8\,\delta - 6 = 0{\cdot}67\,(\delta - 6) = 0{\cdot}67\,\mathrm{B}.
\end{aligned} \tag{192}$$

These values of A′, B′ substituted for A, B in the expression for M give approximately the tabular values of Bernoulli's fourth table, but the numerical coefficients do not exactly agree.

---

* It may be noted here that the expression

$$\sqrt{\delta^2 + (6^2 + 2\,\delta 6 \cos 2\,\theta)},$$

where θ is the angle between the sun and moon, gives, when expanded, all but the last term of M.

† Here A and B denote quantities entirely different from the former.

94. To determine the effect of the declination of either luminary upon the tide at a given place, Bernoulli finds by spherical trigonometry the distance of the place from the pole of the tidal spheroid. Then the radius of the spheroid at the place in question becomes known from the fact, already stated, that in an ellipse the radius vector is the half of the minor axis increased by a quantity proportional to the square of the cosine of the angle between it and the major axis.

Denoting the sine of the moon's polar distance by S and the cosine by C, the sine and cosine of the polar distance of the place by s and c, also the cosine of the moon's local hour angle by y, then the height of the lunar tide above mean lunar low water (i. e., a plane $\frac{1}{2}\delta$ below mean sea level) is

$$(Ssy + Cc)^2\,\delta, \tag{193}$$

which gives for lunar high water height

$$(Ss + Cc)^2\,\delta \tag{194}$$

or

$$(-Ss + Cc)^2\,\delta, \tag{195}$$

according to the transit used. Expression (193) serves to explain most of the peculiarities of the lunar tide due to the moon's declination, and the latitude of the place. The declinational effect of the sun follows from analogy.

Bernoulli shows how the range of the resultant tide of sun and moon is affected by their angular distances apart, their parallaxes, their declinations, and the latitude of the place considered simultaneously, and his final expression for the range of tide involves all these effects. In this formula he assumes that observations at the particular place furnish values for the spring and neap ranges.

From Bernoulli's discussion it would seem that he intended his tables of phase and parallax corrections to be applicable to all places, although, in the instance just referred to, he assumes that spring and neap heights must be found from observation. He does not state whether or not a general table could be prepared showing the effect of declination (on the semidaily tide). But by taking the mean of (194) and (195) also putting $y = 0$ in (193) for low waters, we see that the effect of declination is to decrease the range as the square of the cosine of the moon's declination.*

It would seem that he supposed the age or *retard* of the tide to be about a day and a half for all places, and so capable of being determined once for all. But he was aware of the fact that within a small extent of longitude the *heur du port*, high-water interval, could assume all values from zero to a half lunar day. He notes the use which might be made of the high-water inequality in keeping track of the tide where the coast line is very irregular.† He erroneously attributes the *retard* to the inertia of the water, as does Newton, but suggests that it may be due in part to the time required for the action of gravitation to reach the earth.‡ He agrees with Newton in the mistaken notion that the smallness of the diurnal inequality is due to the oscillation of the sea, so that a large tide would have a tendency to increase the otherwise small tide following it by twelve hours.§ Both writers failed to see that the tides are forced, and not free, oscillations.

Bernoulli knew that recourse to observation would be necessary for ascertaining, at any given place, quantities like the range of tide, the lunitidal interval, and the magnitude of the diurnal inequality.

Near the close of his essay Bernoulli investigates the amount of rise and fall in a small inclosed sea situated upon the equator. He inadvertently makes the amount twice too great, as Lalande‖ points out. Thus corrected his rule may be stated

$$\left.\begin{matrix}\text{range at}\\ \text{extremities}\end{matrix}\right\} : \left.\begin{matrix}\text{range for earth}\\ \text{covered with water}\end{matrix}\right\} = \text{length of sea} : \text{earth's radius}. \tag{196}$$

* General tables for corrections due to parallax and declination based upon Bernoulli's work were prepared by Lubbock, Phil. Trans., 1836, pp. 57–75 and pp. 217–266. These tables still appear in the annual "Tide Tables for the British and Irish Ports." The tables giving the phase orrections are derived from observations made at the ports for which predictions are given. (See §§ 47, 50–61.)

† Traité sur le flux et reflux de la mer, Ch. X, § 14.

‡ Ibid., Ch. VII, § 4.

§ Ibid., Ch. X, § 11. Lalande, Astronomie, Vol. IV, p. 81, upholds this explanation; but Laplace, Méc. Cél., Bk. IV, § 8, Bk. XIII, § 1, points out the fallacy of it.

‖ Astronomie, Vol. IV, p. 120.

[The truth of this is obvious. For, consider the condition of the tide on a sphere covered with water at a given instant. Reckoning distance from the point of instantaneous half-tide level, the height at a neighboring point on the equator is

$$\text{height } 45^\circ \text{ E. or W.} \times \sin 2 \left(\frac{\text{distance}}{\text{earth's radius}} \right). \tag{197}$$

But an inclosed sea will keep its surface parallel to the surface in the supposed case, and its length will be twice the above "distance;"

$$\therefore \text{height at extremities} = \text{height } 45^\circ \text{ E. or W.} \times \sin \left(\frac{\text{length of sea}}{\text{earth's radius}} \right)$$

$$= \text{height } 45^\circ \text{ E. or W.} \times \frac{\text{length of sea}}{\text{earth's radius}}, \text{ when the length is not great.]} \tag{198}$$

95. *Colin Maclaurin* (1698–1746).

Neither Maclaurin nor Euler, in the prize essays already referred to, developed methods for the reduction or prediction of tides, but each added to the theoretical side of the tidal problem. Maclaurin demonstrates for the first time (what Newton had assumed without demonstration) that a homogeneous sphere when disturbed by the moon or sun becomes, upon the equilibrium hypothesis, a prolate ellipsoid. So far as known to the present writer, Maclaurin is the first to call attention to an effect upon the water which the earth's rotation might produce. He says in Proposition VII of his essay:

If water be carried from the south toward the north, either by the general motion of the tide or by any other cause whatever, the course of the water will thereby be deflected little by little toward the east, because the water at a prior time was carried, by the diurnal motion, toward this sea with a greater velocity than pertains to the more northerly place. Conversely, if the water be carried from the north toward the south, the course of the water, on account of a similar cause, will be deflected toward the west. From this source I suspect various phenomena of the motion of the sea to arise.

He suspects the winds are also affected by the same diurnal motion.

This effect of the earth's rotation forms an essential part of Laplace's dynamical theory of tides. A somewhat analogous question, viz., the effect of the earth's rotation upon a body falling freely from a great height, was discussed by Newton and Dr. Hooke in 1679.

96. *Leonard Euler* (1707–1783).

Euler discusses the tidal problem upon the correct assumption that the tides are caused by the horizontal component of the moon's disturbing force. But if the water have a density comparable to that of the earth, it is necessary to take into account the horizontal attraction of the two instantaneous high-water regions. Euler neglects this and obtains a smaller elevating effect than that given by the equilibrium hypothesis where the density of the earth is assumed to be that of water, and the mutual attraction of the water is properly allowed for. He expresses (§ 44 of his essay) the height of the tide due to sun and moon in spherical harmonic functions of their zenith distances, carrying the expression to the fourth power of the parallaxes.

Later on, he attempts to treat the tides as a problem of fluid motion; that is, he attributes to the fluid particles the property of inertia which the equilibrium theory does not imply. It is now known that the fundamental tidal equations (first obtained by Laplace) have reference to the horizontal motions of the fluid, and to the invariability of its volume. None of these equations were obtained by Euler; he took into account the vertical oscillation only, and neglected the condition of continuity. For a somewhat more detailed review of Euler's essay than is here given, the reader is referred to the introduction of Ferrel's Tidal Researches, which includes the most of Laplace's criticisms upon it.

97. *Joseph Jérôme Lefrançais de Lalande* (1732–1807).

In the fourth volume of his Astronomy, pages 1 to 348, Lalande gives an exhaustive survey of all available tidal knowledge up to the close of the year 1780; in other words, his treatise covers nearly all that was known on the subject prior to the investigations of Laplace. His great familiarity with sources of information can be inferred from the fact that he had been paying attention to the subject during the seventeen years preceding 1780. He was the editor of the

Connaissance des Temps from 1759 to 1774, and again from 1794 to 1807, in which publication a few of his tidal papers appear. He contributed also to the academies at Paris and Dijon.

Among the matters treated or included in his astronomy the following may be mentioned:

The knowledge of tides possessed by the ancient Greeks and Romans. Their theories and other theories (or hypotheses) before the time of Newton. Newton's theory. Work of Maclaurin, d'Alembert, Euler, and Bernoulli. The equilibrium theory. Tidal phenomena or inequalities—phase, parallax, diurnal, and declinational. Observed equinoctial spring tides and other remarkably high tides. Cassini's discussions in the memoirs of the French Academy. Tides in closed seas, especially in the Mediterranean. River tides. Observations at Brest—times and heights of two or three tides daily, from June 10, 1711, to December 31, 1712, and from January 1, 1714, to August 31, 1716. Some observations at Toulon, 1777–78; Rochefort, 1771–72; St. Malo, 1775–76; Havre, 1701–2; Dunkirk, 1701–2, and Katwyk (Holland), 1766. A collection of information bearing upon tides the world over, including sources of information. General circulation of the sea. Earthquake waves. Tides in lakes, including seiches. Intermittent springs. Table of establishments for places in all countries, with authorities and dates of determination.

81 A glance at this table will show a sudden activity in tidal observations along the coast of Europe, beginning with the year 1701; one of the great incentives to this work was the hope of connecting in a more satisfactory manner the tides and the law of gravitation.

The treatise of Lalande, like those of Riccioli and Gassendi, is of special interest in connection with the historical side of the subject of tides.

98. *Jacques Henri Bernardin de Saint-Pierre* (1737–1814). In memoirs written about 1790,* this writer accounts for the semidaily and semiannual tides and flowings of the sea by the daily and yearly meltings of the ice caps in the northern and southern polar regions. He believes the heat of the moon may have some small share in melting the snow and ice. He states that certain lakes have tides from the similar periodic melting of snow and ice on the surrounding mountains. The annual flow of water from the polar regions (as evidenced by icebergs always moving toward the equator) results, he thinks, naturally enough from the hypothesis of the elder Cassinis which regards the polar diameter of the earth the greatest diameter instead of the least.

Saint-Pierre's hypothesis is examined by Woods in the Philosophical Magazine, Vol. 8 (1800).

S. Bennett, in a small volume entitled "A new Explanation of the Ebbing and Flowing of the Sea, upon the Principles of Gravitation,"† makes most of the points against the commonly accepted notion that the vertical attraction of the moon produces the tides, and shows that the horizontal attraction is the real cause.

A rather important remark of his is that since the moon's daily period exceeds that of the sun, the lunar tides will get under way better than the solar and so would become greater even if the forces to which they are due were equal.

* Œuvres complètes (1818), Vol. XI, pp. 425–508. Œuvres posthumes (1840), pp. 405–427. † New York, 1816.

CHAPTER VII.

LAPLACE.*

99. Laplace's work upon the tides occurs in Books IV and XIII of his Mécanique Céleste. Near the beginning of the thirteenth book he gives an account of his work in this direction and of the results obtained. For our purpose, it has seemed best to give this account at the outset. It will be noticed that his aim is theoretical rather than practical; i. e., he tries to develop a rational theory for explaining tidal phenomena. In most of his comparisons between theory and observation, however, he virtually falls back upon the equilibrium hypothesis; but his kinetic theory enables him to explain in a general way certain features of the tide which do not accord with conditions of static equilibrium. For instance, the comparatively small diurnal tide at Brest; also the fact that relative sizes of partial tides of nearly but not exactly equal periods may depend in part upon the motion in right ascension of the tidal bodies.

The work of Laplace has been of much practical importance in the treatment of tides. He points out the three different species of oscillations and shows how to obtain the constants involved in them from tidal observations. In fact, up to the present time the predictions for the French ports have been based upon the formula obtained by Laplace for the port of Brest. His principle of forced oscillations has since become, in the hands of Sir William Thomson (Lord Kelvin), the foundation of the most practical as well as the most accurate system known for the treatment of tidal observations.

The following authors have commented upon, expounded, or restated the principal parts of Laplace's tidal theory:

Nathaniel Bowditch, Mécanique Céleste, by the Marquis de la Place, Translated with a Commentary, Vol. I (1829), Vol. II (1832).

George B. Airy, Tides and Waves (c. 1842).

H. Resal, Traité élémentaire de Mécanique Céleste (1885).

Edmond Dubois, Résumé analytique de la Théorie des Marées telle qu'elle est établie dans la Mécanique Céleste de Laplace (1885).

100. The motion of fluids which cover the planets was then a subject almost entirely new, when, in 1774, I undertook to treat it. Aided by the discoveries which had just been made in the calculus of partial differences and in the theory of fluid motion, discoveries in which d'Alembert had a large share, I published in the Mémoirs de l'Académie des Sciences, for the year 1775, the differential equations of the motion of the fluids which cover the earth, when they are attracted by the sun and moon. I first applied these equations to the problem which d'Alembert had tried in vain to solve; viz., that of the oscillations of a fluid which would cover the earth assumed to be spherical and without rotation, supposing the attracting star in motion around our planet. I gave the general solution of this problem, regardless of the density of the fluid and its initial state, assuming that each molecule of the fluid experiences a resistance proportional to its velocity; this made me see that the primitive conditions of the motion are annihilated in the long run by the friction and small viscosity of fluid. But the inspection of differential equations very soon made me recognize the necessity of having regard to the rotary motion of the earth. I therefore considered this motion, and especially strove to determine the oscillations of the fluid which are independent of its initial state, and which alone are permanent. These oscillations are of three classes. Those of the first class are independent of the rotary motion of the earth, and their determination offers few difficulties. The oscillations dependent upon the earth's rotation, and whose period is about one day, form the second class. Finally, the third class is composed of oscillations whose period is about half a day: they are considerably larger than the others in our ports. I determined the diverse oscillations, exactly in such cases as it is possible, and by rapidly convergent approximations in the other cases. The excess of one high water over an adjacent one—the tide-producing body being in the solstice—depends upon oscillations of the second class. This excess is scarcely sensible at Brest, where, according to the theory of Newton, it should be very large. This great geometer and his successors attributed, as I have said, this discrepancy between their formulæ and the observations to the inertia of the waters of the ocean. But analysis made me see that it depends upon the law of depth of the sea. I then sought the law which would

* Pierre Simon, Marquis de Laplace (1749–1827).

render this excess zero, and I found that the depth of the sea for that purpose should be constant. Finally, assuming the figure of the earth elliptical, which also gives the sea an elliptical figure of equilibrium; I gave the general expression for inequalities of the second class, and I concluded therefrom this remarkable proposition; viz., that the movements of the terrestrial axis are the same as they would be if the sea formed with the earth a solid mass. This was contrary to the opinion of geometers, and especially d'Alembert, who, in his important work on the precession of the equinoxes, had advanced the theory that the fluidity of the sea took from it all influence upon this phenomenon. My analysis made me recognize, moreover, the general condition for the stability of the sea's equilibrium. Geometers, considering the equilibrium of a fluid placed upon an elliptic spheroid, had pointed out that in flattening its figure a little it tends to return to its initial state only in the case where the ratio of its density to that of the spheroid should be below $\frac{3}{5}$; and they had made out of this condition the condition of the stability of the equilibrium of the fluid. But it is not sufficient in this research to consider a state of repose of the fluid very near to the state of equilibrium; it is necessary to assume in this fluid any very small initial motion and to determine the necessary condition for having the motion always remain within narrow limits. In looking at this problem from a general point of view, I found that if the mean density of the earth exceed that of the sea this fluid, disturbed by any causes whatever from its state of equilibrium, will never deviate therefrom save by very small quantities; but that the deviations might be very large if this condition were not fulfilled.* Finally I determined the oscillations of the atmosphere as those upon the ocean which it surrounds, and I found that the attractions of the sun and moon can not produce the constant movement from east to west which we observe under the name of *trade winds*. The oscillations of the atmosphere produce in the height of the barometer small oscillations whose extent at the equator is a half millimetre and which deserve the attention of observers.

The preceding researches, although very general, are still far from representing the observations upon the tides in our ports: they assume the surface of the terrestrial spheroid regular and entirely covered by the sea; and we feel that the great irregularities of this surface ought to considerably modify the movement of the waters by which it is covered only in part. In fact, experience shows that accessory circumstances produce considerable varieties in the heights and in the intervals of the tides at ports even when very near one another. It is impossible to submit these varieties to calculation because the circumstances upon which they depend are not known; and even when they are known the extreme difficulty of the problem would prevent its solution. However, in the midst of the numerous modifications of the motion of the sea due to circumstances, this motion preserves, with the forces which produce it, ratios suitable for indicating the nature of these forces and for verifying the law of the attraction of the sun and moon upon the sea. Inquiry into these ratios of the causes to their effects is not less useful in natural philosophy than is the direct solution of the problems, either for verifying the existence of these causes or for determining the laws of their effects; it is of use more frequently, and it is thus, with the calculus of probabilities, a happy supplement to the ignorance and the feebleness of the human mind. In the present question I make use of the following principle, which may be useful on other occasions:

"The state of a system of bodies in which the initial conditions of the motion have disappeared through the resistance which this motion experiences is periodic, like the forces which animate it."

From this I concluded that if the sea is actuated by a periodic force expressed by the cosine of an angle which increases uniformly with the time; there results from it a partial tide expressed by the cosine of an angle increasing in the same manner, but in which the constant involved in this angle and the coefficient of this cosine may be, by virtue of accessory circumstances, very different from the same constants in the expression for the force and can be determined only through observation. The expression for the actions of the sun and moon upon the sea can be developed in a convergent series of similar cosines. Whence arise as many partial tides as, by the principle of the coexistence of small oscillations, being added together, constitute the tide which is observed in a port. It is from this point of view that I have investigated the tides in Book IV. In order to connect with these the diverse constants of the partial tides, I considered each partial tide as produced by the action of a star which moves uniformly in the plane of the equator. The tides whose period is about half a day are due to the action of stars whose proper motion is very slow in comparison with the rotary motion of the earth; and as the angle of the cosine term which expresses the action of one of these stars is a multiple of the rotation of the earth plus or minus a multiple of the proper motion of the star; and as besides the constants of these cosine terms which express the tides produced by two stars should have the same ratios as the constants of the cosine terms which express their actions, provided the proper motions were equal: I have supposed that the ratios varied from one star to another proportionally to the difference of their proper motions. The error of this hypothesis, if there be any, has no sensible influence upon the principal results of my calculations.

The greatest variations in the heights of the tides in our ports are due to the action of the sun and moon, supposed to move uniformly in their orbits and always at the same distance from the earth. But for obtaining the law of these variations it is necessary to so combine the observations that all the other variations disappear from the result. It is this which we obtain in considering the heights of the high waters above the neighboring low waters in the syzygies and quadratures taken in equal number toward each equinox and toward each solstice. By this means the tides which are independent of the rotation of the earth and those having a period of about a day should disappear, as well as the tides produced by the variation of the distance of the sun from the earth. In considering three consecutive syzygies or three consecutive quadratures, and in doubling the intermediate one, we cause the tide which is produced by the variation of the distance of the moon to disappear; because if this star is in perigee in one syzygy it is very near apogee in the syzygy following; and the compensation increases in accuracy proportionately to the greater number of observations employed. By this procedure the influence of the winds upon

*Cf. Kelvin, Popular Lectures and Addresses, Vol. II, p. 328.

the result of observations becomes almost nothing; for if the wind elevates the height of a high water it elevates by nearly the same amount the neighboring low water, and its effect disappears in the difference of the two heights. Thus, by so combining observations that their combination presents only one element, we are able to determine successively all elements of the phenomena. The analysis of probabilities furnishes a method still more sure for obtaining these elements and which we can designate by the name of *the most advantageous method.* It consists in forming between the elements as many equations of condition as there are observation equations. We reduce by the rules of this method the number of these equations to that of the elements which we determine in solving the equations so reduced. It is by this process that Mr. Bouvard has constructed his excellent tables of Jupiter, Saturn, and Uranus. But the observations of the tide being far from attaining the precision of astronomical observations, the great number of them which it is necessary to employ in order that their errors counterbalance one another does not permit us to apply to them the most advantageous method.

101. Upon the invitation of the Académie des Sciences and at the beginning of the last century, observations were made upon the tides in the port of Brest during six consecutive years. It is with these observations, published by Lalande, that I have compared my formulæ in the book cited. The situation of this port is very favorable to this kind of observations: it communicates with the sea by a vast canal, at the head of which the city is built. Thus the irregularities of the motion of the sea come into the port much weakened, nearly as the oscillations which the irregular motion of a vessel produces in the barometer are lessened by a choking made in the tube of this instrument. Moreover, the tides being considerable at Brest, accidental variations are only a small part of them. If we multiply the observations of these tides a little we notice a great regularity which the little river that loses itself in the large bay of this port does not alter. Struck by this regularity, I proposed to the Government to have a new series of observations of the tides made at Brest, and that it be continued at least during one period of the lunar node. Under these circumstances the observations were undertaken. These new observations date from the first of June, 1806, and since that time they have been continued each day without interruption. We have treated those belonging to the year 1807, and the fifteen following years. I owe to the indefatigable zeal of Mr. Bouvard whatever concerns astronomy, and the immense calculations which the comparison of my analysis with the observations has exacted. He has employed in it nearly six thousand observations: the results of these calculations are consigned to the tables which we shall see further on. For obtaining the height of the high waters and their variation, which, near the *maximum* and *minimum* is proportional to the square of the time, we have treated near each equinox and near each solstice three consecutive syzygies between which the equinox or the solstice was comprised: we have doubled the results of the intermediate syzygy in order to destroy the effects of the lunar parallax. We have taken in each syzygy the height of the evening high water above the morning low water of the day which precedes syzygy, of the day of syzygy, and of the four following days; because the *maximum* of the tides falls nearly in the middle of this interval: the choice of hours is based upon the notion that observations made during the day should therefore be more sure and more exact. We have made for each of the sixteen years one sum out of the heights of the tides of the corresponding days in the equinoctial syzygies and a similar sum relative to the solstitial syzygies, and we have therefrom ascertained the *maxima* of the heights of the high waters near the syzygies, either equinoctial or solstitial, and the variations of these heights near their *maxima.* The inspection of these heights and their variations show the regularity of this species of observations in the port of Brest.

In the quadratures we have followed a similar process, save with the difference that we have taken the excess of the morning high water above the evening low water of the day of the quadrature, and of the three days which follow it. The increase of the quadrature tides, starting with their *minimum,* being much more rapid than the diminution of the syzygial tides, starting with their *maximum,* we ought to restrict to a very small interval the law of variation proportional to the square of the times. We have formed for each of the sixteen years tables similar to those of the syzygial tides.

All these tables put in evidence the influence of the declination of the sun and moon, not only upon the absolute heights of the tides, but also on their variations. Several savants, and especially Lalande, have called this influence in doubt, because, instead of considering a great number of observations, they have confined themselves to some isolated observations when the sea, by the effect of accidental causes, was elevated to a great height toward the solstices. But the most simple application of the calculus of probabilities to the results of Mr. Bouvard is sufficient to see that the probability of the influence of the declination of the stars exceeds and is much superior to that of a great number of facts on which we do not permit ourselves to have any doubt.

We have determined from the variations of the tides near their *maxima* and *minima,* the interval by which these *maxima* and these *minima* follow the syzygies and the quadratures, and we have found this interval to be a day and a half, very nearly, which is perfectly in accord with what the old observations gave me in Book IV. The same agreement takes place relative to the sizes of these *maxima* and *minima* and in reference to the variations of the heights of the tides going from these points, so that nature after a cycle is found conformable to herself. The interval of which I have just spoken depends upon constants under the cosine signs in the expression for the two principal tides due to the action of the sun and moon. The corresponding constants of the expression of the forces are differently modified by the accessory circumstances: at the moment of syzygy the lunar tide precedes the solar tide, and it is only a day and a half afterwards, the lunar tide retarding each day upon the solar tide, that these two tides coincide and thus produce the *maximum* of the resultant tides. A similar modification takes place in the constants which multiply the cosines: thence there results an augmentation in the action of the stars on the sea. I have given in Book IV the means of recognizing this augmentation, which I have found about one-tenth from the old observations; but, however the observations of the tides at quadrature agree on this point with the observations of syzygial tides, I have said that so delicate an element would require a very much greater number of observations. The calculations

of Mr. Bouvard have confirmed the existence of this increment and have increased it to about one-fourth for the moon. The determination of this ratio is necessary in order to ascertain from tidal observations the true ratios of the actions of the sun and moon, upon which ratio depends the phenomena of the precession of the equinoxes and of the nutation of the earth's axis. In correcting the actions of the stars on the sea for the increments due to the accessory circumstances, we find expressed in sexagesimal seconds $9''\cdot4$ for the nutation, $6''\cdot8$ for the lunar equations of the tables of the sun, and for the mass of the moon $\frac{1}{75}$ that of the earth. These results are very little different from those given by the discussion of astronomical observations. The accordance of values obtained by methods so diverse is very remarkable. It is in comparing with my formulæ the *maxima* and *minima* of the observed heights of the tides that the actions of the sun and moon upon the sea, and their augmentations have been determined. The variations of the heights of the tides near these points are a series necessary for the purpose; in substituting, then, the values of these actions in my formulæ we should return to very nearly the observed variations. It is this that, in fact, we find. This accordance is a grand confirmation of the law of universal gravitation; it receives a new confirmation from tidal observations taken at syzygies when the moon is near apogee or near perigee. I have considered in Book IV only the difference of the heights of the tides for these two positions of the moon. I consider further the variation of these heights going from their *maxima*, and on these two points my formulæ represent the observations.

The times of the tides and retards from day to day offer the same varieties as their heights. Mr. Bouvard has formed of them tables for the tides which he had employed in the determination of their heights. We there clearly see the influence of the declinations of the stars and of the lunar parallax. These observations, compared with my formulæ, offer the same accordance as the observations upon the heights. Without doubt we might make the small anomalies, which comparisons still present, disappear by properly determining the constants of each partial tide. The principle by which I have united these diverse constants cannot be rigorously exact. Perhaps also the quantities which we neglect in adopting the principle of the coexistence of oscillations might become sensible in large tides. I have here contented myself with noting these small anomalies in order to direct those who would extend these calculations when the observations of the tides which are being made at Brest, and which are deposited at the royal observatory, shall be numerous enough for making certain that the anomalies are not due to errors of the observations. But before modifying the principles which I have made use of, it will be necessary to carry further the analytical approximations.

Finally, I have considered the tide whose period is about a day. In comparing the differences of two consecutive high waters and two consecutive low waters in a great number of solstitial syzygies, I have determined the amplitude of this tide and the hour of its *maximum* in the port of Brest. I found its range to be about one-fifth of a meter and about one-tenth of a day for the time by which it precedes the time of the *maximum* of the semidiurnal tide at Brest. Although its amplitude be not one-thirtieth of the amplitude of the semidiurnal tide, at the same time the generating forces of these two tides are nearly equal, which shows how differently the accessory circumstances influence the amplitudes of the tides. One will not be surprised, if he consider that in the case where the surface of the earth is regular and entirely covered by the sea, the diurnal tide should disappear if the depth of the sea were constant.

The accessory circumstances might even make the semidiurnal inequalities disappear in a port and make the diurnal inequalities very sensible. Then there is only one tide each day, and this disappears when the stars are in the equator. It is this which has been observed at Batsham, a port of the Kingdom of Tonquin, and in some islands of the South Sea. I shall observe, relatively to these circumstances, that some belong to the entire sea and are due to causes very remote from the port where the tides are observed. We cannot doubt, for example, that the undulations of the Atlantic Ocean and South Sea, reflected by the eastern coast of America, which extends almost from one pole to the other, should have a great influence upon the tides of the port of Brest. It is principally upon these circumstances that the phenomena, which are nearly the same in our ports, depend. Such appears to be the retard of the highest tide after the instant of syzygy. Other circumstances nearer the port, such as the neighboring coasts or straits, produce the differences which one observes between the heights and times of the tides in ports near together. From this it follows that a partial tide has not, with the latitude of the port, the ratio indicated by the force which produces it; since it depends upon similar tides corresponding to the latitudes far away and even in another hemisphere. We can, then, determine only by observation the sign and amplitude of this tide.

The phenomena of the tides of which I have just spoken depend upon the terms of the development of the action of the stars divided by the cube of their distances from the earth, the only ones which we have considered heretofore. But the moon is sufficiently near the earth for rendering the terms of the expression of its action divided by the fourth power of its distance sensible in the results of a great number of observations; for we know by the theory of probabilities that the number of observations supplements their want of precision, and put in evidence inequalities much less than the errors of which each observation is susceptible. We may even by this theory assign the number of observations necessary for acquiring a great probability that the error of the result obtained is included within given limits. I have thought, therefore, that the influence of the terms of the action of the moon divided by the fourth power of her distance to the earth might be manifest in the combination of the numerous observations discussed by Mr. Bouvard. The tides corresponding to the terms divided by the cube of the distance give no difference between the tides of the new moons and those of the full moons, but those which have for divisor the fourth power of her distance make a difference between these tides. They produce a tide whose period is about a third of a day. The observations of Mr. Bouvard discussed under this point of view indicate with a great probability the existence of this partial tide. They establish, further, without any doubt, that the action of the moon for producing tide at Brest is greater when her declination is south than when it is north, which can be due only to the terms of the lunar action divided by the fourth power of the distance.

We see by this exposition that the research containing the general ratios between the phenomena of the tides and the actions of the sun and moon on the sea supplement in a happy fashion the impossibility of integrating the differential equations of its motion and the ignorance of the necessary data for determining the arbitrary functions which enter into their integrals; thence results a complete certitude that these phenomena have as their only cause the attraction of these two bodies conformable to the law of universal gravitation.

I have insisted particularly upon the flow and ebb of the sea because it is, of all the effects of the attraction of celestial bodies, the nearest to us and the most sensible; moreover, it appeared to me very proper to show how we could recognize and determine by a great number of observations, although made with little precision, the laws and the causes of the phenomena for which it is impossible to obtain the analytical expressions by the formation and integration of their differential equations. Such are the effects of the solar heat upon the atmosphere in the production of trade winds and monsoons, and in the regular variations, either diurnal or annual, of the barometer and the thermometer.

102. By §§ 1, 3, Bk. IV, the general differential equations of the motion of a perfect fluid upon a sphere may be written

$$y = -\frac{\partial (\gamma u)}{\partial \theta} - \frac{\partial (\gamma v)}{\partial \varpi} - \frac{\gamma u \cos \theta}{\sin \theta}, \tag{199}$$

$$\frac{\partial^2 u}{\partial t^2} - 2 n \sin \theta \cos \theta \frac{\partial v}{\partial t} = -g \frac{\partial y}{\partial \theta} + \frac{\partial V'}{\partial \theta}, \tag{200}$$

$$\sin^2 \theta \frac{\partial^2 v}{\partial t^2} + 2 n \sin \theta \cos \theta \frac{\partial u}{\partial t} = -g \frac{\partial y}{\partial \varpi} + \frac{\partial V'}{\partial \varpi}; \tag{201}$$

or

$$y = \frac{\partial (\gamma u \sqrt{1-\mu^2})}{\delta \mu} - \frac{\partial (\gamma v)}{\partial \varpi}, \tag{202}$$

$$\frac{\partial^2 u}{\partial t^2} - 2 n \mu \sqrt{1-\mu^2} \frac{\partial v}{\partial t} = \sqrt{1-\mu^2} \frac{\partial}{\partial \mu} (gy - V'), \tag{203}$$

$$\frac{\partial^2 v}{\partial t^2} + \frac{2 n \mu}{\sqrt{1-\mu^2}} \frac{\partial u}{\partial t} = -\frac{1}{1-\mu^2} \frac{\partial}{\partial \varpi} (gy - V'), \tag{204}$$

where

1 = the earth's radius;
θ = the polar distance of the particle subject to disturbance;
$\mu = \cos \theta$;
ϖ = the terrestrial longitude of this particle;
y = the elevation of the particle above the surface of the undisturbed sea;
u = the corresponding change in θ;
v = the corresponding change in ϖ;
γ = the depth of the sea, supposing it to be small in comparison with the earth's radius and to be a function of the independent variables θ, ϖ, or μ, ϖ where $\mu = \cos \theta$;
nt = the rotary motion of the earth;
g = the force of gravity;
V (found below) = the potential of the disturbing forces—that is, the function whose partial differential coefficients are the impressed forces in the corresponding directions;
V' = this potential increased by the potential due to the disturbed water; i. e., V' is the entire tide-producing potential on the statical or equilibrium hypothesis.*

If u, v are to represent quantities proportional to the horizontal distances moved over by the disturbed particle, then $u, v \sin \theta$ (of Laplace) should be replaced by u, v; and the above equations

---

* Comparison between notations:

Laplace Méc. Cél.);	θ,	ϖ,	y,	u,	v,	γ,	nt,	V',	g.
Résal (Méc. Cél.);	θ,	ϖ,	z,	u or u/r,	$v/\sin\theta$ or $v/r \sin \theta$,	γ,	nt,	V,	g.
Darwin (Enc. Brit.);	θ,	ϕ,	$\mathfrak{h}/a$	ξ/a,	η/a	γ/a,	nt,	$g\mathfrak{e}/a^2$,	g/a.

Darwin's a denotes the earth's radius which Laplace, and generally Résal, assume to be unity. The character $\mathfrak{e}$ as here used and as used in §§ 11, 12 of article "Tides," Enc. Brit., denotes the equilibrium height of the tide. In § 6 of the same article, this height is $\frac{3}{2} a \mathfrak{e} P_2$, $\mathfrak{e}$ there being a constant.

Laplace's α is here generally omitted.

take the form given by Résal in his Mécanique Céleste, § 132. The equation not involving t is the equation of continuity for an incompressible fluid; the other two equations respectively express relations between the different forces acting along the meridian and parallel. When friction is taken into account and assumed to be proportional to the velocity of the fluid particle (as is done in § 6, Bk. IV), a term of the form $\beta \frac{\partial u}{\partial t}$ must be included in (200) and one of the form $\beta \sin\theta \frac{\partial v}{\partial t}$ in (201) divided by $\sin\theta$, and these two equations will still remain linear; but not so if the resistance due to friction be proportional to some power of the velocity other than the first. Before integrating these equations, something must be known about V' and something assumed in regard to the depth γ.

Let

r = the distance between the earth's center and the center of the disturbing body L;
L = the mass of the disturbing body;
ψ = its right ascension;
v = its declination.

Then by § 1, Bk. IV,

$$V = \frac{L}{\sqrt{r^2 - 2r\delta + 1}} - \frac{L}{r} - \frac{L}{r^2}\delta^*, \tag{205}$$

wherein

$$\delta = \cos\theta \sin \mathrm{v} + \sin\theta \sin \mathrm{v} \cos(nt + \varpi - \psi),$$

and so δ is the cosine of the angle between L and the disturbed particle as seen from the earth's center.

Now by § 23, Bk. III, this may be developed in powers of $1/r$, thus

$$V = \frac{Z^{(2)}}{r^3} + \frac{Z^{(3)}}{r^4} + \frac{Z^{(4)}}{r^5} + \cdots, \tag{206}$$

where $Z^{(i)}$† is a rational integral function of μ (= $\cos\theta$), $\sqrt{1-\mu^2}\sin\varpi$ and $\sqrt{1-\mu^2}\cos\varpi$ of the degree i, such that

$$\frac{\partial\left\{(1-\mu^2)\frac{\partial Z^{(i)}}{\partial\mu}\right\}}{\partial\mu} + \frac{\frac{\partial^2 Z^{(i)}}{\partial\varpi^2}}{1-\mu^2} + i(i+1)Z^{(i)} = 0. \tag{207}$$

By putting $Z^{(i)}/r^{i+1}$ equal to $U^{(i)}$, (206) will be a series of U's. Laplace expresses the part of V' due to the spherical layer by means of a series of Y's, any one of which satisfies equation (207). The combined result is

$$\therefore V' = \Sigma\, U^{(i)} + \Sigma \frac{3g}{\rho}\frac{Y^{(i)}}{2i+1}, \tag{208}$$

ρ being the mean density of the earth, that of water being unity.

[When $i = 2$, which is the case for the tides, total V' is to partial V', or V, as $1 : 1 - \frac{3}{5\rho}$. $\frac{3}{5\rho}$ being a small fraction is sometimes disregarded by Laplace.‡ In fact, his theory generally fails to take into account the attraction of the water in its actual disturbed state.]

103. In treating the three fundamental equations (202)-(204), Laplace confines himself to the case where the depth, γ, is a function of μ without ϖ. He seeks solutions wherein y, u, v, V' are of the following forms:

$$y = a\cos(it + s\varpi + \varepsilon), \tag{209}$$
$$u = b\cos(it + s\varpi + \varepsilon), \tag{210}$$
$$v = c\sin(it + s\varpi + \varepsilon), \tag{211}$$
$$y - \frac{V'}{g} = a'\cos(it + s\varpi + \varepsilon), = y', \tag{212}$$

* Cf. equation (106), Part II.
† $Z^{(i)}$ is proportional to $P_i(\delta)$ or $Y_i(\mu, \varpi)$ in recent notation.
‡ See § 36, Part II.

a, b, c, a', being rational functions of μ and $\sqrt{1-\mu^2}$, s being an integer; y' is the excess of the true height of the tide (y) above the equilibrium height (V'/g or V/g).* These values substituted in (203), (204) give

$$b=\frac{-g(1-\mu^2)\frac{\partial a'}{\partial\mu}+\frac{2\,ngs}{i}\mu a'}{(i^2-4\,n^2\mu^2)\sqrt{1-\mu^2}},\tag{213}$$

$$c=\frac{\frac{2\,ng}{i}\,\mu\,(1-\mu^2)\,\frac{\partial a'}{\partial\mu}-gsa'}{(i^2-4\,n^2\mu^2)\,(1-\mu^2)};\tag{214}$$

and these values of b and c substituted in (202) give

$$a=g\,\frac{\partial}{\partial\mu}\left\{\,z\left[\frac{2\,ns}{i}\,\mu a'-(1-\mu^2)\frac{\partial a'}{\partial\mu}\right]\right\}$$

$$+\frac{2\,ngs\,\mu z}{i\,(1-\mu^2)}\left\{\frac{2\,ns}{i}\,\mu a'-(1-\mu^2)\,\frac{\partial a'}{\partial\mu}\right\}+\frac{s^2g\,z\,(i^2-4n^2\,\mu^2)\,a'}{i^2\,(1-\mu^2)}\tag{215}$$

wherein

$$z=\frac{\gamma}{i^2-4\,n^2\mu^2}.$$

This is known as Laplace's tidal equation for an ocean whose depth is constant along any parallel of latitude. When by its solution a' has been found, b and c become known by the two preceding equations.

He does not attempt to integrate his equation, but merely to satisfy it. The primitive motions of the ocean must long ago have disappeared because of the resistauces which the waters encounter, and so the tide must depend upon the tide-producing bodies; the oscillation of the ocean ought to correspond to the approximately periodic terms of the attraction of these bodies.

Neglecting terms in $1/r^4$, expression (205) becomes

$$\frac{3}{2}\frac{L}{r^3}\left(\delta^2-\frac{1}{3}\right);\tag{216}$$

or, when arranged with reference to $(nt+\pi-\psi)$,

$$\frac{L}{4r^3}\left\{\sin^2\mathrm{v}-\frac{1}{2}\cos^2\mathrm{v}\right\}\ \left\{1+3\cos2\,\theta\right\}$$

$$+\frac{3L}{r^3}\sin\theta\cos\theta\sin\mathrm{v}\cos\mathrm{v}\cos(nt+\varpi-\psi)$$

$$+\frac{3L}{4r^3}\sin^2\theta\cos^2\mathrm{v}\cos2\,(nt+\varpi-\psi).\dagger\tag{217}$$

---

* Comparison between notations:

Laplace (Méc. Cél.); r, a, b, c, a', i, s, ε.

Résal (Méc. Cél.); A, a, b, c, a', i, s, ζ.

Darwin (Enc. Brit.); r, h, x, y, u$=$h$-$e, $2nf$, k, α.

Darwin's r, h, y, x, and u must be divided by his a, the earth's radius, in order to make them comparable with r, a, b, c, and a' of Laplace.

† These terms may be written

$$\frac{L}{4r^3}(1-3\cos^2\theta)(1-3\sin^2\mathrm{v})+\frac{3L}{4r^3}\sin2\,\theta\sin2\,\mathrm{v}\cos(nt+\varpi-\psi)$$

$$+\frac{3}{4}\frac{L}{r^3}\sin^2\theta\cos^2\mathrm{v}\cos2\,(nt+\varpi-\psi),\tag{218}$$

or, using Ferrel's notation,

$$\Sigma_s\,\mathrm{N}_s\cos s\,(nt+\varpi-\psi)\tag{219}$$

where $s=0$, 1, and 2.

Referring to this expression Laplace remarks:

As the three quantities r, v, ψ, vary with extreme slowness, in comparison with the rotary motion of the earth; the three preceding terms produce three different species of oscillations. The periods of the oscillations of the first kind are very long; they are independent of the rotary motion of the earth, and depend wholly upon the motion of the body L in its orbit. The periods of the oscillations of the second species depend chiefly on the rotary motion of the earth nt; their duration is nearly one day. Lastly the periods of the oscillations of the third kind depend chiefly on the angle 2nt; they are completed in about half a day.

The equation (215) is a linear differential equation; hence it follows, he remarks, that these three species of oscillations can exist together, without being confounded with each other; therefore, we may consider them separately. These three kinds of tides are then discussed by putting s successively equal to 0, 1, and 2.

104. *Oscillation of the first species.*—The expression for c indicates that for small values of i as in case of oscillation of the first species, the east-and-west motion of the fluid particle may be great in comparison with the other motions, provided there is no fluid friction involved. Laplace believes that the resistances which the waters encounter will, in the case under consideration, reduce the oscillations to their equilibrium values, especially in the case of the sun [Bk. IV, § 6].* The oscillation of long period thus becomes

$$y=\frac{L\,(\sin^2\mathrm{v}-\frac{1}{2}\cos^2\mathrm{v})\,(1+3\cos 2\theta).}{4\,r^3 g\left(1-\frac{3}{5\rho}\right)} \tag{220}$$

Oscillations of the second species.—Here $i=n$, and $s=1$; γ is assumed to be equal to $l\,(1-q\mu^2)$, l and q being constant for all latitudes.

He finds for the height of the oscillation of the second species an expression which becomes

$$y=\frac{\frac{6L}{r^3}\,lq\sin\theta\cos\theta\sin\mathrm{v}\cos\mathrm{v}}{2\,lgq-n^2}\cos\,(nt+\varpi-\psi) \tag{221}$$

when we neglect the small fraction $\frac{3}{5\rho}$. The coefficient is a, or the amplitude of the oscillation. This coefficient diminished by the (astronomical) equilibrium value is the value of a'. These values of i, s, γ, a and a' satisfy the fundamental equation (215); moreover, y has a form analogous to the corresponding term in the tide-producing potential. This is therefore a solution of Laplace's tidal equation of the required form.

When the depth is such that q is small this oscillation, and so the diurnal inequality becomes small. When the depth is constant all over the sphere $q=0$, and so oscillations of the second species vanish everywhere.†

The coefficients b and c become known by (213), (214) as soon as a' has been determined; the horizontal oscillations become

$$u=-\frac{\frac{3L}{r^3}\sin\mathrm{v}\cos\mathrm{v}}{2lgq-n^2}\cos\,(nt+\varpi-\psi), \tag{222}$$

$$v=\frac{\frac{3L}{r^3}\frac{\cos\theta}{\sin\theta}\sin\mathrm{v}\cos\mathrm{v}}{2lgq-n^2}\sin\,(nt+\varpi-\psi). \tag{223}$$

These do not vanish when $q=0$; i. e., tidal currents of a daily period would exist in the case of an ocean of uniform depth covering the sphere.

In the second as well as in the third species, Laplace finds a value of q which will enable one to determine the oscillation even if i be not exactly equal to n or to $2n$.

* Darwin concludes [Proc. Roy. Soc., Vol. 41 (1886), pp. 339, 342, and Enc. Brit. art. "Tides"] that this hypothesis is untenable unless the period be very long, as in the case of the minute oscillation whose period is nearly 19 years.

† See under Newton and Bernoulli.

105. *Oscillations of the third species, the depth being constant.*

The quantities r, ψ, and v vary slowly in comparison with $2nt$, and so may be treated as constants. The small fraction $1/\rho$, which expresses the ratio of the density of the sea to that of the earth, may be regarded as negligible. Putting $i = 2n$ and $s = 2$, the oscillation corresponding to the semidiurnal term of V', or

$$V_2', = \tfrac{3}{4}\frac{L}{r^3}\sin^2\theta\cos^2 \mathrm{v}\cos 2\,(nt + \varpi - \psi), \tag{224}$$

is

$$y = a\cos(2nt + 2\varpi - 2\psi. \tag{225}$$

$$\therefore a' = a - \tfrac{3}{4}\frac{L}{r^3 g}(1 - \mu^2)\cos^2 \mathrm{v}. \tag{226}$$

The depth being constant, $\gamma = l$. Substituting these values of a' and γ in Laplace's equation (215), it becomes

$$\frac{4n^2}{lg}a(1 - \mu^2)^2 = -\frac{\partial^2 a}{\partial\mu^2}(1 - \mu^2)^2 + (6 + 2\mu^2)\,a - \frac{6L}{r^3 g}(1 - \mu^2)\cos^2 \mathrm{v}; \tag{227}$$

or putting

$$1 - \mu^2, \text{ or } \sin^2\theta, = x^2,$$

$$x^2(1 - x^2)\frac{\partial^2 a}{\partial x^2} - x\frac{\partial a}{\partial x} - 2a\left(4 - x^2 - \frac{2n^2}{lg}x^4\right) + \frac{6L}{r^3 g}x^2\cos^2 \mathrm{v} = 0. \tag{228}$$

To satisfy this equation, replace a by the assumed value

$$a = A^{(1)}x^2 + A^{(2)}x^4 + A^{(3)}x^6 + \ . \ . \ . \tag{229}$$

and compare coefficients of like powers of x.

$$\therefore A^{(1)} = \frac{3}{4}\frac{L}{r^3 g}\cos^2 \mathrm{v}; \tag{230}$$

the comparison of the coefficients of x^4 gives an identity; but all following coefficients can be expressed in terms of $A^{(1)}$ and $A^{(2)}$.

Laplace's determination of the coefficient of x^4 has led to discussions by Airy,* Ferrel,† Thomson,‡ Darwin,§ and G. H. Ling.||

Laplace assigns (Bk. IV, § 10) values to l such that $2\,n^2/lg = 20$, 5, and 5/2 successively; that is, since $n^2/g = 1/289$,

$$l = \frac{1}{2890},\ \frac{1}{722{\cdot}5},\ \frac{1}{361{\cdot}25},$$

the earth's radius being unity. The value of a' becomes a series in powers of x^2, each having a numerical coefficient, and (230) as a general coefficient. For places on the equator the three corresponding spring ranges of the semidiurnal tide are, if $A^{(1)} = 0{\cdot}12316$ meters,

7·34, 11·05, 1·90 meters.

The expressions show that the tides are inverted in the first instance but direct in the second and third. In high altitudes, however, the tides are always direct. For a great depth, the equilibrium spring range is approached which is 0·98528 meter, assuming the tidal effect of the moon thrice that of the sun and disregarding the density of the water.

---

* Tides and Waves, Arts. 110–113.
Phil. Mag., Vol. 50 (1875), pp. 277–279, Tidal Researches, p. 154.
† Phil. Mag., Vol. 1 (1876), pp. 182–187.
The Astronomical Journal, Vol. IX (1889), pp. 41–44.
The Astronomical Journal, Vol. X (1890), pp. 121–123.
The Astronomical Journal, Vol. IV (1856), pp. 173–176.
‡ Phil. Mag., Vol. 50 (1875), pp. 227–242.
§ Encyclopedia Britannica, article Tides.
|| Annals of Math., Vol. 10 (1896), pp. 95–125.

106. In the chapter on the equilibrium of the sea he finds that—

The equilibrium of the sea is stable, if its density be less than the mean density of the earth.
If the density of the sea exceed the mean density of the earth, its figure ceases to be stable in many cases.

In the chapter on particular circumstances of each port he finds that—

The oscillations of the second kind cannot vanish, for the whole earth, except in the single case where the depth of the sea is constant. . . .

No admissible law of the depth of the sea can make oscillations of the third kind vanish in all parts of the earth.

At the end of § 15, Book IV, he says:

The irregularity of the depth of the ocean, the manner in which it is spread over the earth, the position and declivity of the shores, their connexions with the adjoining coasts, the currents, and the resistances which the waters suffer, cannot possibly be submitted to an accurate calculation, though these causes modify the oscillations of this great fluid mass. All we can do is to analyze the general phenomena which must result from the attractions of the sun and moon, and to deduce from the observations such data as are indispensable, for completing in each port the theory of the ebb and flow of the tides. These data are the arbitrary quantities, depending on the extent of the surface of the sea, its depth, and the local circumstances of the port.

At this point he returns to an empirical or modified equilibrium theory, taken in connection with a dynamical principle which he here lays down, viz.:

The state of a system of bodies, in which the primitive conditions of the motion have disappeared by the resistances it suffers, is periodical, like the forces which act on it.

Knowing the terms of the potential to be simply harmonic, and having found the terms of the component forces to be of the same character, Laplace concludes, by an argument based upon the introduction of another equal sun, that the oscillation is simply harmonic. Consequently the expression

$$y=\frac{BL}{r^3}\cos\left(2nt+2\varpi-2\psi-2\lambda\right), \tag{231}$$

B and λ being two constants dependent upon the particular port, gives the law by which the solar tide rises and falls, the sun being in the equator.

By considering a tide propagated up a canal having two mouths, Laplace notes that the phase and amplitude of the resulting tide are dependent upon the rapidity of the motion of the body in its orbit; i. e., upon the period of the oscillation, and so

Therefore the ratio of the coefficients $\frac{BL}{r^3}$ and $\frac{B'L'}{r'^3}$, given by observation of the tides, is not exactly that of the forces $\frac{L}{r^3}$ and $\frac{L'}{r'^3}$. . . . . . .

But since the constant quantities B and λ would be the same for the sun and moon, if the motions of these bodies were equal, it is natural to suppose that their differences are proportional to the differences of these motions; therefore we shall adopt this hypothesis, and we shall find that it satisfies the observations with remarkable exactness. Hence we shall put

$$\lambda=O-mT; \tag{232}$$

$$B=P\,(1-2mQ); \tag{233}$$

*O, T, P and Q being the same for the sun and moon.**

In § 19, Bk. IV, Laplace investigates inequalities in the motions of the sun and moon moving in the equator by means of fictitious bodies. He says:

The most important of these terms is that depending on the angle

$$2\,nt-2\,mt+2\,\varpi, \tag{234}$$

which produces the ebb and flow of the tide, in the case we have just examined, where the sun is supposed to move in the plane of the equator, and to be always at the same distance from the earth. The other terms may be considered as the result of the actions of as many other bodies, moving uniformly in the plane of the equator. Combining together the partial ebb and flow, corresponding to each of these bodies, we shall obtain the total ebb and flow arising from the action of the sun.

* Ferrel early came to the conclusion that the change in the tidal coefficients due to a change of velocity of the disturbing body in right ascension is not generally proportional to the amount of change in this velocity.

If we put l for the mass of the fictitious body, whose action produces the term depending on the angle $2\,nt - 2\,qt + 2\,\varepsilon$, and a for its distance from the centre of the earth; we shall have

$$\frac{3\,l}{2\,a^3} = k, \qquad \text{or } \frac{l}{a^3} = \frac{2}{3}\,k. \tag{235}$$

We have seen in the preceding article, that the sun being supposed to move uniformly in the plane of the equator, with an angular motion equal to mt, the part of the expression of the height of the sea, depending on the angle $2\,nt - 2\,mt + 2\,\varpi$, is equal to

$$P\,(1 - 2\,mQ)\,\frac{L}{r^3}\cos 2\,(nt - mt + \varpi - O + mT). \tag{236}$$

The constant quantities P, Q, O, T, are the same for all the heavenly bodies, whatever be their proper motions; therefore the sum of the partial tides, arising from the action of all the bodies l, l', l'', &c., will be,

$$\Sigma P\,(1 - 2\,qQ)\,\frac{l}{a^3}\cos 2\,(nt - qt + \varepsilon - O + qT); \tag{237}$$

consequently it will be,

$$\tfrac{2}{3}\,P\,\Sigma k \cos 2\,(nt - qt + \varepsilon - O + qT)$$

$$+ \tfrac{2}{3}\,PQ\,\frac{d}{dt}\,\Sigma k \sin 2\,(nt - qt + \varepsilon - O + qT); \tag{238}$$

the differential being taken supposing nt to be constant. But by what precedes, we have

$$\Sigma k \cos 2\,(nt - qt + \varepsilon - O + qT) = \frac{3\,L}{2\,r^3}\cos 2\,(nt + \varpi - \psi - \lambda); \tag{239}$$

the time t being decreased by T, in the variable quantities nt, ψ, r, of the second member of this equation, and $\lambda = O - nT$. Therefore the part of the height of the tide depending upon the action of the sun, and also upon the angle $2\,nt + 2\,\varpi - 2\,\psi$, with the preceding conditions, is represented by

$$P\,\frac{L}{r^3}\cos 2\,(nt + \varpi - \psi - \lambda) + PQ\,\frac{d}{dt}\left\{\frac{L}{r^3}\sin 2\,(nt + \varpi - \psi - \lambda)\right\}. \tag{240}$$

If we transfer to the moon what we have said relative to the sun, we shall find, that the part of the height of the tide depending upon the lunar action, and the rotatory motion of the earth, is

$$P\,\frac{L'}{r'^3}\cos 2\,(nt + \varpi - \psi' - \lambda) + PQ\,\frac{d}{dt}\left\{\frac{L'}{r'^3}\sin 2\,(nt + \varpi - \psi' - \lambda)\right\} \tag{241}$$

in which expression the time t must also be decreased by T. . . . .

Introducing the declination and the part independent of the rotary motion of the earth, the general expression for the height of the tide is found to be

$$\alpha y = -\frac{1 + 3\cos 2\,\theta}{8\,g\,(1 - \frac{3}{5\rho})}\left\{\frac{L}{r^3}\,(1 - 3\sin^2 \text{v}) + \frac{L'}{r'^3}\,(1 - 3\sin^2 \text{v}')\right\}$$

$$+ A\left\{\frac{L}{r^3}\sin \text{v} \cos \text{v} \cos\,(nt + \varpi - \psi - \gamma) + \frac{L'}{r'^3}\sin \text{v}' \cos \text{v}' \cos\,(nt + \varpi - \psi' - \gamma)\right\}$$

$$+ B\,\frac{d}{dt}\left\{\frac{L}{r^3}\sin \text{v} \cos \text{v} \sin\,(nt + \varpi - \psi - \gamma) + \frac{L'}{r'^3}\sin \text{v}' \cos \text{v}' \sin\,(nt + \varpi - \psi' - \gamma)\right\}$$

$$+ P\left\{\frac{L}{r^3}\cos^2 \text{v} \cos 2\,(nt + \varpi - \psi - \lambda) + \frac{L'}{r'^3}\cos^2 \text{v}' \cos 2\,(nt + \varpi - \psi' - \lambda)\right\}$$

$$+ PQ\,\frac{d}{dt}\left\{\frac{L}{r^3}\cos^2 \text{v} \sin 2\,(nt + \varpi - \psi - \lambda) + \frac{L'}{r'^3}\cos^2 \text{v}' \sin 2\,(nt + \varpi - \psi' - \lambda)\right\}. \tag{242}$$

In this expression the differentials must be taken supposing nt to be constant; and the time t must be diminished by a constant quantity T', in the terms multiplied by A, B; and by the constant quantity T, in the terms multiplied by P, Q; these constant quantities, as well as A, B, γ, P, Q, λ, must be determined, in each part, by observation.

107. Now observations made at Brest show that the terms in B and Q are small and may be neglected, and so the preceding formula becomes that resulting from the equilibrium theory.*

* See below; also under Ferrel.

This formula gives the height of the tide at any instant. [Laplace finds for the tide at Brest (Bk. IV, § 41)

$$ay = -0^{m}\cdot 02745\,[i^3\,(1-3\sin^2 \mathrm{v} + 3\,i'^3\,(1-3\sin^2 \mathrm{v}')]$$

$$+0^{m}\cdot 07179\,[i^3 \sin \mathrm{v} \cos \mathrm{v} \cos\,(\upsilon - 66^\circ\,5')^{*} + 3\,i'^3 \sin \mathrm{v}' \cos \mathrm{v}' \cos\,(\upsilon + \psi - \psi' - 66^\circ\,5')]$$

$$+0^{m}\cdot 78112\,[i^3 \cos^2 \mathrm{v} \cos 2\,(\upsilon - 66^\circ\,5') + 3\,i'^3 \cos^2 \mathrm{v} \cos 2\,(\upsilon + \psi - \psi' - 66^\circ\,5')]. \tag{243}$$

Here υ denotes the sun's hour angle; i, the ratio of the sun's mean distance to its actual distance.]

To find the times of high or low water all but the terms in P can be omitted, and we have upon equating $\frac{dy}{dt}$ to zero

$$\mathrm{tang}\,2\,(nt + \varpi - \psi' - \lambda) = \frac{\frac{L}{r^3} \cos^2 \mathrm{v} \sin 2\,(\psi - \psi')}{\frac{L'}{r'^3} \cos^2 \mathrm{v}' + \frac{L}{r^3} \cos^2 \mathrm{v} \cos 2\,(\psi - \psi')} \tag{244}$$

$$\text{Range of tide} = 2\,P\sqrt{\left(\frac{L}{r^3}\cos^2 \mathrm{v}\right)^2 + \frac{2\,L}{r^3}\cos^2 \mathrm{v}\,\frac{L'}{r'^3}\cos^2 \mathrm{v}' \cos 2\,(\psi' - \psi) + \left(\frac{L'}{r'^3}\cos^2 \mathrm{v}'\right)^2} \tag{245}$$

$$\text{Range of tide near the } \begin{matrix}\text{syzygies}\\ \text{quadratures}\end{matrix} = y'' = 2\,P\left\{\frac{L}{r^3}\cos^2 \mathrm{v} \pm \frac{L'}{r'^3}\cos^2 \mathrm{v}'\right\}.$$

$$\mp \frac{4\,P \frac{L}{r^3}\cos^2 \mathrm{v}\,\frac{L'}{r'^3}\cos^2 \mathrm{v}}{\frac{L}{r^3}\cos^2 \mathrm{v} \pm \frac{L'}{r'^3}\cos^2 \mathrm{v}'}\left\{(\psi' - \psi)^2 + \tfrac{1}{8}\,q^2\right\}, \tag{246}$$

q being the variation of the arc $\psi' - \psi$ in the interval of two consecutive high waters near syzygy or quadrature; $\psi' - \psi$ is taken at a time of a low water, and so for the mean of two adjacent high waters

$$\tfrac{1}{2}\,[(\psi' - \psi - \tfrac{1}{2}q)^2 + (\psi' - \psi + \tfrac{1}{2}q)^2] = (\psi' - \psi)^2 + \tfrac{1}{8}q^2. \tag{247}$$

The last term of (246) is the $\begin{matrix}\text{decrease near the syzygies.}\\ \text{increment near the quadratures.}\end{matrix}$

The time of high water is also given by the equation

$$\tan 2\,(nt + \varpi - \psi - \lambda) = \frac{\frac{L'}{r'^3}\cos^2 \mathrm{v} \sin 2\,(\psi' - \psi)}{\frac{L}{r^3}\cos^2 \mathrm{v} + \frac{L'}{r'^3}\cos^2 \mathrm{v}' \cos 2\,(\psi' - \psi)}, \tag{248}$$

which near the syzygies may be written

$$nt + \varpi - \psi - \lambda = \frac{(\psi' - \psi)\frac{L'}{r'^3}\cos^2 \mathrm{v}'}{\frac{L'}{r'^3}\cos^2 \mathrm{v}' + \frac{L}{r^3}\cos^2 \mathrm{v}}; \tag{249}$$

similarly near the quadratures

$$nt + \varpi - \psi - \lambda = \frac{(\psi^1 - \psi)\,\frac{L'}{r'^3}\cos^2 \mathrm{v}'}{\frac{L'}{r'^3}\cos^2 \mathrm{v} - \frac{L}{r^3}\cos^2 \mathrm{v}}. \tag{250}$$

These expressions give the retardation of the tide near the syzygies and near the quadratures.

---

* Sexagesimal.

Observations show that at Brest

$$\frac{L'}{r'^3}=3\frac{L}{r^3}, \tag{251}$$

very nearly (§ 31, Bk. IV). With this assumption, the above formulæ, all of which follow from the equilibrium theory, enable one to verify the theoretical part of the statements here quoted:

We shall now recapitulate in a few words the principal phenomena of the tides, and their relation with the laws of universal gravitation. We have generally considered these phenomena near their *maxima* and *minima*, and we have divided them into two classes; the one relative to the heights of the tides, the other relative to the hours of the tides, and their intervals. We shall now examine separately these two classes of phenomena.

The heights of the tides in each port, at their *maximum* near the syzygies, and at their *minimum* near the quadratures, are the *data* of the observations, which best show the ratio of the actions of the sun and moon upon the tides; and by means of this ratio, the various phenomena of the tides, which result from the theory of universal gravitation. One of these phenomena, which is very proper for the verification of the theory, is the law of the diminution of the tides from the time of *maximum*, or the law of their increase from the *minimum*. We have seen in [2593′, 2716],* that the theory of gravity accords perfectly with the observations in this respect.

These laws of the decrease and increase of the tides vary with the declinations of the sun and moon: we have seen in [2590, 2592], that *their decrease near the syzygies of the equinoxes, is to their corresponding decrease near the syzygies of the solstices, in the ratio of* 13 *to* 8; and that this result is conformable to the theory of gravity.† We have seen likewise, [2717′, 2718′], that *the increment of the tides, counted from the minimum near the quadratures of the equinoxes, is to the corresponding increment near the quadratures of the solstices, as* 2 *to* 1; and that the theory of gravity gives nearly the same ratio.†

According to this theory, *the height of the total tide, at its maximum near the syzygies of the equinoxes, is to the corresponding height near the syzygies of the solstices, nearly as the square of the radius is to the square of the cosines of the declinations of these bodies near the solstices;* and we have seen in [2590, 2592], that this differs but little from the result of observations. By the same theory, *the excess of the heights of the total tides, in their minimum, near the quadratures of the solstices, above their corresponding heights, near the quadratures of the equinoxes, is the same as the excess of the heights of the total tides in their maximum, near the syzygies of the equinoxes, above their corresponding heights near the syzygies of the solstices;* and we have seen [2590, 2592, 2717′, 2718′] that this is exactly conformable to the theory.‡

The influence of the moon upon the tides *increases, by the principle of gravity, as the cube of its parallax;* and by [2608, 2623, &c], this is so exactly conformable to observation, that we might have deduced from observations the law of this influence.

The phenomena of the intervals of the tides accord equally well with the theory, as those of their heights. According to the theory, *the daily retardation of the tides, at their maximum, near the syzygies, is only about half what it is at their minimum, near the quadratures.* In the first case it is nearly 27′ § [2757], and in the last case 55′ [2831]. We have seen in [2745, 2809], that the observations differ but little from this result of the theory.‖

The retardation of the tides varies with the declinations of the bodies. According to the theory, it is greater in the syzygies of the solstices, than in those of the equinoxes, in the ratio of 8 *to* 7. *In the quadratures of the equinoxes, it is greater than in those of the solstices, in the ratio of* 13 *to* 9. We have seen in [2777, 2839′], that the observations give nearly the same ratios.‖

The distance of the moon from the earth has an influence on the retardation of the tides. According to the theory, an increase of one minute in the semi-diameter of the moon, produces an increase of 258″ [2783] *in this retardation, in the syzygies, and only* 90″ [2847] *in the quadratures;* and we have seen in [2847], that this agrees with the observations, and conforms in every respect, relatively to the tides, to the law of universal gravitation.¶

We have treated fully on the ebb and flow of the sea; because it is one of the results of the attraction of the heavenly bodies most obvious to us, and the law which regulates it can be examined at every moment. It is hoped that the theory of the tides here given will induce observers to attend to the subject, in ports which, like Brest, are well situated for such observations. Accurate observations, continued during a period of the revolution of the moon's nodes, might fix with precision the elements of the theory of the ebb and flow of the tide, and perhaps make sensible the small oscillations depending on the inverse ratio of the fourth power of the distance of the moon from the earth, which have heretofore been confounded with the errors of the observations.**

---

* References marked with brackets here refer to Bowditch's Laplace.

† Last term of equation (246).

‡ Equation (246).

§ 27 minutes decimal time $=39^{m}$ ordinary time; and 55 decimal minutes $=79$ ordinary.

1^{h} dec. $=\frac{1}{10}$ day $=2^{h}\cdot4$ ordinary.
1^{m} dec. $=(\frac{1}{100})^{h}=\frac{1}{1000}$ day $=1^{m}\cdot44$ ordinary.
1^{s} dec. $=(\frac{1}{100})^{m}=\frac{1}{100000}$ day $=0^{s}\cdot864$ ordinary.

$1^{\circ}=\frac{1}{100}$ rt. angle $=0\cdot9$ degree, sexagesimal.
$1'=(\frac{1}{100})^{\circ}=\frac{1}{10000}$ rt. angle $=0\cdot54$ minute of arc, sexagesimal.
$1''=(\frac{1}{100})'=\frac{1}{1000000}$ rt. angle $=0\cdot324$ second of arc, sexagesimal.

‖ Equations (248)–(250).

¶ Equations (248)–(250).

** The quotations from Bk. IV follow Bowditch's translation and italicizing.

108. In the thirteenth book, Laplace writes the semidiurnal tides in the form

$$\frac{AL}{r^3}\cos^4 \tfrac{1}{2}\,\varepsilon\cos\,(2\,nt+2\,\varpi-2\,mt-2\,\lambda)$$

$$+\tfrac{1}{2}\,\frac{BL}{r^3}\sin^2\varepsilon\cos\,(2\,nt+2\,\varpi-2\,\gamma)$$

$$+\quad\frac{A'L'}{r'^3}\cos^4\tfrac{1}{2}\,\varepsilon'\cos\,(2\,nt+2\,\varpi-2\,m't-2\,\lambda')$$

$$+\tfrac{1}{2}\,\frac{BL'}{r'^3}\sin^2\varepsilon'\cos\,(2\,nt+\varpi-2\,\gamma) \tag{252}$$

when the constants A, A', B, γ, λ, λ' are known only from observation.*

At a time T when the cosine of the first angle is unity, and the sun is distant mT from the equinox, the range of the solar tide is

$$2\,A\frac{L}{r^3}\cos^4\tfrac{1}{2}\,\varepsilon+B\,\frac{L}{r^3}\sin^2\varepsilon\cos 2\,mT. \tag{253}$$

If p denotes the square of the cosine of the declination at syzygies,

$$\cos^2 \mathrm{v}=p\doteq 1-\frac{\sin^2\varepsilon}{2}+\frac{\cos 2\,mT}{2}; \tag{254}$$

and since

$$\cos^4\tfrac{1}{2}\,\varepsilon\doteq\frac{1+\cos^2\varepsilon}{2}, \tag{255}$$

the above may be written

$$2\,A\frac{L}{r^3}p-(A-B)\,\frac{L}{r^3}\sin^2\varepsilon\cos 2\,mT. \tag{256}$$

Similarly, the high water of the lunar tide may be written

$$2\,A'\frac{L'}{r'^3}p'-(A'-B)\,\frac{L'}{r'^3}\sin^2\varepsilon'\cos\,(2\,m'T-2\delta), \tag{257}$$

δ being the right ascension of the "intersection."

If P, Q denote the sums of the squares of the cosine of the sun's declinations at equinoctial and solstitial syzygy, respectively, and P', Q' similar quantities for the moon, then the value of i ranges at equinoctial syzygies is

$$2\,i\alpha=2\,A\,\frac{L}{r^3}\,P+2\,A'\,1{\cdot}02734\,\frac{L'}{r'^3}\,P'$$

$$-(A-B)\,\frac{L}{r^3}\,(P-Q)-(A'-B)\,1{\cdot}02734\,\frac{L'}{r'^3}\,(P'-Q') \tag{258}$$

where 1·02734 is written instead of unity because the inequality of variation increases the tide-producing force of the moon in the syzygies by 2·734 per cent.

For solsticial syzygies

$$2\,i\alpha'=2\,A\,\frac{L}{r^3}\,Q+2\,A'\,1{\cdot}02734\,\frac{L'}{r'^3}\,Q'$$

$$+(A-B)\,\frac{L}{r^3}\,(P-Q)+(A'-B)\,1{\cdot}02734\,\frac{L'}{r'^3}\,(P'-Q') \tag{259}$$

2α may be supposed to refer to equinoctial syzygy and $2\alpha'$ to solstitial syzygy.

For equinoctial quadratures

$$2\,i\alpha''=2\,A'\,0{\cdot}97266\,\frac{L'}{r'^3}\,Q_1'-2\,A\frac{L}{r^3}\,P_1$$

$$+(A'-B)\,0{\cdot}97266\,\frac{L'}{r'^3}\,(P_1'-Q_1')+(A-B)\,\frac{L}{r^3}\,(P_1-Q_1). \tag{260}$$

*The terms in B are solar and lunar K_2, the orbits being circular. ε, ε' denote inclinations of the orbits to the plane of the equator.

For solstitial quadratures

$$2\,i\alpha''' = 2\,A'\,0{\cdot}97266\,\frac{L'}{r'^3}\,P_1' - 2\,A\,\frac{L}{r^3}\,Q_1$$

$$-\,(A' - B)\;0{\cdot}97266\,\frac{L'}{r'^3}(P_1' - Q_1') - (A - B)\,\frac{L}{r^3}\,(P_1 - Q_1). \tag{261}$$

Suppose the values $2\,i\alpha$, $2\,i\alpha'$, $2\,i\alpha''$, $2\,i\alpha'''$ to be known from observation, and P, Q, P', Q', . . . to be taken from the almanac; the unknown quantities are

$$A\frac{L}{r^3},\ A'\frac{L'}{r'^3},\ B\frac{L}{r^3},\ B\frac{L'}{r'^3}. \tag{262}$$

If the two latter could be found, then the ratio $\frac{L'}{r'^3}\Big/,\frac{L}{r^3}$ would become known; and, taking from astronomy the distances of sun and moon, the relative mass of the moon (L') would become known. But $B\frac{L}{r^3}$, $B\frac{L'}{r'^3}$ cannot be determined because we cannot eliminate the one without eliminating the other. Accordingly Laplace assumes, as in his fourth book,*

$$A = (1 + mx)\,B,\quad A' = (1 + m'x)\,B, \tag{263}$$

wherein m/m' = sun's motion ÷ moon's motion = 0·0748. From these relations and the four observation equations, the four unknown quantities and $m'x$ or mx become known, and so the ratio $\frac{L'}{r^{3\prime}}\Big/\frac{L}{r^3}$.

For Brest Laplace finds

	New observations (1807–1822)	Old observations (1711–1716)
EQUINOCTIAL SYZYGIES.		
	met.	*met.*
$48{\cdot}\alpha$ =	153·711	150·235
$48{\cdot}\beta$ =	3·388	3·163
SOLSTITIAL SYZYGIES.		
$48{\cdot}\alpha'$ =	134·325	132·371
$48{\cdot}\beta'$ =	2·078	1·945
EQUINOCTIAL QUADRATURES.		
$48{\cdot}\alpha''$ =	56·561	58·033
$48{\cdot}\beta''$ =	7·744	7·495
SOLSTITIAL QUADRATURES.		
$48{\cdot}\alpha'''$ =	74·769	75·517
$48{\cdot}\beta'''$ =	3·394	3·410

$$m'x = 0{\cdot}25291$$

$$\frac{L'}{r'^3}\Big/\frac{L'}{r^3} = 2{\cdot}35333$$

[old observations gave about 3]

Mass of moon = 1/74·946.

Age of phase inequality in these four cases:

$$\left.\begin{matrix} \overset{d.}{1{\cdot}48013} \\ 1{\cdot}54684 \end{matrix}\right\}\ \overset{d.}{1{\cdot}51349}$$ = the quantity by which maximum tides follow syzygy;

$$\left.\begin{matrix} 1{\cdot}50964 \\ 1{\cdot}51269 \end{matrix}\right\}\ \overset{d.}{1{\cdot}51116}$$ = the quantity by which minimum tides follow quadrature.

* Cf. equation (233); also Ferrel, United States Coast Survey Report, 1870, pp. 190–199.

He adds that *the interval from syzygy to maximum tides and the interval from quadrature to minimum tides may be regarded as equal.* This is in accord with the old observations, which gave 1·50724 and 1·5077 (Bk. IV, §§ 24, 31).

[The variation in height (of which $\mathfrak{b}$ is a coefficient) near spring or neap tides is as the square of the time from them.]

In the fifth chapter, Laplace writes the diurnal portion of the sun's disturbing force in the form

$$\frac{3L}{2r^3}\sin\theta\cos\theta\left\{\begin{array}{l}\sin\varepsilon\sin(nt+\varpi)\\ -\sin\varepsilon\sin(nt+\varpi-2\phi)\end{array}\right\}, \tag{264}$$

thus bringing to light the two components which have since been styled solar K_1 and P_1. Similarly the lunar attraction gives rise to two waves since styled lunar K_1 and O_1. He determines the amplitude of the diurnal tide by means of observations around the solsticial syzygies. He finds its range to be $0^{met}{\cdot}2134$, while that of the semidaily tide is $5^{met}{\cdot}60$, the equilibrium theory giving $0^{met}{\cdot}674$ and $0^{met}{\cdot}350$, respectively; also that the high water of the dirunal wave precedes that of the semidirunal by $0^{d}{\cdot}095$.* He says:

> Thus, by the effect of the rotation of the earth and accessory circumstances, the diurnal tide is reduced to nearly one-third [of its theoretical value], while the semidiurnal tide becomes multiplied by 16. However, this great difference ought not surprise us, if we consider that, by Book IV, the rotation of the earth destroys the diurnal tide in a sea everywhere of equal depth; and that if the depth of the sea is $\frac{1}{720}$ of the terrestrial radius, or about 9,000 metres, the heights of the semidiurnal tide in the syzygies is 11 metres.

In the sixth chapter he detects a tide depending upon the fourth power of the moon's parallax and having a period of one-third lunar day.

* From recent harmonic analyses, $2(K_1+O_1+P_1)=1.00$ foot $=0.30$ meter; $M_2^\circ-K_1^\circ-O_1^\circ=66^\circ=2.3$ hours $=0.09$ day.

CHAPTER VIII.

WORK SINCE THE TIME OF LAPLACE.

109. *Dr. Thomas Young* (1773–1829). The tidal theory of Dr. Young is contained in the Encyclopædia Britannica, 8th edition; article, Tides.*

He seems to have been the first to distinctly suggest an extensive system of cotidal lines. But in reference to it he adds:

> If, however, we actually make such an attempt, we shall soon find how utterly inadequate the observations that have been recorded are, for the purpose of tracing the forms of the lines of contemporary high water with accuracy or with certainty although they are abundantly sufficient to show the impossibility of deducing the time of high water at any given place from the Newtonian hypothesis, or even from that of Laplace, without some direct observation.

Somewhat after the manner of Euler he first treats the tidal problem by the equilibrium theory, using the horizontal component of the tidal force. He does not fail to notice the necessity of taking into account the attraction due to the high-water regions when the density of the water is considerable. He states as theorems that the (horizontal) disturbing force of a body varies as the sine of twice its altitude; that an oblong spheroid with its axis passing through the disturbing body is a form of equilibrium; that the tide will be propagated with a velocity equal to the velocity acquired by a body falling through half the depth of the fluid (Lagrange's rule); that "the oscillations of the sea and of lakes, constituting the tides, are subject to laws exactly similar to those of pendulums . . . . . ." Besides these he states theorems relating to the disturbing attraction of the meniscus of water, to the reflection of waves, and to certain differential equations.

He introduces fluid friction into his equations, generally regarding it as proportional to the square of the velocity. The arbitrary constants entering into his solution, being assumed at pleasure give possible explanations for phenomena observed in various places: such as unusually large or small tides; also, whether high water or low water should occur at the time of transit. He finds that the "age" of the tide may be explained either by the difference of the velocities of sun and moon, or by the resistance due to friction. In fact, he was the first to mention the latter explanation. His equations refer to a single oscillating particle without reference to the ocean as a whole, and so their solution must be regarded as giving results analagous to nature, but not as being a complete solution of the tidal problem. This manner of treating the subject, while open to criticism from a theoretical point of view, is in line with most working hypotheses and especially that underlying the harmonic analysis. In further anticipation of this analysis, it may be noted that he gives some developments of tidal forces, and that he makes the following significant statement:

> There is indeed little doubt, that if we were provided with a sufficiently correct series of minutely accurate observations on the tides, made, not merely with a view to the times of low and high water, but rather to the heights at the intermediate times, we might by degrees, with the assistance of the theory contained in this article, form almost as perfect a set of tables for the motions of the ocean as we have already obtained for those of the celestial bodies, which are the more immediate objects of the attention of the practical astronomer. There is some reason to hope that a system of such observations will speedily be set on foot by a public authority; and it will be necessary, in pursuing the calculation, on the other hand, to extend the formula for the forces to the case of a sea performing its principal oscillation in a direction oblique to the meridian, as stated in the beginning of this section.

The results of *Weber* brothers' experiments on waves are published in their treatise entitled Wellenlehre auf Experimente gegründet (1825). In this book reference is made to nearly all previous researches on waves.

*This article was written in 1823. The first complete mathematical sketch of his theory was published in Nicholson's Journal, Vol. 35 (1813), pp. 145 aud 217. Both articles may be found in his Miscellaneous Works, Vol. II.

110. *Henry R. Palmer.** This engineer has given an account of a self-registering tide gauge which he designed and used in recording tides in the port of London.

A hollow iron float rises and falls with the tide within a protected well or float box. A chain passes over a barrel or float wheel. Two ends of the chain rest upon the ground, "so that the weight of the chain on each side of the barrel is always equal." Shafting connects the float wheel with a pinion which, working in a rack, moves the recording point or pencil. The record is made upon paper passing over a large drum, and wound up on a smaller one. Hourly impressions, in the figure of an arrow and along the margin of the sheet, are made by means of a punch actuated by a hammer and cam wheel. The direction of the arrow is that of a weather vane on the top of the tide house. He proposes a counterpoise equal to the resistance of the friction of those parts which the clock has to move.

In 1833 *George Rennie* communicated to the British association a paper entitled "Report on the progress and present state of our knowledge of hydraulics as a branch of engineering." The portion of the paper (B. A. A. S. Report, 1834, pp. 415 et seq.) concerning the flow of rivers is of considerable interest, especially from an historical point of view.

111. *Sir John W. Lubbock* (1803–1865).

The writings of Lubbock on tides are, for the most part, contained in the Philosopical Transactions and British Association Reports covering the period 1831 to 1837. Besides these papers may be mentioned his account of Bernoulli's paper, also An Elementary Treatise on Tides (1839).

In his first paper on the tides of London † he discusses high-water observations extending over 19 years, beginning with 1808. He adopts the (half) hour of transit as one of the arguments of tabulation and for the other, according to the inequality concerned, the time of year (month), moon's horizontal parallax, or declination. He also separates the upper and lower transits for the month of June, thus obtaining the diurnal inequality for that month for each hour of transit. In bringing out the various inequalities he makes use of all observed tides and not of certain groups as did Laplace. Tabulations according to his method have been quite extensively used in the formation of the tide tables for the coasts of Great Britain. Whewell, Ferrel, and others have, with certain modifications, followed this method first laid down by Lubbock.

Upon consulting available authorities, he prepares two charts, one for the coasts of Great Britain and one for the world; on these are written the hour of tide (at full and change) in Greenwich time, also in local time (i. e., the establishment). Although he does not actually draw cotidal lines, he introduces the term "cotidal line" which he defines as a "series of points at which it is high water at the same instant."‡

Lubbock early reaches the conclusion that the constant implying the moon's mass so varies from place to place that the moon's mass as determined therefrom and after the manner of Laplace, must be unreliable. He also finds that the variation in the retard from place to place is sometimes difficult to account for.

The London high waters being deficient in diurnal inequality it is from the Liverpool tides (Phil. Trans., 1836) that Lubbock deduces the magnitude of this inequality. He tabulates it with reference to the hour of transit and the time of year (and so for the various declinations of the moon). He finds the maximum value to occur about six days after the extreme declination of the moon and the difference in high-water heights there to be about 1 foot. He says: "The diurnal inequality in the *interval* appears to be insensible [at Liverpool]."

The observed corrections for parallax and declination do not at first agree with the theory of Bernoulli. But Lubbock points out the cause of this disagreement in a paper read before the Royal Society, June 16, 1836. His results previously published were obtained by referring the tides to the immediately preceding transit; but in this and in subsequent papers, such a preceding transit is used as will nearly allow for the retard of the tide. The parallax and declinational corrections then conform much more closely to theory. He says:

The variations in the interval between two successive transits of the moon are, in fact, of the same order in amount as those in the interval between the moon's transits and the time of high water due to the variations in

* "Description of a graphical register of tides and winds." Phil. Trans., 1831, pp. 209–213. Read March 10, 1831. It was recording the London tides as early as 1828. (Fig. 40, Airy, Tides and Waves.)

† Phil. Trans., 1831, pp. 379–415.

‡ Ibid., p. 382.

magnitude of the attractive forces; and when the interval between the time of high water and the moon's transit *immediately preceding* is considered (at least on our coasts) the variations from both these causes are mixed up together.

Concerning the connection between the height of the barometer and that of the tide, he says:

M. Daussy has ascertained that at Brest the height of high water varies inversely as the height of the barometer, and that the ocean rises 0·223 metre, or 8·78 inches, for a depression of 0·0158 metre, or 0·622 inch, in the barometer. . . . . . .

We may say roughly that at Liverpool a fall of one tenth of an inch in the barometer raises the tide an inch, *cœteris paribus*. The time of high water appeared not to be much affected.

It appears that north-easterly winds at Liverpool depress the tide, and south-westerly winds raise it. As northerly winds raise the barometer and southerly winds depress it, it will be difficult, if not impossible, to separate the effect of winds and that of the variation in the pressure of the atmosphere from each other.

But if the tide originates at a very remote distance on the surface of the earth, the atmospheric pressure *there* has probably more influence upon the phenomena than the pressure in our vicinity. This difficulty is diminished by the circumstance that the great fluctuations of the barometer are not rapid, and that the variations in the pressure of the atmosphere are extremely extensive. It will still, however, I apprehend, be very difficult to distinguish between the effects arising from variations in the atmospheric pressure, and those arising immediately from the effect of wind, as I have before remarked.*

Lubbock resumes this subject in the Transactions for 1837, p. 97, and for 1838, p. 103.†

In a paper contained in the Transactions for 1837, Lubbock assigns the letters A, B, C, D, E, F to the transits upper and lower in the order of their occurrence, F marking the one immediately preceding London high water.

112. *Dr. William Whewell* (1794–1866).

Whewell wrote extensively on tides and their attendant phenomena. His most important papers on these subjects are contained in the Philosophical Transactions and the reports of the British Association between the years 1833 and 1854. They contain much information about tides all over the world. His discussions or reductions are generally along the lines laid down by Lubbock. He was the first to draw cotidal lines,‡ although Lubbock had already marked the cotidal hours upon two charts.

He was at the head of a movement to obtain simultaneous observations over as large a portion of the earth as possible. In 1834 he succeeded in obtaining a fortnight of such observations around the British Isles. But between June 8 and June 28, 1835, simultaneous observations were made from Louisiana to Nova Scotia and from the Strait of Gibraltar to North Cape. The results are published on pages 308 to 341 of the Transactions for 1836. These observations were upon low as well as high water. Later, in the Transactions for 1847 and 1851, he strongly recommends a similar procedure for the Pacific Ocean. The simultaneous observations threw much light upon the form of cotidal lines, showing, among other things, that in general these lines meet the shore at very acute angles.

Several descriptive terms, not all of which have come into general use, are due to Whewell. Of these may be mentioned the following, together with the year of publication in which they first occur: Points of convergence, points of divergence, time of slack water *vs.* time of high water, semimenstrual inequality, corrected establishment, vulgar establishment, age of the tide (Phil. Trans., 1833); coefficient of the semimenstrual inequality (Phil. Trans., 1834); retroposition, lunitidal interval, mean lunitidal (Phil. Trans., 1836); incidence [of diurnal inequality] (B. A. A. S. Report, 1851).

Whewell treats at some length the question of diurnal inequality. He points out the fact that the age of this inequality may vary very rapidly from place to place and that it may be quite different from the age of the phase or parallax inequality. Hence no one age is common to all inequalities. The simultaneous observations of June, 1835, already alluded to, afforded a large variety of tides, and showed the prevalence of this inequality on the eastern coast of America and the outer coast of Europe. His work pertaining to this subject is not very accurate, because

* Phil. Trans., 1836, pp. 219–221.

† Cf. T. G. Bunt, B. A. A. S. Report, 1841, pp. 30–33; J. C. Ross, Phil. Trans., 1854, pp. 285–291; Airy, Tides and Waves, Arts. 572, 573; Ferrel, U. S. Coast Survey Report, 1871, pp. 93–99; Schmick, Das Flutphänomen; Lentz, Fluth und Ebbe; T. L. Ortt, Nature, Vol. 56 (1897), pp. 80–84.

‡ Phil. Trans., 1833, 1836, 1848. See under Dr. Young.

he considers the diurnal inequality dependent almost wholly upon the moon, and so the sun's effect is not properly allowed for.

Whewell describes (Phil. Trans., 1837) a graphic method for obtaining the diurnal wave. The method is applicable only when its range is small in comparison with the mean range of tide. It will be readily recognized as a forerunner of Pourtales's method. With the times of the tides as abscissæ, and the inequalities in the individual high or low water heights as ordinates, points are located through which a sinuous curve is passed by inspection. The high-water inequality curve properly combined with a similar one for the low waters gives the diurnal wave as it appears from day to day. The following are some of his conclusions in regard to the diurnal inequality:

The diurnal inequality depends upon the moon being north or south of the equator; its maximum *corresponds* to (but is not necessarily simultaneous with) the moon's greatest declination; and the period of its vanishing corresponds in like manner with the time of the moon passing the equator. Between periods corresponding to two such passages, the inequality increases from 0 to a maximum, and decreases to 0 again; after which it again increases.*

The different epoch [age] of the diurnal inequality in different parts of the world is a very curious fact; and the more so, since it is inconsistent with the mode hitherto adopted of explaining the circumstances of the tides by conceiving a tide-wave to travel to all shores in succession . . . It would seem as if the tidal phenomena on this side of the Atlantic corresponded to an epoch (of the equilibrium-theory) two or three days later than the same phenomena in America; and we may perhaps add, that different kinds of phenomena do not appear to travel at the same rate.†

The diurnal inequality [in time] is also very large in places where the tide has to run far inland, as in the Sound of Christiania in Norway, and in the Zuyder Zee in Holland.‡

We cannot know, except by observation, to what transit of the moon any tide *belongs;* but it is manifest that if we begin with any tide, the tides must belong alternately to south and north transits, and therefore the above alternation of greater and smaller tides, as the moon has north or south declination, must come into view.§

On the east coast of America, the changes of this inequality appear to be contemporaneous with the corresponding changes of the moon's declination, and the epoch is *zero*. On the coasts of Spain, Portugal, and France, it is successively *two* and *three days*. And this is quite consistent with the fact that this epoch is *four* days on the coast of Cornwall and Devonshire, *five* days at Bristol, *six* at Liverpool, and *twelve* at Leith.‖

It is easy to conceive the diurnal inequality carried a little further than it is at Singapore; so that at a certain stage of it the alternate tides would vanish. This is equivalent to supposing the highest low water and the lowest high water to have the same height.

There are statements of navigators respecting various places at which there is "only one tide in twenty-four hours." From what has been said it appears that this may happen during a part of each semilunation [half-tropical month] by the effect of the inequality now under consideration, but that it cannot in this way be constantly the case.¶

Using the phrases which have previously been employed on this subject, the *epoch* of the diurnal inequality is zero, and the effect of the moon's declination reaches Petropaulofsk without any delay or retardation.

But this view of the laws of the tides at this place, which might otherwise be accepted without difficulty, is extremely perplexed and interfered with by the other parts of the diurnal inequality,—the inequality of heights at high water and of times of low water. For though these inequalities are not so large or so regular as the others, still they are sufficiently marked and steady to allow their laws to be seen beyond doubt. And it appears, not only that the epoch of these parts of the inequality does not agree with that of the others, but that these two inequalities *alternate* with the other two, vanishing when the other reach their maxima, and showing their maxima when the others vanish.

This is a very perplexing circumstance; for we cannot doubt that the diurnal inequality depends upon the moon's not moving in the equator; and therefore, how this inequality should affect the height of high water and the time of low water most, at that period at which the moon is in the equator, it is difficult to conceive.**

As I have shown in former memoirs, we may represent the usual diurnal inequality at any place as the effect of a tide wave arriving at the shore once a day and superimposed upon the semidiurnal tide wave. We are naturally led to ask whether such a mode of representation is applicable to the tides now under consideration. The features which the diurnal inequality exhibits at Petropaulofsk are not, for the most part, inconsistent with such a representation. Thus, that the inequality should affect high water and low water very differently, is easily explained. Nor is there any difficulty in accommodating the hypothesis of the diurnal wave to one of the most curious of the laws which we have discovered; namely to this, that the inequality affects in the largest degree the *time* of high water and the *height* of low water.††

---

*Phil. Trans., 1836, pp. 301–302.
†Ibid., p. 304.
‡Ibid., 1836, p. 305.
§Ibid., 1837, p. 76.
‖Ibid., 1837, p. 81.
¶Ibid., p. 83.
**Ibid., 1840, p. 162.
††Ibid., p. 165.

On page 84, Phil. Trans., 1837, Whewell reaches the important conclusion that the mean height of the sea (half-tide level), as determined by the four tides of a day, is nearly constant, even where the diurnal inequality is large; also that this plane is preferable to such a plane as mean low water or mean high water.

It appears that in all these cases the mean height of the sea is very nearly constant. This is most remarkable at Singapore, where, though the successive low waters often differ by six feet, the mean level only oscillates through a few inches. At Plymouth the mean level is not quite so steady. The fact is, that at that port the low water varies more by the difference of springs and neaps than the high water does; and hence the mean level slightly follows the low water, and is lowest at spring tides, and highest at neap tides, or perhaps more exactly a day or two later.

"The level of the sea at low water," a phrase sometimes used by surveyors, is altogether erroneous, and may lead to material error. From the instances just quoted (and indeed from the nature of the case) it is certain that the mean height of the sea is far more nearly constant than low or high water, under whatever assumed conditions. A *level surface* drawn from any point (that is a surface of *stagnant* water) would probably be nearly parallel to the points of mean water at different places. This becomes more manifest when we consider that at places near each other the tide often differs greatly in amount. At St. Davids Head in Pembrokeshire the range of the tides is near thirty feet; on the opposite coast of Ireland it is only two or three: if the sea were level at low water the difference of the mean heights on the two sides of the Channel (which is only about fifty miles) would be fourteen feet. Such an average elevation of one side of a narrow sea above the other is quite inconsistent with the laws of fluids.*

In the Transactions for 1838, Whewell notes the effect of taking various anterior transits; and, although he finds that Transit B does well for the coast of Europe, he says:

We may, however, observe that we do not in this way obtain an exact agreement of observation and theory, even with regard to the semimenstrual inequality. It has appeared from Mr. Lubbock's researches respecting the Liverpool tides,† that while the Transit A gives a very exact agreement of the theoretical and observed times, we must take a still earlier transit if we would obtain this agreement with respect to the heights. Nor does that selection of a transit which best represents the semimenstrual inequality, bring out an agreement with theory in the parallax and declination corrections, as we shall see. We must allow, therefore, that though there appear to be, in the actual laws of the tides, inequalities *corresponding* to all these which arise from the supposition of the equilibrium-tide of an anterior epoch transmitted along the ocean to our shores, we cannot so assume the epochs to produce all the inequalities at once. The epoch is of one value for the times, of another for the heights; different again for the parallax correction, and again different for the effect of declination.

In the second volume of his History of the Inductive Sciences, Whewell gives a section on the application of the Newtonian theory to the tides,‡ and earlier in the same volume some account of the equilibrium and motion of fluids.§

In the Philosophical Transactions for 1850, Whewell notices a graphic method for obtaining the range of tide corresponding to each hour of the moon's transit. A triangle having two of its sides proportional to the ranges of the solar‖ and lunar¶ waves, respectively, and the supplement of the enclosed angle twice the difference in right ascension of sun and moon as given by the hour of transit, the remaining side of the triangle is proportional to the resultant range of tide. This method is given in the Manual of Scientific Enquiry, together with the angles showing the priming and lagging of the resultant tide.

The later writings of Whewell place little confidence in cotidal lines except near the shores. The B. A. A. S. Report for 1854, II, page 28, says:

The result is that we are led to consider whether the oceanic tides may not be produced by a great oscillation of the ocean, the littoral tides being derived from them and propagated by cotidal lines like waves along canals. [This view was proposed by Captain Fitz Roy as well as Dr. Whewell several years ago.]

It may be added that Dr. Young had already held a similar view regarding the oscillation of the ocean. In fact a bodily swinging of the ocean is in accordance with the views of Plato, Galileo, and others.

113. In the appendix to Volume II of the Narration of the Surveying Voyages of His Majesty's Ships Adventure and Beagle, between 1826 and 1836, *Capt. Robert Fitz Roy* has set down general notions about the tides, and he has employed them to help explain what is observed in diverse places.

---

* Phil. Trans., 1837, p. 84.
† Ibid., 1836, Part II.
‡ Ed. 1847, pp. 253–259.
§ Pp. 62–72; 116–128.
‖ $\frac{Sg-Np}{2}$
¶ $\frac{Sg+Np}{2}$

He believes the tides to be largely due to the swinging of a sea when slightly displaced from its position of equilibrium, the primary causes of such displacement being the sun and moon acting in accordance with Newton's law.

This theory naturally opposes the idea that the principal part of the Atlantic tide progresses northward along the axis of the ocean; for, the swinging of the sea due to the diurnal motions of the sun and moon must be chiefly east-and-west and not north-and-south. Observation, he notes, shows the range of tide at Ascension and St. Helena islands, as well as at many places upon the shores, to be a very small quantity; again, the tide is almost simultaneous all along the coast from the Cape of Good Hope to the Congo. These facts render very difficult the conception of a great and sufficient tide wave progressing northward in accordance with the cotidal lines of Whewell.

Prof. J. Challis has written numerous papers on hydrodynamics, most of which are to be found in the Philosophical Magazine since 1851; a "Report on the present state of the analytical theory of hydrostatics and hydrodynamics" is given in the British Association Report for 1833. His papers on tidal theory are found in the Philosophical Magazine for the years 1870 and 1875.

The principal writings of *Sir John Scott Russell* are contained in the British Association Reports from 1834 to 1850. They relate mainly to hydrodynamical experiments and applications of the same to marine engineering. He became popularly known through the part he played in the construction of the steamship Great Eastern. The results of his experiments on waves are contained in the Reports for 1837 and 1844.

He classifies waves in the following manner, but deals mostly with the "wave of translation:"

System of Water Waves.

Orders.	First.	Second.	Third.	Fourth.
Designation ..	Wave of translation.	Oscillating waves.	Capillary waves.	Corpuscular wave.
Characters....	Solitary.	Gregarious.	Gregarious.	Solitary.
Species......	Positive. Negative.	Stationary. Progressive.	Free. Forced.	
Varieties	Free. Forced.	Free. Forced.		
Instances....	The wave of resistance. The tide wave. The aërial sound wave.	Stream ripple. Wind waves. Ocean swell.	Dentate waves. Zephyral waves.	Water-sound wave.

One of his most important results is that the velocity of the "wave of translation," which he supposes to include the tide wave, is

$$v = \sqrt{g\,(h+\eta)} \tag{265}$$

where η is the height of the crest of the wave, reckoned from the surface of the undisturbed fluid, and h is the undisturbed depth; i. e., $h+\eta$ is the height of the free surface above the bottom, and so this formula is readily suggested by that of Lagrange. Airy's work makes, for high-water phase,

$$v = \sqrt{g\,(h+3\,\eta)}. \tag{266}$$

In Articles 393 et seq. of his Tides and Waves he compares values resulting from this and other formula with Russell's experiments and concludes this value of v to be justified. In the Report for 1844 Russell does not accept Airy's statement, but says:

> Later discussions of the experiments not only confirm this result [$v=\sqrt{g\,(h+\eta)}$], but are themselves established by such further experiments as have been recently instituted, so that this formerly obtained velocity may now be regarded as the phænomenon characteristic of the wave of the first order.

Russel's result has reference to a "solitary wave," while Airy's refers to a long wave which is periodic. See Rayleigh, Phil. Mag., Vol. 1 (1876), p. 262.

Thomas Kerigan, in a book entitled The Anomalies of the Present Theory of the Tides,* contends that the tides are due to "the *negative influence* of the moon" because he finds the attraction of the sun, and especially of the moon, to be wholly inadequate for raising them.

Sir George B. Airy (1801–1892).

* London, 1847.

114. Airy's work on tides consists chiefly of an essay entitled Tides and Waves (c. 1842), forming an article in the Encyclopædia Metropolitana, of four papers on particular tides, appearing in the Philosophical Transactions (1842, 1843, 1845, 1878), and one in the Proceedings of the Royal Society (1877). To these may be added a paper entitled "On a controverted point in Laplace's theory of the tides" (Phil. Mag., 1875).

Stokes has given some account of Airy's work in an article entitled "Report on recent researches in hydrodynamics," B. A. A. S. Report, 1846; also found in his Mathematical and Physical Papers.

D. D. Heath, in the Philosophical Magazine, Vol. 33 (1867), has a paper entitled "On the dynamical theory of deep-sea tides, and the effect of tidal friction." In this he gives a restatement of a part of Airy's work.

Airy outlines the plan of his Tides and Waves as follows:

I. We shall describe cursorily the ordinary phænomena of Tides.

II. We shall explain the *Equilibrium-Theory of Tides*, including the first tidal theory given by Newton, and the more detailed theory of his successors, especially Daniel Bernoulli.

III. We shall give a sketch of Laplace's investigations, (founded essentially on the theory of the *motion* of water,) in the general form in which he first attempted the theory, as well as with the arbitrary limitations which he found it necessary to use for practical application.

IV. We shall give an extended Theory of Waves on water, applying principally to the motion of water in canals of small breadth, but with some indications of the process to be followed for the investigation of the motion of Waves in extended surfaces of water.

V. The results of a few Experiments on Waves will be given, in comparison with the preceding theory.

VI. We shall investigate the mathematical expressions for the Disturbing Forces of the Sun and Moon which produce the Tides, and shall use them in combination with the theory of Waves to predict some of the laws of Tides.

VII. We shall advert to the methods which have been used, or which may advantageously be used, for Observation of Tides, and for the Reduction of the Observations.

VIII. We shall give the results of extensive observations of the Tides, as well with regard to the change of the phænomena of tides at different times in the same place, as with respect to the relation which the time and height of tide at one place bear to the time and height at other places, and shall compare these with the results of the preceding theories, as far as possible.

And as Conclusion, we shall point out what we consider to be the present Desiderata in the Theory and Observations of Tides.

115. Passing over the first three sections, wherein he has given the equilibrium theory in a form subsequently used by Haughton and others, also the principal parts of Laplace's theory in a more intelligible form, we proceed at once to Section IV, which deals mostly with waves in canals:

We have already stated that the Equilibrium-Theory of Tides, though curious in its relation to the history of the science, and valuable for the coincidence of the algebraic form of its results (under certain modifications) with those of more accurate theories, and with the laws deduced from observations, does not deserve the smallest attention as representing the state of the ocean at any time. We have also stated that Laplace's theory of the movement of the sea, supposing the globe completely covered by water, whose depth is uniform, or follows a very simple geographical law, though based upon sounder principles, has far too little regard to the actual state of the earth to serve for the explanation of the principal phænomena of tides. We now come to a third theory: that of the motion of the tidal waters, supposing them to run in the manner of ordinary waves in canals. It is evident that this theory will not apply to every part of the sea, and therefore it must, to a certain extent, be considered imperfect. Still it will apply strictly to many cases (to rivers without exception; and to arms of the sea where their breadth is smaller than their length, and where the irregularities of the coasts are not very remarkable), and it will apply without sensible modification to other cases of open seas, where the whole may be conceived divided into parallel canals in which the circumstances are nearly similar. For these reasons we are inclined to think that this mode of considering the subject, in the present imperfection of mathematics, deserves special notice among the various Theories of the Tides.

It is necessary for our present purpose to enter into a pretty general investigation of the Theory of Waves of water; and we shall therefore commence without any obvious reference to the subject of Tides.

We shall, for convenience, divide this Section into the following parts:

Subsection 1.—General explanation of waves; and general theory of waves, supposing the motion of the particles small.

Subsection 2.—Theory of waves in canals of uniform depth and uniform breadth, whether the waves be short or long, the motion of the particles being supposed small.

Subsection 3.—Theory of long waves in which the elevation of the water bears a sensible proportion to the depth of the canal.

Subsection 4.—Theory of waves when the water is acted on by horizontal and vertical forces, the motions of the

particles being small; including also the theory of a single wave, and the theory of waves in canals of variable depth and variable breadth; with the introduction of the ideas of *free-wave* and *forced-wave*.

Subsection 5. Method of introducing the limits of the canal in general; and application of the doctrine of *free-wave* and *forced-wave*.

Subsection 6. Theory of waves, as affected by friction.

Subsection 7. Theory of waves in water of three dimensions, or where the horizontal extent of the surface in two dimensions is taken into account.

116. Having obtained the equation of continuity and the equation of equal pressure, Airy next supposes the motion to be oscillatory and proposes the problem—

To examine whether it is possible that a system of waves, depending upon oscillatory motion of the particles of water, can move along a canal of uniform breadth, but of variable depth: gravity being supposed uniform, and no other force being supposed to act.*

His conclusion is—

It would appear, therefore, that when the depth is variable, it is impossible that there can be a series of waves which consist of oscillatory motion of the particles, and which satisfy the two equations of continuity and of equal pressure.

The following physical interpretation of this mathematical result appears to be correct, and is worthy of attention. It appears that, if the water is moving in the manner of waves, one at least of the two conditions (continuity and equal pressure) must fail. While the continuity holds, the equal pressure will exist, from the nature of the fluid. Therefore the continuity must cease, or the water must become *broken*. This appears to be the explanation of the broken water which is usually seen upon the edge of a shoal or a ledge of rocks, although the whole is covered, perhaps deeply, by the water.

He then takes up the case of a canal of uniform depth, and determines the fundamental relation [equivalent to equation (29) § 18, or (42) § 23] between the length of the wave, depth of water, and its period. These are tabulated in his first two tables;† his third table gives the velocity of a free tide-wave for various depths. His fourth table gives the relative displacements of the fluid particles at various depths, the length of the wave having a given ratio to the depth of the fluid.

117. In Subsection 3, Airy proposes the problem—

To investigate the motion of a very long wave, as the tide-wave, in a canal whose depth is so small that the range of elevation and depression of the surface bears a considerable proportion to the whole depth.

The problem of a long wave propagated in a canal of rectangular cross-section was originally solved (to the first approximation) by Lagrange;‡ that is, the velocity of propagation was found to be $\sqrt{gh}$, h being the depth of the undisturbed fluid. The long wave was subsequently treated by Green,§ who found for a triangular canal with one side vertical, the velocity of propagation to be the same as that in a rectangular canal of half the depth. Kelland ‖ found, for a uniform canal of any cross section whatever, the velocity of propagation to be $\sqrt{\frac{gA}{b}}$, A denoting the area of the cross section and b the breadth of the fluid at the surface. This formula was found to agree with results obtained from Russell's experiments. Airy derives and uses the exact equation of equal pressure (for a uniform canal of rectangular cross-section), viz.

$$\frac{\partial^2 X}{\partial t^2} = gh \frac{\partial^2 X/\partial x^2}{\left(1+\frac{\partial X}{\partial x}\right)^3} \tag{267}$$

He obtains an approximate solution upon the assumption that $\frac{\partial X}{\partial x}$ is small but not negligible. This gives for X, K, or $\frac{\partial K}{\partial x}$,¶ besides a single sine or cosine term, other similar terms having

* Tides and Waves, Art. 154.

† Tables 47 and 48 are taken from Airy's first and second tables, respectively

‡ Berlin Memoirs, 1781, 1786, Œuvres, Vol. 1, p. 747.

§ Trans. Camb. Phil. Soc., Vol. VI (1837); Vol. VII (1839).

‖ Trans. Roy. Soc. of Edinburgh, Vol. XIV (1840). In this paper Kelland makes brief mention of the workers on wave motion since the time of Newton.

¶ X, K correspond to ξ, η of § 18. We have written h instead of k for the undisturbed depth, also the conventional ∂ for partial differentiation.

double the argument of the original term. Moreover, a coefficient belonging to one of these additional, or harmonic, terms does in each case involve the factor x. He therefore concludes that the height of the secondary wave continually increases as it travels along the canal. Again, he says:

When the wave leaves the open sea, its front slope and its rear slope are equal in length, and similar in form. But as it advances in the canal, its front slope becomes short and steep, and its rear slope becomes long and gentle. In advancing still further, this remarkable change takes place in the rear slope: it is not so steep in the middle as in the upper and the lower parts; at length it becomes horizontal at the middle; and, finally, slopes the opposite way, forming in fact two waves.*

As McCowan† has pointed out, this tendency to subdivide does not follow from an exact solution of the above equation, but from applying Airy's approximation to stations situated so far from the sea that it ceases to be applicable.

Airy finds that in a shallow canal the duration of fall should exceed the duration of rise:

Excess of the time of water falling above the time of water rising $= 6\,b \times$ time occupied by the tide-wave in passing from the open sea to the station under consideration.

$$\text{Where } b = \frac{\text{rise of tide above the mean state}}{\text{mean depth of water}}. \tag{268}$$

Thus in any part of the canal far from the sea, the times of high water and of low water, and the interval between them, will on different days depend on the extent through which the surface of the water oscillates up and down, or upon the magnitude of the whole rise of tide. And in places on the canal at different distances from the sea, the inequality of the times of water rising and water falling will, on the same day, depend upon the distance of the places from the sea.‡ . . . . . .

Therefore the phase of high water has travelled along the canal with the velocity . . . $\sqrt{gh(1+3b)}$ nearly. The velocity with which a shallow wave of great length would travel along the surface of water, whose depth = depth here at high water, would, by (172.),§ be $\sqrt{g \times \text{depth at high water}} = \sqrt{g \times h\,(1+b)}$. Consequently, the phase of high water travels along the canal with a velocity greater than that of a shallow wave on water of the same depth as the high water. In like manner, the phase of low water travels along the canal with the velocity $\sqrt{[gh(1-3b)]}$ nearly, which is less than that of a long shallow wave on water of the same depth as the low water. The following theorem will be easily remembered. If D_2 be the depth at low water, D_3 that at high water, and if D_1, D_2, D_3, D_4, are in arithmetical progression; then the phase of low water travels with the velocity due to the depth D_1, and the phase of high water with the velocity due to the depth D_4.‖

After showing that the ebb-stream should be swifter than the flood-stream, and also giving a solution to the third approximation, he takes up the problem—

To investigate the motion of the tide-wave under the same circumstances, when the water of the canal is supposed also to have a current-flow (independent of fluctuations of tide) towards the sea.

He finds that the duration of fall exceeds the duration of rise by a quantity greater than in the case of no current.

The subsection concludes with an investigation for long waves in a canal whose section is invariable, but of any form, and here the velocity of propagation is found to agree with the rules of Kelland and Green.

118. Subsection 4 supposes the water acted upon by an extraneous force and has applications to a solitary wave, tides, and wind waves.

Thus it appears, that a single discontinuous wave of any degree of complexity may travel on water without any force to maintain it, provided, in the first place, that it satisfies the conditions laid down with regard to the differential coefficients at its terminations, and in the next place, that the wave is so long that a succession of simple waves, each of that length, would travel sensibly with the velocity due to waves of infinite length.¶

If the single wave is moderately long, a small force will maintain it as a discontinuous wave: but if it be short, the force must be (in proportion to the various pressures acting on the water) considerable. In fact, each of the different terms in the wave-function represents a wave of different length; and, when the waves are short, each of these would tend to travel on with its own peculiar velocity, which velocities are very different for the different waves. But when the waves are long, the peculiar velocities are very nearly the same for the different waves.

---

* Tides and Waves, Art. 203. The tides at Wilmington, N. C., show signs of this.

† Phil. Mag., Vol. 33 (1892), pp. 251, 265.

‡ Tides and Waves, Art. 207.

§ Numbers thus inclosed, in quotations from Airy, refer to articles or paragraphs in his Tides and Waves.

‖ Ibid, Art. 208. See under Russell.

¶ Tides and Waves, Art. 234.

When a long wave is propagated along a canal of non-uniform depth, Airy's investigations show that the amplitude of the horizontal displacement varies inversely as the fourth root of the cube of the depth, while the amplitude of the vertical displacement (i. e., the rise and fall of the tide) varies inversely as the fourth root of the depth.*

For a canal of non-uniform breadth, the amplitudes of the horizontal displacement will be inversely as the square root of the breadth of the canal; and the same law holds for the amplitude of the vertical displacement.†

In the case of long waves in shallow water, where the depth diminishes, the water is sensibly elevated above its mean height when the flow ceases; and in like manner it is sensibly depressed below its mean height when the ebb ceases.‡

I. e., slack-before-ebb or flood occurs earlier because of this shoaling.

Where the breadth diminishes, the water is sensibly elevated above its mean height when the flow ceases; and in like manner, the water is sensibly depressed below its mean height when the ebb ceases.

I. e., slack-before-ebb or flood occurs earlier because of this contracting.

H and G being the amplitudes, or coefficients, of the horizontal and vertical periodic forces acting upon the waters, he finds the following result agreeing with the equilibrium theory as well as with Laplace's theory:

If we consider G and H to be quantities not very dissimilar in magnitude, (which we shall find to be true,) the term depending on G in each of these expressions is wholly insignificant in comparison with that depending on H; and thus we arrive at this remarkable conclusion, that *the effect of the vertical disturbing force upon the phænomena of the tides is insignificant,* the whole of the sensible effect being due to the horizontal force.§

Near the end of this subsection he says:

The preceding conclusions are very important, as showing that the amount of elevation of the water under the action of forces depends in a most remarkable degree upon other circumstances than the magnitude of the forces. One is, the depth of the sea: another is, the periodic time of the forces. As depending upon the former, it appears that, if there were two parallel canals of different depths acted on by precisely the same forces, there might be high water in one when there was low water in the adjacent part of the other: or there might be elevations and depressions at the same time in both, but their magnitudes might have any proportion whatever. As depending upon the latter, it appears that, if there were two forces acting simultaneously upon the water in the same canal, the periods of those forces being different, (as, for instance, the forces depending upon the action of the Sun and the Moon,) the high water produced by one force might bear the same relation to the phases of that force which the low water produced by the other bears to the phases of that other force: (thus low water of the solar tide might accompany the transit of the Sun, and high water of the lunar tide might accompany the transit of the Moon, in the same canal.) Or the phases of the two tides might stand in the same relation to the phases of the two forces, but the proportion of their magnitudes to the magnitudes of the forces might differ in any degree whatever.

119. In subsection 5, Airy investigates the tides produced by the moon in a canal, friction being still left out of consideration. He finds, for a canal bounded at both ends,—

If the length of the canal is any multiple of half the length of the free tide-wave, this expression‖ becomes infinite. In reality the wave will become so large that the amount of friction, &c., will be so great as to neutralize the moon's force.¶

But when such a canal is short he concludes from the expressions for the horizontal and vertical displacements—

The first of these expressions shows that the horizontal motion will be greatest in the middle of the canal's length, and will diminish gradually both ways to the ends, where it is 0. The second shows that there is no variation of level at the middle of the canal's length, but that the variation of level in other parts is proportional to the distance from the middle, elevation taking place on one side of the middle at the same time as depression on the other side, so that the surface of the fluid remains sensibly plane, though inclined to the horizon. The law of motion as regards the time is the usual oscillatory law expressed by cos *it;* but the motion of every particle differs in this respect from the motion of particles in an open sea affected by the tide: that here, the greatest horizontal displacement happens at the same time as the greatest vertical displacement; whereas, in an open sea, the greatest horizontal displacement happens when the vertical displacement is 0, and *vice versa.*

For a canal of *any* assumed length, and bounded at both ends, the expressions for the displacements are generally complicated.

---

* Tides and Waves, Art. 247.
† Cf. Lamb, Hydrodynamics, § 181; or § 33; Part I, this manual.
‡ Tides and Waves, Art. 256.
§ Tides and Waves, Art. 279.
‖ Vertical displacement.
¶ Tides and Waves, Art. 299.

Airy next supposes the case of a canal closed at one end whose waters are acted on by the forces of the moon and which communicates with a tidal sea.

The result of this supposition is complicated; but if the moon's force in the canal is insensible, it follows that all the oscillations in different parts of the canal take place at the same time.*

When the elevation of the water bears a sensible proportion to the whole depth he finds for a canal opening at one end into a tidal sea—

The law of the rise and fall of the water, at every part of the canal except its mouth, is now different from that which holds on the supposition that the oscillation is small in proportion to the depth of the canal. But the times of high water and of low water are still the same as before, and the high water and the low water are still simultaneous through the whole length of the canal.†

Brief mention is then made of a canal connecting two seas, both tided, or one may be tideless.

120. In subsection 6 friction is taken into account; it is assumed to be proportional to the velocity of the fluid particles, or $-f\frac{\partial X}{\partial t}$, since the motion is chiefly horizontal.

In an indefinite canal, friction shortens the horizontal displacement; it causes the horizontal disturbing force to become zero at a point farther east, and so accelerates the times of the tides. Tides of longer period are more accelerated.

Considering the coefficients of the tidal force as variable, it appears that the greatest tide follows the greatest force by the time $f \times (\text{constant})^2$. He says:

This appears to us an important result, and one which no other theory has obtained. The equilibrium-theory of tides necessarily makes the tides to be greatest upon the same day on which the force is greatest. Laplace's theory, and the theory of waves in canals without friction, give the same result. But here we find a retardation accounted for by friction; and moreover this retardation is considerable.‡

For a tide propagated up a river of indefinite length, he finds that, because of friction, the vertical and horizontal motions of the particles diminish continually as the wave travels up the river; also that the flow ceases before the water has dropped to its mean height, and so turns earlier than in the case of no friction.

In a tidal river stopped by a barrier, he finds that the slack before ebb is not simultaneous with the time of high water, but somewhat later. This interval may be considerable near the mouth, but it is small near the head.§

Also that when a canal bounded at both ends is acted upon by an external force, the rise and fall of the tide is greater at the ends than at the middle.

In regard to a river of indefinite length running on a declivity toward a tidal sea, he concludes that—

The circumstance that low water on a tidal river may be higher than high water on the sea, paradoxical as it may appear, is therefore a simple consequence of theory.

121. Subsection 7 contemplates the motion of water in three dimensions. The equation of continuity is symmetrical in X, x and Z, z; there are two equations of equal pressure, the one in X, x; the other is similar in Z, z.

He finds solutions for annular and parallel waves, noting the effect of reflection from a straight boundary.

Leaving for the present the consideration of the motion of the waves as determined by the differential equations, we shall consider one case in which we seem to derive some assistance from general reasoning.

Suppose that a tide-wave is travelling along a canal of large dimensions, and of variable depth in its cross section, the depth diminishing gradually to both shores. (We may suppose the dimensions to be such as those of the English Channel, or any similar arm of the sea.) It is evident that the investigation of (218.)‖ does not apply here: for, on account of the shallowness of the water at the sides, the velocity of flow towards both sides to produce the elevation of water there must be comparable with, perhaps equal to, the velocity of flow at mid-channel in the

* Cf. § 30, Part I, this manual.

† Tides and Waves, Art. 309.

‡ Tides and Waves, Art. 329.

§ In the Philosophical Magazine, Vol. 12 (1856), pp. 184–188, C. Marret gives a popular explanation of how high-water occurs before the turn of the current, and of how the current near the shore turns before it turns in the offing. See Art. 507 of Tides and Waves.

‖ Numbers thus inclosed, in quotations from Airy, refer to articles or paragraphs in his Tides and Waves.

direction of the canal's length. Moreover, as the slope of the bottom is exceedingly small, the waves in every part of the channel will be travelling in nearly the same manner as if the extent of sea of the same depth were infinitely great, and will therefore travel with the velocity due to that depth: and, therefore, the ridge of wave cannot possibly stretch transversely to the channel, and travel along with uniform velocity lengthways of the channel. The state of things, then, will be this: the central part of the wave will advance rapidly (171.) along the middle of the channel; the lateral parts will not advance so rapidly; and the whole ridge will assume a curved shape, its convex side preceding. When this form is once acquired, it may perhaps proceed with little alteration; for if . . . we suppose two such curves exactly similar, but one a little in advance of the other, the space which separates the wings of the two curves, measured perpendicularly to the curves, (the direction in which that part of the wave must really travel,) is much less than the space which separates the centres of the curves, and by proper inclination may be less in any proportion; and, therefore, may represent exactly the space travelled over by the wave at that depth while the wave at the greater depth travels over the greater space. That part of the ridge of the wave which is nearest to the coast will, therefore, assume a position nearly parallel to the line of coast.

Now the wave whose ridge is nearly parallel to the coast, or which advances almost directly towards the coast, will be a wave of the same character as that treated of in (307.). For the slope of the beach adds to the surface of the sea a very insignificant quantity, as compared with the breadth of the tide-wave, and the general effect is the same as if a perpendicular cliff terminated the sea on that side. Therefore, for those parts of the sea which are near to the coasts the law of (307.) holds; namely, the greatest horizontal displacement of the particles occurs at the same time as the greatest vertical displacement; and, therefore, when the sea is rising, the water is, for some distance from the coast, flowing towards the coast, and when it is falling, the water is flowing from the coast.

In mid-channel, the motion of the water will be such as is described in (184.), &c.; that is, the water will be flowing most rapidly up the channel at the time of high water, and its motion upwards will cease when the water has dropped to its mean height.

From this there follows a curious consequence with regard to the currents at an intermediate distance from the shore, where the effects of these two motions may be conceived to be combined.

At high water the water is not flowing to or from the shore, but is flowing up the channel.

When the water has dropped to its mean elevation, the water is ebbing from the shore, but is stationary with regard to motion up or down the channel.

At low water, the water is not flowing to or from the shore, but is running down the channel.

When the water has risen to its mean height, the water is flowing to the shore, but is stationary with regard to motion up or down the channel.

Consequently, in the course of one complete tide, the direction of the current will have changed through 360°, the water never having been stationary. And the direction of the change of current will be of such a kind that, if we suppose ourselves sailing up the mid-channel, the tide-current will turn, in those parts which are on the *left* hand, in the *same* direction as the hands of a watch; and in those parts which are on the *right* hand, in the direction *opposite* to that of the hands of a watch.*

Beyond this we can add little to the Theory of Waves upon a sea extended in both dimensions. But the following remarks will be found important with reference to the method of determining from observations some of the phenomena of tides:

In tracing the progress of the tide across an extended sea, we cannot observe the different waves as we can those upon a small piece of water. We can do nothing but make observations of the time of the rise and fall of the sea at many different points along the shores of the bounding continents, or at islands in different parts of the sea: and when we have thus ascertained the absolute time of high water at many different points, if they are sufficiently numerous, we may draw lines over the surface of the sea passing through all the points at which high water takes place at the same absolute instant. These lines (adopting the word introduced into general use by the highest authority on the discussion of tide-observations) we shall call *cotidal lines*. The tracing out the cotidal lines in different seas is the greatest advance that has yet been made in the discussion of the phenomena of the tides in open seas.

Now when the series of waves is single, the cotidal lines correspond exactly with the lines marking the position of the ridge of the wave at different times. But when the series of waves is compound, it may happen that the form of the cotidal lines will not present to the eye the smallest analogy with the forms of the ridges of the mingled waves.†

The fifth section of the essay is devoted to an account of experiments on waves and to comparisons with theory. He finds a general agreement between the theoretical and observed velocity of propagation, but attributes the want of close agreement to the fact that Mr. Russell neglected to observe the length of the waves in his experiments.

122. In the sixth section he applies Laplace's equations of motion to tides in narrow canals. In these cases it is unnecessary to consider the forces arising from the earth's rotation. The problem thus simplified admits of solutions which take into account the motion in right ascension of the tidal body. The cases especially considered are a canal along a parallel of latitude, and a great circle in any position. For an equatorial canal, the tide is equal throughout its whole extent, and

* Cf. § 13, Part I, this manual.

† Tides and Waves, Arts. 358–366.

the depth will decide whether high water or low water occurs under the moon.* For a canal passing through the poles the tide wave is a stationary wave.

In considering the effect of sun and moon he concludes:

1st. If the depth of the sea is less than 14 miles, the mass of the moon inferred from the tides is inevitably too great.

2d. The error will be different (or the moon's mass will appear different) in canals of different depths.†

Having introduced the effect of friction, he says:

Thus it appears that for computing the time of high water it is necessary to use, not the positions of the sun and moon at the true time of the tide, nor the positions at that anterior time which is employed in computing the height of high water, but a time later than that which is used for computing the height, and therefore a time which is nearer to the true time of high water. . . . . . .

If we investigated the effect of the passage up the shallow river upon the time of low water, we should find that the positions of the sun and moon corresponding to an earlier time than that used for the height of the high water must be employed; but we should still find that the mass of the moon inferred from the variations of the time of low water as referred to the moon's transit is too small.

We shall here close our exposition of the Wave-Theory as applied to the tides. As nearly the whole of this theory is published for the first time in the present treatise, we shall not remark upon it at great length. We think it right, however, to point out to the reader its great and important defect as applied to the explanation of tides upon the earth, namely, that in the case of nature the water is not distributed over the surface of the globe in canals of uniform breadth and depth, or in any form very nearly resembling them. In this regard its fundamental suppositions are probably as much, or nearly as much, in error as those of Laplace's theory. But we also think it right to point out that in regard to the completeness of detail with which the principles can be followed out, there is no comparison between the two theories. This will be seen by the reader who has remarked the facility with which the results of "difference between the angular velocities of the sun and moon," "variable coefficients of force," and "friction," are obtained in finite form. For these, Laplace's theory is quite useless. And though (as we have stated) the fundamental suppositions differ much from the real state of the seas, yet no one can hesitate to admit that the same general conclusions will apply:—for instance, that the moon's mass inferred from the height of the tides is too great, and by different degrees in different places: that the effect of friction will be a retroposition of tides in reference to the places of the sun and moon, &c. The peculiarities of river-tides, which no other theory has touched upon, are almost completely mastered by this.

123. In the seventh section is described Bunt's self-registering tide gauge; the methods of discussion adopted by Laplace, Lubbock, and Whewell; but of special interest is his description of a process of harmonic analysis which he had already applied to the tide curves at Deptford,‡ and which he is about to apply to the tide curves at Southampton and Ipswich.§

In brief, he expresses the depression of the surface of the sea below a fixed mark for any given phase of the tide in the form

$$\begin{aligned} &A_0 + A_1 \cos. \text{phase} + A_2 \cos. 2\,\text{phase} + \&\text{c.}, \\ &\quad + B_1 \sin. \text{phase} + B_2 \sin. 2\,\text{phase} + \&\text{c.}, \end{aligned} \tag{269}$$

which, it is well known, is sufficient for the representation of a function which is periodical for 360° of phase.

Then follow directions for determining the A's and B's.

124. In Section VIII Airy brings the tidal theories (equilibrium, Laplace's, and wave theories) to bear upon many questions connected with tides the nature of which are indicated by the following topics:

Variation in range and shape of the tide as it progresses; the bore; tides in small seas; revolution of tidal currents; races; mean level of the sea little affected by the range of tide; the ratio of the solar to the lunar wave (coefficient of semimenstrual inequality) varies from place to place, and also depends upon whether it is obtained from heights or from times; similarly for the age of this inequality; the necessity of using different transits for different inequalities; the diurnal tide; and cotidal lines.

The essay concludes with a statement of the present desiderata in the theory and observation of tides.

125. In the Philosophical Transactions for 1845 Airy makes a study of the tide at about twenty stations scattered around the coast of Ireland.

He ascertains the times when the high and low water inequalites become zero and when a

---

* Cf. § 41, Part I, this manual.

† Tides and Waves, Art. 455.

‡ Phil. Trans., 1842, pp. 1–8.

§ Ibid., 1843, pp. 45–54.

maximum. He is the first writer to make special use of the diurnal wave at the time of its maximum amplitude. Its amplitude, and position with respect to the semidiurnal wave are found from the height inequalities.* He shows that the diurnal and semidiurnal waves do not travel alike either in direction or in velocity.

The high water at Kingstown coincides *precisely* with the low water at Dunmore East, and *vice versâ*. Moreover, between these two stations occurs the station Courtown; and here . . . the semidiurnal tide is nearly insensible. The difference in the times at Dunmore East and Kingstown does not therefore arise from a slow transmission of tide; but arises from a sudden *inversion* of the wave, the point which separates elevation from depression being not far from Courtown. And the question now is, whether, on the supposition that the tide-wave enters the Irish Sea by this southern entrance, it is possible to explain the existence of this neutral point and the inversion of the tide beyond it.

This he believes may be explained by a result established in "Tides and Waves," where a uniform canal closed at one end communicates with a tidal sea; and which is, that the oscillation is simultaneous throughout the canal. In case of the Irish Sea representing such a canal, the open end is to the south and the closed end to the north; and if the depth be such that the coefficient (amplitude) of the simultaneous oscillation have opposite signs at Kingstown and Dunmore East, the phenomenon is, in a general way, explained. (See §30, this manual.)

He has some further discussion upon the coefficient and age of the semimenstrual inequality as determined from time and height.

He determines the coefficients of each individual tide at the various stations, the period covered being two months.†

Having discussed the tides at Courtown, he says:

Both the semidiurnal tides are very much diminished, the lunar so much that its range is rather less than that of the solar tide. The quarto-diurnal tide exists in nearly its greatest magnitude. The geometrical representation is perfect; the mechanical explanation is not complete. In both respects, as regards what is reduced to law and what is yet incomplete, the Courtown tides must be regarded as the most remarkable that have ever been examined.

He is inclined to believe that the tertio-diurnal tide is not sensible on the coast of Ireland.

At the close of his discussion of the tides at Malta (Phil. Trans., 1878), Airy gives some account of the seiches as observed at that place.

126. Among the Mathematical and Physical Papers (1880) of *Prof. G. G. Stokes* several relate to the subject of wave motion. He treats the "long wave" in the paper entitled "Recent researches in hydrodynamics" (B. A. A. S. Report, 1846), and in the one entitled "Notes on hydrodynamics. IV—On Waves" (Camb. and Dub. Math. Jour., 1849). Airy's work upon tides in canals and certain of Russell's experiments are considered in this connection. Besides these may be mentioned a third paper entitled "On the theory of oscillating waves" (Trans. Camb. Phil. Soc., 1847).

Capt. F. W. Beechey, Phil. Trans., 1848, discusses the tidal currents in the Irish Sea and English Channel. He draws upon a series of maps lines indicating the direction of the current (lines of flow) at stated hours and lines of equal range, the moon being new or full. Upon a chart of the Irish Sea showing "the set and rate of the flood stream" he has indicated a region of no current, which is caused by the meeting of the waters from north and south. Upon the chart of ranges of tide, the range nearly vanishes at Courtown while on the opposite coast of Wales it is 15 or 16 feet. In the Philosophical Transactions for 1851 he gives charts of the English Channel showing the lines of flow for each hour before and after high water at Dover. These charts have been copied in several publications. Quite recently (1891) M. Hédouin of the *Service hydrographique de la marine* has designed similar charts for this region, the currents being referred to the tides at Cherbourg.

$$*\ 2D_1 = \sqrt{HWQ^2 + LWQ^2}$$

$$\tan(\text{HW phase}) = \frac{LWQ}{HWQ}$$

Of course the amplitudes obtained have not been corrected for the time of year or the longitude of the moon's node, i. e., each value is really $D_1 \div F_1$, Table 32.

† In the mean, these coefficients nearly coincide with M_2, M_4, M_6, and M_8, excepting at Courtown, where the lunar tide does not predominate. The angular constants correspond to 0, $2\,M_2° - M_4°$, $3\,M_2° - M_6°$, $4\,M_2° - M_8°$.

127. *Prof. Alexander D. Bache* (1806–1867).

As Superintendent of the Coast Survey, Bache caused many tidal observations to be made all along the coasts of the United States. He also gave his personal attention to the discussion of the observations; and among those who assisted in this work were L. F. Pourtales, L. W. Meech, Henry Mitchell, Charles A. Schott, and R. S. Avery.

His writings on tides are contained in the Coast Survey Reports (1851–1866) and in the Proceedings of the American Association (1850–1857). As shown by these writings, the chief purpose of his work was the construction of cotidal lines and the obtaining of suitable elements for prediction of tides. This implied, besides many and extended observations, suitable modes of classification according to proper astronomical arguments. The principal numerical results are given in the Reports (1853–1864) under the title "Tide tables for the use of navigators," the most complete of these tables being found in the Report for 1864, pp. 58–90. The constants given for nearly all except Gulf stations are: Mean high-water interval, extreme phase inequality in time, mean range of tide, spring range, neap range, duration of rise, of fall, and of stand. For the Gulf stations the constants are: The average range, the range at greatest declination and at zero declination. For numerous stations the phase inequality is tabulated according to the single argument—the time of the moon's transit. For several Pacific stations tables of double argument are provided—the time of the moon's transit and the number of days from her extreme declination. Tables of single argument—the number of days from greatest declination—are also given. The double-argument table must give predictions superior to those given by means of two tables of single argument successively applied. And so it seems that the predictions for the Pacific Coast issued by the Survey for the years 1867–1870, should, on this account, be more accurate than the predictions for the years 1871–1884 where single-argument tables were used, in accordance with Avery's paper published in the Report for 1868 and entitled "Mode of forming a brief tide table for a chart." In fact, if a series of such double-argument tables were prepared for the different years or portions of the node-equinox period, the resulting predictions must accord well with the tides in nature, especially if the heights are corrected for parallax. Bache contemplated corrections for "the solar and lunar parallax and declination," but they were never extensively introduced into the computed predictions of the Coast Survey tide tables.

The work of *Meech* was largely directed along this line which had already been opened up by Bernoulli, Laplace, Lubbock, Whewell, and Airy. Some account of his work is given in the Proceedings of the American Association, 1856, I, pp. 166–170, and in the Coast Survey Report, 1856, pp. 249–251. An obvious fault of his treatment, where the diurnal inequality is large, is the neglecting of the motion of the moon's node.

Bache paid considerable attention to the diurnal inequality along the Pacific coast.* His manner of treatment was essentially that of Lubbock and Whewell, and he shows in the report for 1854 that even where the diurnal inequality is large, the height inequalities are nearly proportional to the sine of twice the moon's declination.

For the Gulf tides, Bache constructed two sets of cotidal lines, one for the semidiurnal tide, and one for the *diurnal at the times of extreme declination.* In the report for 1856, p. 254 and sketch 35, he shows the semiannual variation of the lunitidal interval of the diurnal wave at extreme declination. This variation he finds to accord in a general way with that given by a formula of Airy's† for the displacement of the lunar diurnal tide by the solar. In fact the variation is zero at the equinoxes and solstices, the interval being longest in February and August. But he infers that extreme variation differs greatly for different places. This conclusion is doubtless based upon too few determinations. In fact the variation in interval is nearly alike in amount the world over, or at least wherever the age of the diurnal inequality is small. It is simply the perturbation in the K_1 O_1 wave, at the time when K_1 and O_1 conspire, due to P_1, or it is very nearly the perturbation in K_1 due to P_1 (Table 31) multiplied by $K_1/(K_1 + O_1)$. This gives the extreme variation for mean years as about $\pm$ 46^m.

In the same report Bache notices the important fact that for several days at the time of

* See also discussions of Gulf tides, U. S. Coast Survey Reports, 1851, pp. 127–136 or 1866, pp. 113–119; 1852, pp. 111–122.

† Tides and Waves, Art. 46.

maximum diurnal tides, the lunitidal interval of the diurnal wave varies slowly, but that it varies rapidly from day to day as the moon approaches the equator. This is shown by a diagram on "Sketch 35." In a general way, this accords with a remark made in § 13, Part III, or with results obtainable from Table 27.

In the Survey Report for 1857, also in the Proceedings of the American Association for the same year, Bache shows, by aid of diagrams, the effects which the three great bays of the Atlantic coasts of the United States have upon the range of tide; also how the range of tide increases in passing up the Bay of Fundy.

In the reports for 1856 and 1858 he discusses the tidal currents near Sandy Hook and their effect upon the growth of the hook. His explanation of the fact that the velocity of the ebb stream generally exceeds that of the flood is as follows:*

> Since the tide wave is propagated most rapidly in deep water, it follows that the fall of the tide takes place earlier in the channel than upon the shore; hence the water tends to flow laterally *from* the shore *towards* the channel. In this way a convergence of the ebb streams may be expected, especially in shallow bays. With the flood streams the reverse must be true, and the tide wave rising earlier in the channel a flow of water takes place toward the shore. In consequence of these distinctive characteristics the ebb and flood assume an unequal share in the molding of sandy coasts. The ebb current, with its concentration of forces, is a far more powerful agent than the flood; its scouring capacity along its normal course must be more considerable, and it creates more extensive draft currents . . . . But the ebb is the primary working agent, and the characteristic features of all channels and basins, on alluvial tidal coasts, must, as a rule, reflect the effects of the ebb current.

128. An account of *Pourtales'* method for finding the diurnal wave is given by Charles A. Schott in a discussion of Kane's tidal observations in the Arctic Seas, Smithsonian Contributions to Knowledge, Vol. XIII (1863), p. 78:

> The process of decomposition in use in the U. S. Coast Survey was at first an analytical one, by computing sine curves; since 1855, however, a graphical process, equivalent thereto, was substituted; this latter method, as introduced by Assistant L. F. Pourtales, may be briefly explained as follows: After the observations are plotted and a tracing is taken, the traced curves are shifted in epoch 12 (lunar) hours forward, when a mean curve is pricked off between the observed and traced curves; this process is repeated after the tracing paper has been shifted 12 hours backward; the average or mean pricked curve thus obtained represents the semi-diurnal wave. On an axis parallel with that on which the time is counted, the differences between the originally observed and the constructed semi-diurnal wave were laid off; this constitutes the diurnal curve. In the case in hand I have simplified the process of separation by blackening the under surface of the tracing paper with a lead pencil, and running in with a free hand; the intermediate curves by the pressure of a style, an average of the two traces thus left on the lower paper, gave the semi-diurnal wave in quite an expeditious manner. On the diagram, the diurnal curve with its epoch of high water nearly coinciding with that of the semi-diurnal wave, appears plainly with its variation in size depending on the moon's declination.

Besides the Arctic tides just referred to and those observed by Dr. Hayes, *Schott* has discussed, in the Coast Survey Report for 1854, the tidal currents around Nantucket and Marthas Vineyard, also the tides and currents of Long Island Sound.

Largely through the exertions of *Avery*, the Survey commenced the annual publication of tables of predicted tides. These tables, already alluded to, began with the year 1867 and continue up to the present time. In a paper entitled "Methods of registering tidal observations," found in the Survey Report for 1876, Avery gives a considerable amount of practical information in regard to observing tides; also a description of a self-registering gauge of his own design.

The principal writings of *Mitchell* are found in the Coast Survey Reports from the year 1854 to the year 1887. They deal mostly with the effects of tidal currents upon harbors and shore lines. Incidental to such work, he devised a tide gauge for exposed stations,† and an apparatus for observing currents below the surface.‡ The localities treated at some length are: Marthas Vineyard and Nantucket, New York Harbor, Monomoy Peninsula, Portland Harbor, Greytown and Uraba, the Lower Mississippi River, the Delaware River, and the Gulf of Maine.

His principal papers of a general character are: "On the reclamation of tide lands and its relation to navigation" (Report, 1869); "Location of harbor lines" (Report, 1871); "Notes concerning alleged changes in the relative elevation of land and sea" (Report, 1877). They contain

* Cf. Mitchell, United States Coast Survey Report, 1869.
† United States Coast Survey Report, 1854, pp. 190, 191; 1857, pp. 403, 404.
‡ Ibid., 1859, pp. 315–317.

numerous rules and practical suggestions, which belong to the art of hydrographic engineering rather than to the study of the tides. To these we may add his pamphlet, issued by the Navy Department in 1868, entitled "Tides and tidal phenomena."

Hydrographic work of a similar character has been since carried on by *Henry L. Marindin.* His papers upon the same are to be found in the Survey Reports since 1880; in particular those for 1888 and 1892.

129. *Rev. Samuel Haughton's* principal writings upon tides are to be found in the Philosophical Magazine (1856, 1863), the Philosophical Transactions (1863, 1866, 1875, 1877, 1878), and the Transactions of the Royal Irish Academy (1854, 1893, 1895). Besides these may be mentioned brief notices in the Proceedings of the Royal Society of London (1860–1877) and a small book entitled Manual of Tides and Tidal Currents (1870).

The tides discussed by him are those around the coasts of Ireland and in the Arctic Seas. He has, in a general way, followed the methods of Airy, and among the quantities worked for are the mass of the moon, the eccentricity of the lunar orbit, the mean depth of the Atlantic Ocean regarded as a canal running north and south.

William Parkes, in the Philosophical Transactions for 1860, treats the high and low waters at Bombay and Karachi, where the diurnal inequality is large. He combines the two waves of variable amplitudes—the diurnal and semidiurnal—and obtains results agreeing fairly well with observation. In the British Association Report for 1870 (I, p. 150), Thomson makes some mention of Parkes' work, comparing with observations predictions made by the latter's method, those made by Thomson's method, and those according to the Admiralty method.

James Croll has written upon the influence of the tidal wave on the earth's rotation, and upon the causes and climatic effects of ocean currents. These writings are found in the Philosophical Magazine, American Journal of Science (1864–1876), and British Association Report (1876).

T. K. Abbott has contributed brief papers on tidal theory to the Philosophical Magazine (1870–1872), the Quarterly Journal (1872), and Hermathena (1882). His small book, based upon the foregoing papers, entitled Elementary Theory of Tides (1888), gives a popular treatment of a few fundamental questions in the kinetic, or rather the canal theory.

E. Lacy Garbett, somewhat after the manner of Abbott, gives a popular exposition of several difficulties in the kinetic theory of tides. He says (Phil. Mag., 1870):

> It appears that, without supposing the remark to be in anywise new, I happened in 1853 to make the first English mention that tidal friction must increase the length of the day . . . and to suggest (what Delaunay is now considered to have verified) that this cause might have counteracted and masked the shortening due to contraction, so as to account for the non-diminution (or, as now admitted, lengthening) of the day since Hipparchus's time. [See under Ferrel, "Questions of priority."]

In the Philosophical Magazine for 1874 *Alfred Taylor* has a paper entitled "On tides and waves,—deflection theory." He advocates the view "that the level of the ocean is nearly represented by high-water mark on coasts and bays where there is free access of the tide and a channel without a sudden taper," instead of being about half-tide level as it would be natural to suppose. He does not believe that tidal action has the smallest effect on the rotation of the earth. His "Deflection theory" takes its name from a supposed deflection (refraction?) which the attractive rays experience in passing through the earth. He deduces from experiments by J. S. Russell and by Darcy a new formula for wave propagation in any depth, p, viz.: $v=3\sqrt{p}$ feet per second.

E. J. Chapman, in the Philosophical Magazine for 1874, proposes the theory that the tides result from the compression of the earth's nucleus, which is surrounded with a layer of incompressible water.

130. *J. Heinrich Schmick* is the author of a book entitled Das Flutphänomen und sein Zusammenhang mit den säkularen Schwankungen des Seespiegels (1874). Besides treating the matter indicated by the title one part is devoted to earthquake, sea, or ocean waves.

Hugo Lentz is the author of a book entitled Fluth und Ebbe und die Wirkungen des Windes auf den Meersspiegel (1879), gives among other things an intelligent account of tidal inequalities, and shows that the notion of the "age" of the tide is quite untenable. As indicated by the title, a portion of the work is devoted to wind effects on the height of the sea, the stations considered being along the North and Baltic seas.

Comoy, in his book entitled Étude pratique sur les Marées Fluviales et notamment sur le Mascaret (1881), gives an account of wave motion, particularly waves of translation, as propagated up tidal rivers; a study of the mascaret in the rivers of France; and the effects of river improvements.

J. C. Houzeau and *A. Lancaster* in their Bibliographie générale de l'Astronomie,* Vol. II, pp. 626-635, give a complete list of papers upon the theory of tides, including the effect of lunar attraction upon gravity, the effect of the tides upon the earth's rotation, also atmospheric or aerial tides. The list begins with the writings of Wallis and continues through the year 1880.

131. *A. B. Basset* is the author of a treatise on hydrodynamics (1888), the seventeenth chapter of which is upon liquid waves. Chapter XIX is devoted to the theory of tides. He treats in brief the equilibrium theory, gives Darwin's development of Laplace's theory, and also portions of Airy's canal theory.

Prof. Horace Lamb in A Treatise of the Mathematical Theory of the Motion of Fluids (1879), in addition to a general exposition of his subject, discusses "waves of small vertical displacement" (i. e., "long waves" or "waves of translation"), and illustrates by examples drawn from Airy's treatment of tides in canals. Near the close of his book is a "List of memoirs and treatises" pertaining to fluid motion.

In his Hydrodynamics (1895) the tidal theory is set forth in a concise and masterly manner. It is the best exposition of the theory known to the writer. The chapters entitled "Viscosity" and "Equilibrium of rotating masses of liquid," involve the principal discoveries along these lines made by Stokes, Rayleigh, Kelvin, Darwin, Love, Lamb, Poincaré, and others.

132. *Prof. Wm. Harkness*, Washington Observations for 1885, App. III, gives a collection of determinations of the mass of the moon since the time of Newton, adding thereto determinations made by himself from the harmonic constants of over thirty stations. His concluded value from the tides is $0{\cdot}012714 \pm 0{\cdot}000222 = 1/78{\cdot}653 \pm 1.374$ $(= \frac{1}{80} + \delta\mu)$. At the close he gives a "List of works consulted in the preparation of the foregoing paper," and here are given numerous references to recent papers on tides.

Maj. A. W. Baird is the author of a book entitled Manual for Tidal Observations (1886). In the first part are practical directions for locating stations, setting up and caring for tide gauges, and auxiliary (meteorological) instruments. In the second part are directions for carrying out the harmonic analysis in accordance with the system of Thomson and Darwin. The appendix consists of auxiliary tables used in connection with the analysis.

L. d'Auria has contributed several articles to the Journal of the Franklin Institute, among which are the following: "On the measurement of tidal heights" (1879); "On the force of impact of waves," etc. (1890); "A new theory of the propagation of waves in liquids" (1890); "Analytical discussion of the tidal volume admitted into bays and rivers," etc. (1891); "The law of variation of the theoretical amplitude of tidal oscillation," etc. (1891).

In these the meaning of the author is not always clearly set forth; consequently it seems impossible to ascertain just what he has in mind, and why he believes that certain relations obtain. The subjects of these papers are important and his treatment is suggestive; for these reasons they may be worth consulting.

Prof. William Ferrel (1817-1891).

133. Ferrel's Tidal Researches, published by the Coast Survey in 1874, include the greater part of his theoretical work. One of the principal objects of these investigations is the determination of the effects resulting from fluid friction when assumed to vary according to a power of the velocity greater than the first (friction $= -f\,V^{e}$). Laplace had generally altogether ignored friction, and Airy had assumed it to be proportional to the first power of the velocity. In either of these cases the fundamental differential equations of motion are linear, but upon Ferrel's assumption they no longer remain so. Dr. Young had assumed friction to be as the square of the velocity, but his treatment is imperfect, inasmuch as it does not involve the equation of continuity. The important and then new subject of shallow-water components is treated at some length in the Tidal Researches, but much more fully in his "Discussion of tides in Penobscot Bay."†

* Brussels, 1882.

† United States Coast and Geodetic Survey Report, 1878.

Ferrel was the first to give, in 1868, any considerable development of the tide-producing potential. This development he reproduces, with some modifications, in his Tidal Researches. Confining our attention to this later treatment, we may describe it as follows:

Laplace's expression for this potential, when developed in multiples of the body's hour angle, gives rise to several distinct parts or classes of terms, which may be written in the general form $N_s \cos s\,(nt + \varpi - \psi)$, or $N_s' \cos s\,(nt + \varpi - \psi')$, according as the moon or sun is considered, s taking the values 0, 1, 2, ... The first part does not contain the hour angle of the disturbing body, the moon, say; the second class has a period a lunar day in length; the third class a half lunar day; the fourth class one-third of a lunar day; on account of the smallness of the last it may be disregarded, for the present at least.

The coefficients of these periodic functions of the moon's hour angle have, at a given place, two elements of variability; the one being the factor $1/r^3$, the other some sine or cosine function of δ, the moon's declination. $1/r^3$ is equivalent to a constant quantity (which is slightly greater than $1/\rho^3$, ρ being the mean value of r) plus comparatively small periodic terms whose arguments or periods readily follow from the expression for the moon's parallax. Here and elsewhere Ferrel employs circular arguments which vary uniformly with the time, or nearly so. The arguments of the principal periodic terms in $1/r^3$ are the moon's mean anomaly, the argument of evection, of variation, twice the moon's mean anomaly, and the mean anomaly of the sun. The circular arguments belonging to the functions of δ, already referred to, are the longitude of the moon and of the lunar node. When $1/r^3$ is multiplied by these functions of δ, terms naturally arise whose arguments are simple combinations of those in the two factors. In this manner the coefficients of the three principal parts of the moon's tide-producing potential are each developed into a number of terms constant or periodic. The periodic terms in that part (N_0) of the potential which does not involve the moon's hour angle, and which give rise to oscillations in the sea level of long period, really constitute a harmonic development. The coefficient (N_1) of the function whose period is one lunar day has no constant term, but its principal term has as argument the longitude of the moon reckoned from the solstice. This coefficient or amplitude is therefore negative during half of each month. The coefficient (N_2) of the function whose period is a half lunar day consists of a constant term together with numerous periodic terms. The most important of these have as arguments the moon's mean anomaly, twice the longitude, the arguments of evection and of variation.

Of course the tide-producing potential of the sun admits of a similar development.

The tide producing potential due to the attraction of both sun and moon may likewise be developed. The terms which do not involve the hour angle of either body are simply added together algebraically. The parts having a half-day period, $N_2 \cos 2\,(nt + \varpi - \psi)$ and $N_2' \cos 2\,(nt + \varpi - \psi')$, give, when combined, a resultant amplitude of the form obtained when two cosine curves are combined into one. The angle or argument, which is twice the moon's hour angle, becomes in the resultant somewhat altered; but this, too, is in accordance with the combination of two simple cosine curves. The expansion of the resultant amplitude (N_2) gives rise to a constant term and to numerous periodic terms, the chief of which has as argument twice the angle between the sun and moon. The arguments of several others have already been mentioned.

If in the diurnal part, the sidereal (or, more properly, tropical) day had been used instead of the lunar, then the coefficient would have had a constant term, and numerous periodic terms; the arguments of the two principal periodic terms being twice the longitude of the moon, and twice the longitude of the sun, both reckoned from the solstice, say.

This nonharmonic development of the potential is in a form for application to observations made upon high and low waters. It shows the theoretical proportions between the various inequalities in the tide. The non-harmonic or inequality methods of Ferrel form an extension to the works of Laplace and Lubbock. He makes use of all observations, and not of certain groups selected for particular purposes as did Laplace; he distributes the observations according to the inequality sought, usually dividing its period into 12 or 24 nearly equal parts; he analyzes the corresponding 12 or 24 values of the ranges or intervals, thereby determining the most probable value of the amplitude and position of the inequality; he compares the ratio of the coefficient to the range of tide with the corresponding ratio in the tide-producing potential; the failure of these to agree implies the existence of what Laplace calls "accessory circumstances," or an

incorrect assumed mass of the moon, or both. The greater the number of inequalities treated, the more of these constants can be determined. Ferrel usually determines two besides the correction to the mass of the moon, using therefor the three largest inequalities in the (semidaily) tide.*

As some account of his method of determining the coefficients and epochs of the inequalities appears in § 54, also in § 46, Part III, no further notice will be taken of it here than to refer to Chapter VI of the Tidal Researches, where the tides at Brest are discussed; to his "Discussion of tides in Boston Harbor;"† and particularly to his "Discussion of tides in New York Harbor."‡

In regard to Ferrel's harmonic development of the potential of the tide-producing forces, we will only remark that it is the first ever made—at least with any tolerable completeness; that a number of lunar nodal terms are given which arise from the varying inclination of the lunar orbit to the plane of the equator; and that his numerical values of the coefficients of the sun's tide-producing potential are each affected by a term in $\delta\mu$.§ For, the coefficients of the tide-producing potential of the sun, when expressed as fractions of certain parts of the tide-producing potential of the moon, must involve some assumption regarding the mass of the moon relative to the mass of the earth or sun. Ferrel assumes the mass of the moon 1/80 that of the earth plus another very small fraction $\delta\mu$ of the earth's mass.

134. The fundamental tidal equations are satisfied by assuming the vertical and horizontal displacements of the fluid particle which result from a harmonic term of the potential to be simple harmonic functions with constant coefficients and coperiodic with the term of the potential, friction being ignored or taken to be proportional to the first power of the velocity. But if friction be as a higher power of the velocity, then, although the water be deep, the simple harmonic functions just referred to no longer satisfy the tidal equations, and simple harmonic functions of one-third the period of the others must be included in the expressions for the displacements.

Hence we have obtained as a first result of the effect of friction, which must be regarded as new and important, that *when friction is as a higher power than the first power of the velocity, it produces, in either diurnal or semidiurnal tides, small oscillations with a period which is one-third of that of the principal tide.*

In case of very shallow water, where the amplitude of the vertical oscillation bears a sensible proportion to the depth, quarter-day oscillations must be included in the expressions for the displacements of the particle, the resistance due to friction being either included or ignored.||

135. *Ferrel's method of determining the moon's mass from harmonic components.*

By the equilibrium theory the amplitudes of all components of the same class (long-period, diurnal, or semidiurnal) should have fixed ratios to one another and so to any one of them. The epochs of all components of a class should be equal to one another. If the speeds of the components were very nearly equal this would, undoubtedly, be very nearly the case; and constant use is made of this fact in inferring one component from another. In passing from one component to another of sensibly different speed, Laplace assumed that the amplitude is altered by a small quantity, proportional to the difference in their speeds. As will presently be seen, this agrees with Ferrel's work only to the first approximation.

Ferrel assumes that the change in the tidal coefficient due to a change of velocity of the disturbing body in right ascension, is not generally proportional to the amount of change in this velocity, as Laplace had assumed.¶

Let i_0 denote the speed per day, expressed in radians, of a component A_0; let i_e or i, the speed of another component A_e, to be compared with A_0; and so

$$i = i_0 + u_e \tag{270}$$

where u_e denotes the daily difference in speeds expressed in radians. Let the coefficients of the corresponding terms of the tide-producing potential be H_e and H_0 $(=1)$. Let the ratio A_e/A_0 be denoted by R_e; the question arises, how does R_e differ from H_e, because i_e is not exactly equal to i_0?

* Tidal Researches, §§ 19, 25, 56, 73, 182–196.
† United States Coast Survey Report, 1868.
‡ Ibid., 1875.
§ Ibid., 1878, p. 270; Tidal Researches, §§ 28, 29.
|| Cf. Airy, Tides and Waves, Art. 198; or see under Airy.
¶ United States Coast Survey Report, 1868.

By the equilibrium theory

$$\frac{R_e}{H_e} = 1. \tag{271}$$

But $\frac{R_e}{H_e}$ being a function ϕ of $i, = (i_o + u_e)$,

$$\phi\,(i) = \phi_o + u_e\phi_o' + \frac{u_e^2}{2}\phi_o'' + \quad . \quad . \quad . \quad , \tag{272}$$

where the accent denotes differentiation with respect to i_o. But $\phi_o = R_o/H_o = 1$, and $\phi_o', \frac{1}{2}\phi_o''$, are constants which Ferrel denotes by E, E';

$$\therefore R_e = H_e\,(1 + u_eE + u^2_eE'). \tag{273}$$

Similarly for the epoch,

$$\varepsilon_e = \varepsilon_0 + u_eG + u^2_eG'. \tag{274}$$

In case of the semidiurnal components, Ferrel's equations for determining E, E', $\delta\mu$ are, adding equivalents in the harmonic notation,*

$$S_2/M_2 = R_1 = (0{\cdot}4582 - 36{\cdot}2\,\delta\mu)\,(1 + 0{\cdot}4255\,E + 0{\cdot}181\,E'), \tag{275}$$

$$\mu_2/M_2 = R_2 = 0{\cdot}0240\,(1 - 0{\cdot}4255\,E + 0{\cdot}181\,E'), \tag{276}$$

$$K_2/M_2 = R_3 = (0{\cdot}1256 - 3{\cdot}2\,\delta\mu)\,(1 + 0{\cdot}4599\,E + 0{\cdot}212\,E'), \tag{277}$$

$$L_2/M_2 = R_4 = -\,0{\cdot}0286\,(1 + 0{\cdot}288\,E + 0{\cdot}052\,E'), \tag{278}$$

$$N_2/M_2 = R_5 = 0{\cdot}1922\,(1 - 0{\cdot}228\,E + 0{\cdot}052\,E'), \tag{279}$$

$$\left[\begin{array}{l}\text{lunar}\\ \text{nodal}\end{array}\right] \begin{array}{ll} R_6 = -0{\cdot}0359\,(1 - 0{\cdot}001\,E), & (280)\\ R_7 = 0{\cdot}0359\,(1 + 0{\cdot}461\,E + 0{\cdot}212\,E'). & (281)\end{array}$$

"Where the amplitudes of all the principal components are determined from observation, we get R_e by dividing A_e by A_0, and hence A_0 is thus eliminated . . . from the preceding equations. The first members being thus determined from observation, these equations, or a sufficient number of them for the purpose, can be used in determining the unknown constants in the case of nature, and the correction of the moon's mass. It is readily seen that in these equations, . . . the determination of $\delta\mu$ depends almost entirely upon the first and third, and that . . . the neglect of the terms depending upon E', unless they are large, can have no sensible effect upon the value of $\delta\mu$, and that the effect of neglecting them is thrown almost entirely upon the value of E. When, therefore, the principal object is to obtain the correction of the moon's mass, and a very accurate value of E is not desired, the terms in the equations depending upon E' may be neglected, and then the first and third equations are sufficient for the purpose. All, however, can be used and the most probable values obtained by the method of least squares."

The equations between the amplitudes of the diurnals for the determination of the constants A_0, E, E', and $\delta\mu$ are

$$K_1 = A_1 = (0{\cdot}5306 - 13{\cdot}1\,\delta\mu)\,(1 + 0{\cdot}230\,E + 0{\cdot}053\,E')\,A_0, \tag{282}$$

$$O_1 = A_2 = 0{\cdot}3813\,(1 - 0{\cdot}230\,E + 0{\cdot}053\,E')\,A_0, \tag{283}$$

$$P_1 = A_3 = (0{\cdot}1730 - 13{\cdot}6\,\delta\mu)\,(1 + 0{\cdot}196\,E + 0{\cdot}040\,E')\,A_0, \tag{284}$$

$$\left[\begin{array}{l}\text{lunar}\\ \text{nodal}\end{array}\right] \begin{array}{ll} A_4 = 0{\cdot}084\,(1 + 0{\cdot}231\,E + 0{\cdot}053\,E')\,A_0, & (285)\\ A_5 = 0{\cdot}070\,(1 - 0{\cdot}231\,E + 0{\cdot}053\,E')\,A_0. & (286)\end{array}$$

In §§ 197–228 of his Tidal Researches, Ferrel applies these formulæ to the tides at Liverpool, Portland, Fort Point (Cal.), and Kurrachee.

* Tidal Researches, pp. 91, 92.

In his "Discussion of the tides in Penobscot Bay," United States Coast Survey Report for 1878, he replaces the two lunar nodal components of the diurnal group by the lunar elliptic components Q_1 and J_1, and omits E'. These formulæ become

$$K_1 = (0{\cdot}5306 - 13{\cdot}1\ \delta\mu)\ (1 + 0{\cdot}230\ E)\ A_0, \tag{287}$$

$$O_1 = 0{\cdot}3813\ (1 - 0{\cdot}230\ E)\ A_0, \tag{288}$$

$$P_1 = (0{\cdot}1730 - 13{\cdot}6\ \delta\mu)\ (1 + 0{\cdot}196\ E)\ A_0, \tag{289}$$

$$J_1 = 0{\cdot}011\ (1 + 0{\cdot}458\ E)\ A_0, \tag{290}$$

$$Q_1 = 0{\cdot}052\ (1 - 0{\cdot}458\ E)\ A_0. \tag{291}$$

In a paper by Prof. William Harkness (q. v.), entitled "The solar parallax and its related constants,"* the author has put Ferrel's equations for the moon's mass, etc., into forms better adapted to computation.

136. *On inferring small components.*

Of course the solution of the equations in $\delta\mu$, E and G (E', G' being neglected) render it theoretically possible to infer the amplitudes and epochs of other small components which may be required in the representation of the tide. To illustrate, suppose we wish the amplitude and epoch of Q_1. From the equations in K_1, O_1, and P_1 the quantities $\delta\mu$, E, and A_0 are obtained. These values for E and A_0 being substituted in the equation for Q_1 give its theoretical amplitude. On page 448 of the United States Coast and Geodetic Survey Report for 1882, Ferrel thus finds Q_1 for Port Townsend and Astoria. On account of the large positive value of E, the formula

$$Q_1 = 0{\cdot}052\ (1 - 0{\cdot}458\ E)\ A_0$$

gives in each case a value for Q_1 much smaller than that obtained from harmonic analysis. In fact, Q_1 could have been inferred from O_1 or K_1 by means of its equilibrium ratio much closer than by Ferrel's process. Consequently his remark that his small inferred value of Q_1 is due to a certain shallow water component combining with Q_1 can hardly seem probable.

The epoch of a small diurnal component like Q_1 is inferred by putting

$$\varepsilon_0 = L = \tfrac{1}{2}\ (K_1{}^{\circ} + O_1{}^{\circ}) \tag{292}$$

$$G = \frac{K_1{}^{\circ} - O_1{}^{\circ}}{k_1 - o_1} = 0{\cdot}911\ (K_1{}^{\circ} - O_1{}^{\circ}) \text{ hours} \tag{293}$$

$$= 0{\cdot}038\ (K_1{}^{\circ} - O_1{}^{\circ}) \text{ days.} \tag{294}$$

Then

$$Q_1{}^{\circ} = L - 26{\cdot}25\ G \tag{295}$$

where G is expressed in days. On the next page of the Report (l. c.) Ferrel thus determines $Q_1{}^{\circ}$ for Port Townsend, Astoria, and San Diego. The agreement with the analysis is very satisfactory.

Similarly he makes use of the equations in S_2, K_2, and N_2, determining $\delta\mu$ and E from the semidiurnal group, but with the modification noted below.†

It is readily seen from an inspection of these equations that they can be satisfied only very imperfectly for Pulpit Cove, within any determined values of $\delta\mu$ and E, and that they can be much better satisfied by multiplying the first members of the equations by an unknown constant. This constant is introduced upon the hypothesis that the tidal components are diminished by the effect of friction which is as a higher power than the first power of the velocity, as I have at various times explained. Upon this hypothesis large tides are diminished by friction more than small ones in proportion to their amplitudes, and hence where there is one large component, as the mean lunar, and a number of much smaller ones, since the amplitudes of the latter are obtained by analysis from the differences between the larger and smaller resultant tides, the smaller components are diminished more than the larger ones in proportion to their magnitudes, unless friction is as the first power of the velocity. If we take the first, third, and fifth of the preceding equations for Pulpit Cove, and introduce a constant factor c, we have—

$$0{\cdot}1574c = (0{\cdot}4582 - 36{\cdot}2\ \delta\mu)\ (1 + 0{\cdot}4255\ E) \tag{296}$$

$$0{\cdot}0469c = (0{\cdot}1256 - 3{\cdot}2\ \delta\mu)\ (1 + 0{\cdot}4599\ E) \tag{297}$$

$$0{\cdot}2082c = 0{\cdot}1922\ (1 - 0{\cdot}228\ E) \tag{298}$$

*Washington Observations for 1885, App. III.

† United States Coast and Geodetic Survey Report, 1878, p. 297.

The solution of these equations gives—

$$\delta\mu = 0{\cdot}00263 \quad E = -1{\cdot}164 \quad o = 1{\cdot}166 \tag{299}$$

The solution of all the equations by the method of least squares would give values for these constants differing but little from those above on account of the smallness of the amplitudes in the neglected equations, which gives them little weight. The value of $\delta\mu$ above gives for the moon's mass $\mu = \frac{1}{66}$, which is much too large, as is usually the case where the relations differ much from those of the equilibrium theory. The equations for Liverpool give $\mu = \frac{1}{70}$, and for Kurrachee, where the relations approximate more nearly to those of the equilibrium theory, $\mu = \frac{1}{78{\cdot}6}$, which is perhaps not very much in error.

In regard to the epoch, we have

$$L = M_2{}^{\circ}, \tag{300}$$

and G is determined from the values of $S_2{}^{\circ}$, $N_2{}^{\circ}$, and $K_2{}^{\circ}$.

In regard to the effects of shallow-water components upon such quantities as these and $\delta\mu$, he says:*

From the preceding investigation of the shallow-water tides, I think that we can now see clearly why it is that satisfactory and consistent values of the moon's mass have not in general been obtained from the relations of the semidiurnal tides; for these relations are disturbed by the various shallow-water components, which do not enter into the theory of deep-water tides, which has been used in determining the moon's mass. The perfection of the tidal theory, so as to represent accurately the results of observation at all tide stations, and give a correct mass of the moon, depends now mainly upon the study of the shallow-water terms.

With regard to the determination of the moon's mass, from the results so far as obtained the relations of the diurnal tides promise better success in the future than those of the semidiurnal tides. The diurnal tides are not affected by so many of the shallow-water components, and it is probable that these can be determined from the analysis of the observations, since there are two comparatively quite large components with periods differing from those of any others, and hence can be determined by analysis of the observations; and then from the theoretical relations given in Schedule III the others can be, at least approximately, determined, and the components of deep-water tides which they affect can be corrected for their effect. The relations of these corrected results, obtained from the analysis of the observations, should then agree with the theoretical relatives, and give a correct mass of the moon.

137. Chapter IV of the Tidal Researches treats of tides in canals. He naturally goes over much of the ground previously gone over by Airy. As already stated, Ferrel assumed a more general law of fluid friction so that many of Airy's results follow as special cases of Ferrel's. The subjects here considered are canals extending east and west along the equator or parallels of latitude; canals coinciding with a meridian; and shallow canals extending from the sea inland.

Under east and west canals Ferrel notices that:

In the case of friction, . . . the oscillations of each separate component cannot in general vanish, and give rise to a complete nodal point.

There cannot . . . be in general any place in the canal where the vertical oscillations completely vanish.

We might . . . have two canals near each other, extending east and west, of the same length and depth, such as to satisfy (206)† approximately for either the moon or sun, or both, if the canals were not very long and shallow, and if we should suppose the lunar forces to act upon the one and the solar forces upon the other, the lunar and solar tides in the two canals would not only not be at all in proportion to the forces, which is the effect of a large value of E, but also the epochs might be very different in the two, upon which the value of G depends. If we therefore suppose the lunar and solar forces to act upon the same canal, we have the two tides coexisting without interference, at least when friction is as the first power of the velocity, but the epochs of the two differing, that is, the times of high water occurring at different intervals from the times of transit of the moon and sun over some assumed meridian, the high waters of the two do not coincide generally at the times of the syzygies of the moon and sun, and cause the greatest tides, but some time before or after. . . . It is evident from a mere inspection of the expressions, that it depends entirely upon circumstances whether E and G are positive or negative, that is, whether the lunar or the solar tide is the greater in proportion to the forces, and whether the maximum of the resultant tide happens before or after the syzygies. This will be also shown in a subsequent part of the chapter by means of actual computations in various assumed cases.

In § 145 Ferrel gives a table for various assumed conditions, or rather constants, relating to tides in canals, such as the length, depth, and friction constants. From these he computes the constants A_0, L_0, E, G, F, F'. From the computed values he notices that friction may increase the amplitude of the tide; the amplitudes for different assumed conditions may vary widely; it may be high water at one end of a canal while it is almost low water at the other end; the values

* United States Coast and Geodetic Survey Report, 1878, p. 299.

† I. e., such length and depth as will give very large vertical oscillations.

of E, G, and F' may be either positive or negative, and their values may vary greatly for the different assumed conditions.

The equations belonging to a canal coinciding with the equator apply to shallow canals extending from the sea inward, "by neglecting the forces in these equations, and regarding ϖ as expressing distance in terms of the earth's radius, instead of longitude."

This renders the fundamental equations of motion very simple, especially if friction be also neglected.

Ferrel finds, in § 143—

That the equation of continuity in a shallow canal cannot be satisfied without a change of mean level, and that the periodic vertical oscillations are about this disturbed mean level, instead of the undisturbed in the case of no oscillations. This is a new and important result, and shows that where the water is shallow the true undisturbed level cannot be obtained from any number of tidal observations taken at equal intervals through all parts of the phase of the tide, but that to the level thus obtained a correction must be applied . . . to reduce it to the true level, which . . . is in some places positive and in others negative.*

In §§ 248–253, Ferrel notes instances taken from nature to which these statements seem to apply.

138. Chapter V is devoted to the theory of tides upon an ellipsoid of revolution, and is largely devoted to Laplace's solutions of tidal equations.

In case of the diurnal tide everywhere vanishing if the depth of the water were uniform, Ferrel contends that although Laplace's result is correct, his manner of solution is incomplete in that it fails to show that the problem remains indeterminate until the proper assumption is made regarding the form of a', the excess of the height of tide over the equilibrium height.

Ferrel first published his views on this subject in Gould's Astronomical Journal, Vol. IV (1856).

In regard to the indeterminate coefficient of x^4, i. e. A_4, he contends that, since ever so little friction must destroy the initial conditions, thus making the oscillations depend entirely upon the disturbing force and vanish with it, the value of A_4 to be used must be zero. Finally, for various depths assumed by Laplace, Ferrel computes anew the corresponding ranges of tide, and finds them, he believes, much more conformable to nature. For references to the late papers of Ferrel and others on this question, see footnote under Laplace.

139. Chapter VI is devoted to the discussion of high and low waters by the inequality method already referred to. The next chapter gives various comparisons between theory and observation, the most of which have likewise been referred to.

Chapter VIII gives some account of the tides of the North Atlantic Ocean (whose size he attributes to the fact that a canal extending from Europe to America, thus closed at both ends, and having the depth of the Atlantic, has for its free period approximately a half lunar day); † the tides of the Gulf of Mexico, of the Island of Tahiti, and of Lake Michigan; §§ 248–253, already referred to, are devoted to observed variations in sea level from place to place. The chapter closes with an account of different forms of tidal curves.

140. Chapter IX is upon the tidal retardation of the earth's rotation. This brings the author back to his first scientific paper, and which was published in Gould's Astronomical Journal, Volume III (1853), pages 138–141.

In a note entitled "Questions of priority," published in the Journal, Volume IX (1890), page 189, and quoted below, Ferrel again makes reference to this subject.

In this 1853 paper he makes the first numerical estimate of tidal retardation in the earth's axial rotation based upon mathematical principles, although Kant had in 1754 made a rough estimate of it, and it seems that J. R. Mayer ‡ and others had published something upon the subject. At the time of writing this paper, Ferrel supposed that the then observed secular acceleration of the moon was fully accounted for by Laplace's theoretical expression for the same. He was, therefore, led to believe that the tidal retardation was counteracted by a gradual cooling and shrinking of the earth. He points out that if earth and moon are similarly constituted, the retardation in the moon's axial rotation due to the earth must be to the moon's effect upon the earth's rotation as the square of the mass of the earth is to the square of the mass of the moon.

Bertrand was the first to show, about 1866, that the real motion of the moon in her orbit

* Cf. Tidal Researches, § 183.

† See § 31.

‡ See Phil. Mag., Vol. 25 (1863), p. 403.

(not merely the motion as estimated by the period of the earth's rotation) was affected because of the tides upon the earth. This gives a retardation more than one-third as great as the apparent acceleration. In his Tidal Researches Ferrel puts these two effects together and finds, from the value of the moon's secular acceleration known by other means, that the tidal displacement due to friction ought to be about 2°. As noted below, he had previously (in 1864) shown tidal friction to be the probable cause of the small outstanding acceleration of the moon.

His note entitled "Questions of priority" is as follows:

It is well known that there is a discrepance between the times of the computed and observed phenomena of certain ancient eclipses, which indicates that there has been a retardation of the earth's rotation. A plausible explanation of such a retardation is, that it is due to the effect of friction upon the tidal wave. The first suggestion of this explanation is usually attributed to Delaunay. In Thomson and Tait's Natural Philosophy it is stated that "About the beginning of 1866 Delaunay suggested that the true explanation of the discrepance might be a retardation of the earth's rotation by tidal friction." Delaunay's note on this subject was communicated to the Academy at Paris on December 11, 1865 (*Comptes Rendus*, Vol. CI, p. 1023). My "Note on the Influence of the Tides in Causing an Apparent Acceleration of the Moon's Mean Motion" was read before the American Academy of Arts and Sciences on December 13, 1864 (*Proc.* Vol. VI, pp. 379–383). In this note it was shown that upon the hypothesis of only a very moderate displacement of the vertex of the tidal wave by friction, the resulting amount of retardation of the earth's rotation would furnish an explanation of the discrepance between the computation and observation of the ancient eclipses.

The writer also claims that the first suggestion that the cause of the exact equality between the time of the moon's rotation in its orbit and on its axis is due to the effect of the attracting forces upon the lunar tides, was given in his paper "On the Effect of the Sun and Moon upon the Rotary Motion of the Earth" (*Astr. Jour.*, 1853, III, pp. 138–142).*

G. F. Becker, Am. Jour. Sci., Vol. 5 (1898), p. 108, states that Laplace, in the 1824 edition of the Systém du Monde, refers the equality of the moon's periods of rotation and revolution to tidal action caused by the earth's attraction in the still fluid moon; and that Kant considered the tidal retardation in the moon's axial rotation as well as that in the earth's.

In a paper published in the Astronomical Journal, Vol. V (1858), pp. 97–100, ~~he~~ Ferrel examines the deflected course taken by a body moving upon or near the earth's surface because of the earth's rotation. It has an important bearing upon the general circulation of the ocean and atmosphere. Among other things he establishes that a moving body in the northern hemisphere is always deflected to the right, and in the southern to the left. The radius of curvature of the path is always inversely as the sine of the latitude. For a small range of motion the path is circular, but for a large range it is not; the path is, however, self-returning.

On pages 113, 114 of the same volume is a note supplementary to the preceding paper. He remarks that the deductions from theory have been verified by some delicate experiments of Foucault.†

Besides devising numerous methods for the prediction of tides,‡ Ferrel in 1880 invented a tide-predicting machine. His published account of the machine and its use is found upon pp. 253–272, of the Survey Report for 1883. It was designed with special reference to the prediction of high and low waters, thereby differing from Thomson's machine which simply gives the continuous curve. The theory of the machine is also given in §§ 58 and 60 of Part III.

His paper entitled "Report of meteorological effects on tides," found in the Survey Report for 1871, refers to observations at Boston.

141. *Sir William Thomson* (*Lord Kelvin*).

Thomson seems to have been led to the study of tides through his work upon certain physical problems which involve their consideration. Among these problems is that of the rigidity of the earth, which he considers in the Philosophical Transactions of the Royal Society for the year

* Cf. D. Vaughan, "Secular variation in lunar and terrestrial motion from the influence of tidal action," B. A. A. S. Report, 1857. Thomson, Phil. Mag., Vol. 31 (1886), p. 533, says that Ferrel was the first to evaluate tidal retardation. Abbott, Elementary Theory of Tides, pp. 22 et seq. Kelvin, Popular Lectures and Addresses, Vol. II (1894), pp. 10–44, 64–72. Ball, Time and Tide (1895), pp. 58–68. See under Thomson and under Garbett.

† See Am. Jour. of Science and Arts, Vol. XV, p. 263, and Vol. XIX, p. 141.

‡ United States Coast Survey Report, 1868, pp. 87–95; 1875, pp. 215–221; United States Coast and Geodetic Survey Report, 1878, pp. 299–304.

1863.* He finds that the earth's mass must have an effective rigidity at least as great as that of steel, otherwise the effect of its yielding would have been noticeable upon the amount of the precession or the nutation. Moreover, the earth must be for the most part solid and not fluid as had generally been maintained; for, a thin crust would have to be of fabulous rigidity to prevent tides in the molten matter within. The effect of any elastic yielding is, of necessity, to diminish the range of tide. Calling this range unity for an ocean covering a rigid sphere, the elastic yielding of the nucleus would cause the range to become $\frac{2}{5}$ or $\frac{3}{5}$ according as the rigidity of the nucleus is assumed to be that of steel or of glass.

Supposing the long-period tides to nearly conform to the equilibrium theory, Thomson and subsequently Darwin were led to the careful study of such oscillations. A discussion by the latter of tides observed in European and Indian ports is given in Thomson and Tait's Natural Philosophy.† Poincaré notices that the results of this discussion would be in error by a $\frac{3}{22}$ part, for a sea covering the entire earth, because the attraction of the disturbed water is there disregarded.‡ Darwin concludes that because of the water's inertia these tides (the small nineteen-yearly one excepted) do not conform to the equilibrium theory sufficiently close for making valid the earth's rigidity derived from them.§

Thomson's paper upon the tidal retardation of the earth's rotation appears in Volume 31 (1866) of the Philosophical Magazine; also in Thomson and Tait's Natural Philosophy.||

In Vol. II (1894) of Thomson's Popular Lectures and Addresses entitled Geology and General Physics will be found a popular treatment of the tidal retardation of the earth's rotation, given at the close of the address entitled "On geological time." In this volume are papers which treat of the internal constitution and the rigidity of the earth, viz., "Review of evidence regarding the physical condition of the earth" and "The internal condition of the earth; as to temperature, fluidity, and rigidity." Another paper is entitled "Polar ice-caps and their influence in changing sea levels."

In about 1867 Thomson devised the harmonic analysis for tidal observations. In perfecting it he has been aided by J. C. Adams, E. Roberts, and more particularly by G. H. Darwin. A historical sketch of this subject is given beyond.¶

In about 1872 he invented the tide-predicting machine, although the first machine for actual work was not constructed until about 1876. For a brief account of tide-predicting machines and references to writings connected therewith, see § 57, Part III, of this manual. It is hardly necessary to say that this invention has proved to be thoroughly practical.

The Thomson harmonic analyzer was invented in about 1878. Some account of this machine, along with references pertaining thereto, is given in § 56, Part II.

Among the statical problems given in Thomson and Tait's Natural Philosophy are the equilibrium theory of tides, and the effect of lunar and solar attraction on apparent terrestrial gravity.

If a sphere be but partially covered with water, its surface of equilibrium, even if the sphere turn upon its axis very slowly, cannot generally coincide with that of a sphere entirely covered with water to the same depth. The surface (or portions of surface) will, however, be *parallel* at any given instant to the instantaneous surface of the covered sphere. Upon this fact rests Thomson's "corrected equilibrium theory." Bernoulli treated the case of a small inclosed body of water upon this assumption, but Thomson was the first to suggest its application to the ocean. The effect of the land is to modify the amplitudes and epochs in the expressions for the lunar and solar tides. Since the equilibrium theory is not concerned with depths, these modifications depend upon surface integrals or, rather, quadratures.** The work of making these quadratures

* "On the rigidity of the earth," pp. 573–582. "Dynamical problems regarding elastic spheroidal shells and spheroids of incompressible liquid," pp. 583–616. Cf. Proc. Roy. Soc., Vol. 12 (1862–63), pp. 103, 104; Phil. Mag., Vol. 25 (1863), pp. 149–151.

† Ed. 1883, §§ 847, 848.

‡ Journal de Mathématiques pures et appliquées, Vol. 2 (1896), p. 80.

§ Proc. Roy. Soc., Vol. 41 (1886), pp. 339, 342. See B. A. A. S. Report 1886, pp. 56–58; also under Laplace.

|| Section 830.

¶ See B. A. A. S. Reports, 1868, I, pp. 489–510; 1870, I, pp. 120–151; 1871, I, pp. 201–207; 1872, I, pp. 355–395; 1876, I, pp. 275–307.

** Natural Philosophy, Ed. 1867, or 1883, § 808.

has been performed by Darwin for a long-period oscillation,* and by H. H. Turner for a diurnal and a semidiurnal oscillation.†

It was the intention of Thomson to give the dynamical treatment of tides in a subsequent volume of the Natural Philosophy. The continuation of this work beyond the first volume has, however, been abandoned.

It seems that Darwin's restatement of Laplace's theory is in accordance with suggestions by Thomson.‡ Thomson has worked out additional solutions for Laplace's tidal equation § and defended Laplace's solution in the case of a semidiurnal tide when the ocean is of uniform depth.||

142. *Prof. George H. Darwin.*

In the Philosophical Transactions for 1863 Thomson gives, with important physical deductions, an independent solution of a problem previously solved by Lamé, viz., the state of strain of an elastic sphere under given stresses. In the Transactions for 1879 ¶ Darwin treats the case of a viscous sphere or spheroid instead of an elastic sphere. He finds that the equations of flow in an incompressible viscous fluid are analogous to those of strain for an incompressible solid. He therefore finds it possible, in a measure, to adapt Thomson's work upon bodily tides in the elastic sphere to his case of a viscous sphere. His results regarding the effective rigidity of the earth are in the main confirmatory of Thomson's. He finds a remarkably simple rule for making a comparison between tides in a fluid sphere and in a viscous sphere, also the effect of the internal yielding on oceanic tides. **

Results from Darwin's paper on the precession of a viscous spheroid are subsequently adapted by him to the making of a numerical estimate of the retardation of the earth's axial rotation. ††

In the Proceedings of the Royal Society for 1879 he gives a paper entitled "The determination of the secular effects of tidal friction by a graphical method," where first appear his well-known diagrams illustrating the evolution of the earth-moon system. ‡‡

Outline of Darwin's theory of tidal evolution.—Suppose the earth to be in liquid or semiliquid condition and to be rotating rapidly upon its axis, the period of rotation being from two to four hours. The centrifugal force may be sufficient of itself to cause the matter now constituting the moon to become detached from the earth, whether as one body or as a chain of meteorites constituting a ring, is immaterial, provided the latter soon come together and make up the moon. If the centrifugal force be not sufficient for the accomplishment of this, it may happen that the length of a half day approximately coincide with the period of free bodily oscillation of the earth, which is probably a little less than two hours. The periodic tidal forces from the sun will cause the successive tides to rise higher and higher, until finally a portion of matter will be detached. At first earth and moon revolve nearly as a single rigid body about their common center of gravity; the earth-day and the moon-day are each equal to their "month."

At the time here considered, the energy of the earth-moon system is a maximum; for as yet no energy has been dissipated by tidal friction due to tidal currents, which each body is to set up in the other just as soon as their periods of axial rotation differ from their "month" or period of revolution about their common center of gravity.

It is a principle of dynamics that the sum of the moments of momentum of all rotations and revolutions of a system not influenced by extraneous forces is constant, however the distances, velocities, and the amount of energy may vary. For simplicity of conception, the rotation of the moon upon her axis can at first be ignored or lost sight of. The moon produces tidal currents in the earth, thereby slowing down the earth's axial rotation and lengthening the earth-day. By the

---

* Ibid., Ed. 1883, §§ 810, 848.

† Proc. Roy. Soc., Vol. 40, 1886, pp. 303–315.

‡ Phil. Mag., Vol. 50 (1879), pp. 388–402.

§ Phil. Mag., Vol. 50 (1875), pp. 279–284, 388–402.

|| See under Laplace.

¶ "On the bodily tides of viscous and semi-elastic spheroids, and on the ocean tides upon a yielding nucleus," pp. 1–35. "On the precession of a viscous spheroid, and on the remote history of the earth," pp. 447–538. "Problems connected with the tides of a viscous spheroid," pp. 539–593. See Proc. Roy. Soc., Vols. 27 (1878), pp. 419–424; 28 (1878–79), pp. 194–199.

** Phil. Trans., 1879, pp. 15, 28. B. A. A. S. Report, 1882, pp. 472–475. †† Thomson and Tait, Nat. Phil., App. [G. a].

‡‡ Also found in Thomson and Tait, Nat. Phil., App. [G. b], and in Enc. Brit., Art. "Tides."

principle just referred to this necessitates a retreating of the moon and so, by Kepler's third law, an increase in the length of the "month." The length of the day will go on increasing until day and month shall become equal, and computation shows this day or month to be about two of our present months in length. The energy of the system will then be a minimum, for no tides or tidal friction can then exist. The earth-moon system will then be in stable equilibrium, and not in unstable equilibrium as it was when it possessed a maximum amount of energy. The energy curve of the diagrams already referred to, has orbital momenta as abscissæ and axial momenta as ordinates. It has one real maximum and one real minimum corresponding to the two critical periods already described.

There is one stage in the evolution when the month has a maximum number of days. For the earth-moon system, as here considered, this number is 27, or about the present number. In the paper on the precession of the viscous spheroid (Phil. Trans. 1879), where account is taken of solar tidal friction and the obliquity of the ecliptic, this number is found to be 29. Consequently we have passed through the stage of the greatest number of days in the month, although the month now is really longer than ever before, owing to the increase in the length of the day.

The tides in the moon, due to the earth's attraction, have already caused her day to be one month in length, and tides in the earth due to the sun must finally cause the earth to revolve upon its axis once in a year, whatever length the year may then have.

A popular treatment of tidal evolution is given by *Ball* in a book entitled Time and Tide.

Darwin's work upon the harmonic analysis has been largely in the nature of perfecting methods for its application. He drew up the reports of the Tidal Committee, which appear in the British Association Reports for the years 1883, 1884, 1885, and 1886.

The report for 1883 is intended "to systematize the exposition of the theory of harmonic analysis, to complete the methods of reduction, and to explain the whole process."

The report for 1886 contains, among other things, a method devised by Darwin for analyzing a short series of hourly ordinates, and a method of prediction; these were designed for the Admiralty Manual, where they may also be found.

His more extended method of tidal prediction appears in the Philosophical Transactions for 1891.

He devised a method for the harmonic analysis of high and low waters, which is published in the Proceedings of the Royal Society, Volume 48 (1890).

A concise treatment of the subject of tides by Darwin is contained in the Encyclopædia Britannica, ninth edition.

The brothers George and Horace Darwin have made a series of interesting experiments for a committee appointed by the British Association on the measurements of the lunar disturbance of gravity, for the purpose of throwing some light on the elastic yielding of the earth. These are described in the Association Reports for 1881 and 1882; they are also briefly mentioned in Thomson and Tait's Natural Philosophy, § 818′.

Historical sketch of the harmonic analysis.

143. In the harmonic treatment it is supposed, as indicated by the name, that the tide at any given place consists of simple harmonic oscillations, whose periods and amplitudes remain constant—at least for a considerable time. It now seems almost as natural to adopt a series of periodic terms for the expression of the tide as for the "equations" of the motions of the sun and moon. The reasons why tidal workers before Thomson (in about 1867) did not have recourse to such a series seem to be, 1st, the fact that upon the coasts of Europe, where the tides important to navigation were first carefully studied, the tide wave is almost wholly semidiurnal in its character; that is, the two high waters or the two low waters of a day are almost equal in every respect, and so the phenomenon of rise and fall is comparatively simple; 2d, the custom of observing only the high and low waters (in many cases only the former) instead of the entire tidal curve; and, 3d, the idea that tidal work meant rough work, and so did not necessitate an elaborate scheme which, upon its face, seemed to involve more labor than did the less systematic methods.

Accordingly the tide was assumed to be composed of two simple waves, one due to the moon and one to the sun. Each had a variable amplitude and period due to the body's varying parallax and declination. The resultant tide was given (as now in the British Tide Tables) by means of a series of tables, based in part upon observations at the port, which generally had the hour of the moon's transit as one of their arguments. Predictions obtained by means of such tables usually made no distinction between the two tides of a day although we find the diurnal inequality described by Colepresse and Sturmy in the Philosophical Transactions for 1668, and Laplace had pointed out that it was due to oscillations of approximately daily periods. But at places where the diurnal inequality is large, the want of a systematic procedure became strongly felt. For, the greater the number of important inequalities in the tide, the greater the difficulty in disentangling them; and, moreover, a long series of observations becomes necessary.

The foundations of the harmonic analysis were laid by Laplace. For, he enunciated the principle of forced oscillations; he introduced tidal bodies having uniform motions; he showed how to develop the tide-producing potential into a series of periodic terms, and pointed out the more important harmonic constituents of the astronomical tide; he developed the method of least squares sufficiently far for making it applicable to the determination of the coefficients of a sine-and-cosine function of an angle and its harmonics. But he did not attempt an analysis of equidistant ordinates based upon this knowledge, nor did he completely develop the tide-producing potential.

Dr. Thomas Young suggested the importance of observing and analyzing the entire tidal curve, rather than the high and low waters merely.

Airy showed that in shallow water the difference between the duration of fall and of rise is due to the presence of an oscillation having half the period of the tide wave. Moreover, he applied an harmonic analysis to the tide wave, thus determining, from day to day, the fundamental oscillation and its numerous harmonics. His method made use of the entire curve, and not the points of maxima and minima merely.

144. In the year 1867 the British Association, upon the motion of Sir William Thomson, appointed a committee for the purpose of promoting the extension, improvement, and harmonic analysis of tidal observations. Thomson's statement to the other members of the committee, with some corrections and additions, is given in the British Association Report for 1868, and from this the following, including footnotes, is taken:

The chief, it may be almost said the only, practical conclusion deducible from, or at least hitherto deduced from, the dynamical theory is, that the height of the water at any place may be expressed as the sum of a certain number of simple harmonic functions * of the time, of which the periods are known, being the periods of certain components of the sun's and moon's motions.† Any such harmonic term will be called a tidal constituent, or sometimes, for brevity, a tide. The expression for it in ordinary analytical notation is $A \cos nt + B \sin nt$; or $R \cos (nt - \varepsilon)$, if $A = R \cos \varepsilon$, and $B = R \sin \varepsilon$; where t denotes time measured in any unit from any era, n the corresponding angular velocity (a quantity such that $\frac{2\pi}{n}$ is the period of the function), R and ε the amplitude and the epoch, and A and B coefficients immediately determined from observation by the proper harmonic analysis (which consists virtually in the method of least squares applied to deduce the most probable values of these coefficients from the observations).

The chief tidal constituents in most localities, indeed in all localities where the tides are comparatively well known, are those whose periods are twelve mean lunar hours, and twelve mean solar hours respectively. Those which probably stand next in importance are the tides whose periods are approximately twenty-four hours. The former are called the lunar semidiurnal tide, and solar semidiurnal tide; the latter, the lunar diurnal tide and the solar diurnal tide.‡ There are, besides, the lunar fortnightly tide and the solar semiannual tide.§ The diurnal and the semidiurnal tides have inequalities depending on the eccentricity of the moon's orbit round the earth, and of the earth's round the sun, and the semidiurnal have inequalities depending on the varying declinations of the two bodies. Each such inequality of any one of the chief tides may be regarded as a smaller superimposed tide of period approximately equal; producing, with the chief tide, a compound effect which corresponds precisely to the discord of two simple harmonic notes in music approximately in unison with one another. These constituents may be

* See Thomson and Tait's 'Natural Philosophy', §§ 53, 54.

† See Laplace, 'Mécanique Céleste', liv. iv. § 16. Airy's 'Tides and Waves', § 585.

‡ See Airy's 'Tides and Waves', §§ 46, 49; or Thomson and Tait's 'Natural Philosophy', § 808.

§ See Airy's 'Tides and Waves', § 45, or Thomson and 'Tait's Natural Philosophy', § 808.

called for brevity elliptic and declinational tides. But two of the solar elliptic diurnal tides thus indicated have the same period, being twenty-four mean solar hours. Thus we have in all twenty-three tidal constituents:

		Coefficients of t in arguments. Lunar.	Solar.
The lunar monthly and solar annual (elliptic).	2	$\sigma - [\varpi]$	η
The lunar fortnightly and solar semiannual (declinational).	2	2σ	2η
The lunar and solar diurnal (declinational).	4	γ $\gamma - 2\sigma$	γ $\gamma - 2\eta$
The lunar and solar semidiurnal.	2	$2(\gamma - \sigma)$	$2(\gamma - \eta)$
The lunar and solar elliptic diurnal.	7	$\gamma + \sigma - \varpi$ $\gamma - \sigma + \varpi$ $\gamma - \sigma - \varpi$ $\gamma - 3\sigma + \varpi$	$\gamma + \eta$ $\gamma - \eta$ $\gamma - \eta$ $\gamma - 3\eta$
The lunar and solar elliptic semidiurnal.	4	$2\gamma - \sigma - \varpi$ $2\gamma - 3\sigma + \varpi$	$2\gamma - \eta$ $2\gamma - 3\eta$
The lunar and solar declinational semidiurnal.	2	2γ	2γ

Here γ denotes the angular velocity of the earth's rotation, and σ, η, ϖ those of the moon's revolution round the earth, of the earth's round the sun, and of the progression of the moon's perigee. The motion of the first point of Aries, and of the earth's perihelion, are neglected. It is almost certain that the slow *variation* of the lunar declinational tides due to the retrogression of the nodes of the moon's orbit, may be dealt with with sufficient accuracy according to the equilibrium method; and the inequalities produced by the perturbations of the moon's motion are probably insensible. But each one of the twenty-three tides enumerated above is certainly sensible on our coasts. And there are besides, as Laplace has shown, very sensible tides depending on the fourth power of the moon's parallax,* the investigation of which must be included in the complete analysis now suggested, although for simplicity they have been left out of the preceding schedule. The amplitude and the epoch of each tidal constituent for any part of the sea is to be determined by observation, and cannot be determined except by observation. But it is to be remarked that the period of one of the lunar diurnal tides agrees with that of one of the solar diurnal tides, being twenty-four sidereal hours; and that the period of one of the semidiurnal lunar declinational tides agrees with that of one of the semidiurnal solar declinational tides, being twelve sidereal hours. Also that the angular velocities $\gamma - \sigma + \varpi$ and $\gamma - \sigma - \varpi$ are so nearly equal, that observations through several years must be combined to distinguish the two corresponding elliptic diurnal tides. Thus the whole number of constituents to be determined by one year's observation is twenty. The forty constants specifying these twenty constituents are probably each determinable, with considerable accuracy, from the data afforded in the course of a year by a good self-registering tide-gauge, or from accurate personal observations taken at equal short intervals of time, hourly for instance. Each lunar declinational tide varies from a minimum to a maximum, and back to a minimum, every nineteen years or thereabouts (the period of revolution of the line of nodes of the moon's orbit). Observations continued for nineteen years will give the amount of this variation with considerable accuracy, and from it the proportion of the effect due to the moon will be distinguished from that due to the sun. It is probable that thus a somewhat accurate evaluation of the moon's mass may be arrived at.

The methods of reduction hitherto adopted,† after the example set by Laplace and Lubbock, have consisted chiefly, or altogether, in averaging the heights and times of high water and low water in certain selected sets of groups. Laplace commenced in this way, as the only one for which observations made before his time were available. How strong the tendency is to pay attention chiefly or exclusively to the times and heights of high and low water, is indicated by the title printed at the top of the sheets used by the Admiralty to receive the automatic records of the tide-gauges; for instance, "Diagram, showing time of high and low water at Ramsgate, traced by the tide-gauge." One of the chief practical objects of tidal investigation is, of course, to predict the time and height of high water; but this object is much more easily and accurately attained by the harmonic reduction of observations not confined to high or low water. The best arrangement of observations is to make them at equi-distant intervals of time, and to observe simply the height of the water at the moment of observation irrespectively of the time of high or low water. This kind of observation will even be less laborious and less wasteful of time in practice than the system of waiting for high or low water, and estimating by a troublesome interpolation the time of high water, from observations made from ten minutes to ten minutes, for some time preceding it and following it. The most *complete* system of observation is, of course, that of the self-registering tide-gauge which gives the height of the water-level above a fixed mark every instant. But direct observation and measurement would probably be more *accurate* than the records of the most perfect tide-gauge likely to be realized.

In this paper the short-period tides treated are K, L, M, N, O, and S, each of which has a fictitious moon dividing time into component days and hours. As the heights are read upon the mean solar or S hours, a factor (afterwards called the augmenting factor) slightly greater than

[* The chief effect of this at any one station is a *ter-diurnal* lunar tide, or one whose period is eight lunar hours. A probable indication of this has been obtained from the Ramsgate tidal diagrams of 1864 . . . .]

† See 'Directions for reducing tidal observations,' by Staff-Commander Burdwood, London, 1865, published by the Admiralty; also Professor Haughton on the 'Solar and Lunar Diurnal Tides on the Coast of Ireland,' Transactions of the Royal Irish Academy for April, 1854.

unity has to be applied to all sums (or rather to the amplitudes or component amplitudes) except those in the S summations. The sums belonging to any component summation are assumed to be capable of being represented by the Fourier series,

$$\begin{aligned} A_0 + A_1 \cos nt + A_2 \cos 2\,nt + \quad . \quad . \quad . \quad + A_8 \cos 8\,nt \\ + B_1 \sin nt + B_2 \sin 2\,nt + \quad . \quad . \quad . \quad + B_8 \sin 8\,nt \end{aligned} \qquad (301)$$

The most probable values of these coefficients are found by Laplace's method of least squares.

The tabular forms and rules given by Mr. Archibald Smith, and published by the Admiralty, to be used for the harmonic reduction of the deviation of ship's compasses, have been adopted *mutatis mutandis*, and have proved very conveuient.

In regard to eliminating the effects of other components, he says:

The next step followed was to find corrections upon each summation for the influence of the tides determined by the other summations, these corrections, for a second approximation, being calculated on the supposition that the first approximate values of A_1, B_1, A_2, &c., already found, are correct.

Having called attention to the shallow-water components brought out from the year's analysis at Ramsgate, he says:

The shallow-water tides referred to above depend on the rise and fall of the tide, amounting to some sensible part of the whole depth of the water, or, which comes to the same, the horizontal velocity of the water being sensible in comparison with the velocity of propagation of a long wave, through some considerable portion of the sea which sensibly influences the tides at the point of observation. Helmholtz's explanation of compound sounds, according to which two sounds, each a simple harmonic, having mt, nt for their arguments, give rise, if loud enough, to sounds having for their arguments $(m + n)\,t$, $(m - n)\,t$, suggests that the compound action of the solar and lunar semidiurnal tides, must give rise to shallow-water tides whose arguments are $2\,(\sigma - \eta)\,t$ and $2\,(2\gamma - \eta - \sigma)\,t$. It is intended with the least possible delay to perform averagings with a view to determine these tides. The great influence of the British Channel, and the large extent of it through which the shallow-water condition specified above is fulfilled, makes it probable that the new tidal constituents now anticipated will be found sensible.

In a supplementary report E. Roberts determines, simultaneously by successive approximations the coefficients of five long-period tides, viz.: monthly,* fortnightly (declinational), fortnightly (synodical), annual, and semiannual.

In the next report of the Tidal Committee, published in the British Association Report for 1870, the additional components λ, μ, and ν are included in the schedule. A test is made of the harmonic method upon the Karachi tide, which has large diurnal components.

In the report for the committee, drawn up by Roberts and found in the British Association Report for 1871, the components J, Q, R, and T have been added to those already mentioned.

In the report found in the British Association Report for 1872, certain tides have the symbols 2SM, MS, 3MS, 3SM assigned them. A brief development of the tide potential is given.

In the report of the Tidal Committee for the year 1876, drawn up by Thomson, are tables showing the relative magnitudes of the components according to the equilibrium theory. These are obtained by developing the tide-producing potential of the sun and moon into a series of sine or cosine terms whose arguments increase uniformly with the time.

In this connection it should be noted that Ferrel gives, in the Coast Survey Report for 1868, a development of this potential, not into simple harmonic terms, but into a series of terms suitable for representing the inequalities in the high or low waters. In his Tidal Researches (1874) a harmonic development is also given. He naturally introduces lunar nodal components to account for the varying obliquity of the lunar orbit to the plane of the equator. Thomson dispenses with these by making the theoretical coefficients and epochs slightly variable, in accordance with the longitude of the moon's node. This is justifiable because tidal components whose speeds are equal, or very nearly so, must preserve, very nearly, their (equilibrium) theoretical relations to each other, as the works of Laplace and Airy go to establish. In fact, the case would be the same for any reasonable law of fluid friction.

In the Coast and Geodetic Survey Report for 1878, Ferrel gives some description of the harmonic analysis in general. Here he supplements the work on harmonic analysis found in his Tidal Researches by giving a number of schedules for the shallow-water components. These

* I. e. monthly (elliptic) not monthly (declinational) as implied in this report; pointed out by Roberts as stated in B. A. A. S. Report 1870, I, p. 121.

show how the amplitude, speed, argument, and epoch of a given component are related to like quantities of other components. He writes out elimination formulæ for a series a year in length. He gives formulæ and tables for the effect of the lunar nodal components upon the amplitudes and epochs of the principal components having almost the same speeds.

A report of a tidal committee consisting of Profs. G. H. Darwin and J. C. Adams, drawn up by the former and published in the British Association Report for 1883, gives a complete working manual of the system. The tide-producing potential is developed into a series of cosine terms as in §§ 38–44, Part II. The amplitudes and epochs as obtained from the analysis are to be so treated as to make the results from different years (at the same station) comparable with one another. Tables for this purpose accompany the report. Baird's Manual for Tidal Observations, 1886, gives more extensive tables, together with practical directions in carrying out the analysis. "Computation forms for the reduction of tidal observations" were prepared by Darwin and published in 1884 These indicate how the hourly heights are to be copied for the different kinds of summation. They also show how the partial or hourly sums are to be treated in the analysis.

145. In the year 1885 L. P. Shidy, of the Coast and Geodetic Survey, devised a set of stencils, or perforated sheets, fully described in Part II, which has done away with the copying process at this office.

In the Proceedings of the Royal Society, Volume 52 (1892), pages 345 et seq., Darwin describes a system of strips or scales for indicating how the various summations are to be made. He also mentions Dr. Börgen's tracing-paper sheets, which are substantially the stencils just referred to—the tracing paper being transparent answers the purpose of the holes cut through the stencil sheets.

The Thomson harmonic analyzer and other mechanical aids to analysis and prediction are given in Parts II and III.

At present the harmonic analysis is the working system in nearly all countries where tidal work is carried on. The published results are already extensive. We may here refer to some of the principal collections of harmonic constants:

Proceedings of the Royal Society of London, 1885, 1889. Reports of the Survey of India. Van der Stok's recent work entitled Wind and Weather, Currents, and Tidal Streams in the East Indian Archipelago. Tide Tables and Reports of the United States Coast and Geodetic Survey.

Note on Fourier series.—In connection with the problem of a vibrating string arbitrarily displaced, Daniel Bernoulli was led, in about 1753, to the belief that its solution could be expressed in the form of a trigonometric series. Euler, D'Alembert, and Lagrange made use of such series, but failed to determine the coefficients by means of definite integrals. This was done by Fourier in 1807 in a memoir presented to the French Academy. His treatment of trigonometric series, including the important fact that the function so represented need not be continuous, may be found in his Théorie analytique de la Chaleur which appeared in 1822, and which constitutes the first volume of his works as recently edited by Darboux. Although such series came into general use, the question of their convergence was not definitely settled until 1829, when Dirichlet pointed out the conditions under which convergence would be assured.*

---

*See Byerly, An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics (1893), pp. 61, 268, 269. Fourier, Œuvres, Vol. I, *Avant-Propos* and p. 208, note.

TREASURY DEPARTMENT

U. S. COAST AND GEODETIC SURVEY

W. W. DUFFIELD

SUPERINTENDENT

PHYSICAL HYDROGRAPHY

MANUAL OF TIDES

PART II

By ROLLIN A. HARRIS

APPENDIX No. 9—REPORT FOR 1897

WASHINGTON

GOVERNMENT PRINTING OFFICE

1898

TREASURY DEPARTMENT

U. S. COAST AND GEODETIC SURVEY

W. W. DUFFIELD

SUPERINTENDENT

PHYSICAL HYDROGRAPHY

MANUAL OF TIDES

PART II

By ROLLIN A. HARRIS

APPENDIX No. 9—REPORT FOR 1897

WASHINGTON
GOVERNMENT PRINTING OFFICE
1898

APPENDIX NO. 9—1897.

MANUAL OF TIDES.

Part II.

TIDAL OBSERVATION, EQUILIBRIUM THEORY, AND THE HARMONIC ANALYSIS.

By ROLLIN A. HARRIS.

Submitted for publication November 15, 1897.

PREFACE TO PART II.

The object of Part II is to give a sufficient amount of instruction for enabling a person to make reliable observations upon the tides and to reduce them by the harmonic analysis.

The system of analysis is that given by Darwin in his report to the British Association for the Advancement of Science at its Southport meeting (1883). The mathematical developments are chiefly those embraced in his report, and its notation has been generally followed.

The tables appended to this part have been so numbered as to form a continuation of those appended to Part III, Appendix No. 7, Report for 1894.

I have to acknowledge the assistance received from members of the Tidal Division, in the way of suggestions, computations, and the preparation of tables.

CONTENTS OF PART II.

CHAPTER I.

OBSERVATION OF TIDES.

CHAPTER II.

ASTRONOMY; TIDAL COMPONENTS SUGGESTED; ETC.

CHAPTER III.

THE TIDE-PRODUCING POTENTIAL.

CHAPTER IV.

DEVELOPMENT OF THE TIDE-PRODUCING POTENTIAL.

CHAPTER V.

THE HARMONIC ANALYSIS OF TIDAL OBSERVATIONS.

TABLES.

APPENDIX NO. 9—1897.

MANUAL OF TIDES—PART II. TIDAL OBSERVATION, EQUILIBRIUM THEORY, AND HARMONIC ANALYSIS.

By Rollin A. Harris.

CHAPTER I.

OBSERVATION OF TIDES.

1. *Selection of sites for tidal stations.*

The selection of a site for the observation of tides depends upon the object in view. If a knowledge of the tides at a given point is required, then there is little or no choice in the matter; if, on the other hand, and this is usually the case, a station is to be selected which shall tolerably well represent a considerable area, the following desiderata may govern the selection: Ready communication with the sea, deep water at low tides, shelter from storms, freedom from freshets, and non-proximity to the head of a bay or tidal river. At stations located along straits or upon islands, the tide is liable to be peculiar and so not representative for any considerable region.

Freshets may cause great irregularities in the tide, particularly in mean water level.* Near the head of a bay various shallow-water phenomena may occur;† also seiches in the lesser bays or coves.‡ Far up a tidal river the duration of rise may be several times less than the duration of fall; and the range of tide may be nearly obliterated.§ A wave cannot be propagated through a very narrow strait into a large body of water without losing its original form and altering its amplitude. For, there is no cause at work which will impart to the particles of water sufficient velocity for supplying or taking away the volume of water necessary to maintain the wave form unchanged.‖ Waves coming around an island from different directions generally produce some kind of interference in certain localities.¶

Where the water is deep, it is not likely that the type of tide change rapidly from point to point, although the shore may be cut up by bays, canals, and straits.**

The selection of stations for the prediction of tides may be governed by considerations like the above.

2. *Staff gauges or tide staves.*

A *tide staff* (*tide pole*) is a graduated rod, usually made of wood, but sometimes of metal. It is essential to any series of tidal observations, whether the tides are observed directly upon it or not. It should be carefully divided into feet and tenths, by aid of a steel tape or otherwise, and fixed in a truly vertical position to some object affording a steady and permanent support; for example, to a solid wall or pile. Its zero should be set below the lowest low water likely to occur, and its length should be more than sufficient for measuring the height of the highest tides known at the station.†† Where the beach slopes very gently, or where there are great changes in

* E. g., the lower Mississippi; the Delaware.

† E. g., at Providence, R. I., there are double-headed tides. At the heads of the arms Petit-Coudiac and Avon, Bay of Fundy, bores sometimes occur.

‡ E. g., Bristol, R. I.; Karwar and Beypore, India.

§ E. g., Rivers Adour, Dordogne, Garonne, Charente, Loire, and Seine, of France; also Cape Fear River, North Carolina.

‖ E. g., Strait of Gibraltar; the East River, New York.

¶ E. g., east of Ireland, and Nantucket Island, Massachusetts.

** E. g., southern portion of Alaska.

†† Sometimes the graduations increase downward, the zero being a fixed point above the surface of the water; e. g., Airy, Phil. Trans., 1842, p. 1.

level, as in rivers, several staves may be required for obtaining all readings. They should, of course, be so set, by means of levels, as to virtually constitute a single staff. If made of wood, the staff should be an inch or more in thickness and four or more in width—say not less than $\frac{1}{20}$ part of its length. To prevent the staff from becoming coated and hard to read, it is sometimes well to provide a ready means of removing the same from its support when not in actual use. This is easily done by leaving such small projections and definite marks upon the support as will enable one to readily return the staff to the same position whenever it is to be read. Metal staves, coated with porcelain, are more durable than those made of wood, and do not require removing from their supports when not in use.

Bench marks.—In order to detect any settling or rising in the support of a tide staff, and to enable a person to recover the plane of reference at any future time, several tidal bench marks of a permanent character and situated at various distances from the staff should be established. These marks usually consist of the bottoms of holes drilled into rocks; projections or markings upon rocks or walls; a certain portion of a step, door, or window sill: in the absence of such objects, a buried stone, or a deeply driven stake, pile, or iron pipe may be used. All bench marks should be carefully described (usually by aid of diagrams) for identification. The zero of the staff should be referred to the bench marks by sets of levels run from time to time while tidal observations are in progress. Ordinarily the levels should be reliable to about $\frac{1}{100}$ of a foot.

Tidal bench marks should, whenever feasible, be connected with transcontinental and other long lines of levels. It is particularly desirable to have all tidal stations which are located upon the same body of water accurately connected with one another in order that all heights may eventually be referred to a common datum.

Directions for observing.—The staff and bench marks established, the observer should read the height of the tide at even intervals of time. Readings at the exact hours throughout the twenty-four hours of each day are preferable for most purposes. The kind of time used is immaterial, provided that it be the same throughout the series of observations. It should always be specified in the record. In making such observations it is of importance to know the time to within about one minute. In high and low water observations readings should be made every ten minutes, say, for about forty minutes before to forty minutes after each of the four tides of the day. For tides of large range, less than forty minutes will suffice, while for tides of small range more time will be required.

In reading a height upon the staff, unless the surface of the water be perfectly smooth, note a point midway between the crest and trough of the waves. A glass tube, partially closed at the ends by notched corks, and held alongside the staff, will facilitate making these readings; or glass tubing may be fastened alongside the staff. In either case the opening in the lower end should not be too large. A small floating body will be found serviceable in marking the surface of the water, or a few enclosed drops of colored oil may be dropped into the tube.

3. *Box gauges.*

A box gauge consists of a long vertical box inclosing a float which rises and falls with the tide. By this arrangement observations may be made when the sea is comparatively rough. The bottom of this box may be pointed or funnel-shaped or, for ease of construction, simply slanted, with a small opening at the lowest part, in order to prevent the accumulation of mud or sand. Besides this, other openings should be made near the lower end of the box. These should be provided with slides for closing such a number of them as will give steady motion to the float without causing the level of the confined column of water to differ sensibly from the mean level of the water surface on the outside. The area of the holes left open should usually be between $\frac{1}{200}$ and $\frac{1}{100}$ of the cross-section of the float box, and the lower end of the box should be several feet below the lowest low water. Of course the farther the box extends below the surface of the water the larger may be the openings, as the amplitudes of wind waves decrease rapidly in going downward. (See Fig. 2, Part I.) In some cases the float carries a vertical rod which may itself be graduated,* or it may simply point to graduations upon a fixed scale;† in other cases the float is attached to a wire or varnished cord which passes over one or more pulleys, moves an index

---

* United States Coast Survey Reports, 1854, pp. 190, 191; and 1876, p. 131.

† Baird's Manual for Tidal Observations, p. 5. Phil. Trans., 1893, pp. 55, 56.

along a graduated scale,* or rotates a drum carrying a pointer,† and terminates in a counterpoise; in still other cases the cord is replaced by a flexible tape upon which graduations are made.‡ Various combinations and modifications of these styles readily suggest themselves.§

A simple staff gauge should always be located near a box gauge, and the readings of the two should be frequently compared: for, it is obvious that the line of flotation may in time become altered; or the access of water may be clogged by sand or marine growths, so that the box gauge does not give the true range of tide. In such gauges as have a graduated vertical rod or tape attached to the float, the *reading* point—i. e., the point of the float box opposite which the movable graduated rod or tape is read—should be referred to a bench mark on the shore. The vertical distance of the line of flotation from the zero of the rod or tape should be ascertained and given in the description of the gauge. When the graduations are upon a fixed vertical rod or the float box itself, one of these graduations should be referred to bench mark. The distance of the movable pointer above the line of flotation should be given. The obvious rule covering all cases, even when the graduations are upon a horizontal or oblique scale, is: *Measure the vertical distance from the bench mark to the water, at the same instant noting the reading of the gauge.* The difference between the two values should remain constant. Such reference will detect variations in the working of the gauge, and enable one to recover the plane of reference determined from the series of observations, if such plane should be required in the future.

4. *Other non-self-registering gauges.*

A *siphon gauge* consists of a box gauge upon the shore communicating with the off-shore water by means of a pipe laid along the bottom forming a siphon.|| While observations are in progress, care must be taken that all air in the highest part of the siphon be frequently expelled; otherwise the flow in the pipe will be decreased. This may be accomplished by there inserting a stopcock to be opened at high water or whenever the surface of the water is above it. A closed standpipe or other vessel, preferably of glass, attached to the highest point and filled with water, will serve as a reservoir for the accumulated air. The advantage of this arrangement is that it needs filling with water only occasionally, because the accumulated air does not then immediately decrease the cross-section of the pipe.¶

A *pressure gauge*** is an instrument, somewhat analogous to a barometer, for measuring the pressure of the water at the bottom of a harbor, in order to ascertain the depths of water and so the height of the tide. The pressure, when the depths are not too great, may be exerted upon a bag filled with air which communicates by means of a hose with a manometer located on board a vessel or on the shore. Such gauges have heretofore generally proved unsatisfactory for long-continued records, owing to the difficulty of preventing sand or shellfish from increasing the normal pressure of the water.**

Thomson's depth recorder, which indicates depths by the compression of air, can be used for measuring the tides in very deep water.††

A *spar gauge*‡‡ consists of a long spar bolted at the foot with a universal joint to a block or stone, having attached to the portion above the surface of the water an arc of a circle over which passes a plummet line which indicates the inclination of the spar to the vertical. The graduations upon the spar and this angle of inclination give, by aid of a table of sines, the depth of the water at any time. This gauge may be used for off-shore observations and where the current is strong. Owing to changes which are likely to occur in the sea bottom, the readings of this gauge should be frequently referred to a bench mark upon the shore.

---

*United States Coast and Geodetic Survey Bulletin No. 12 (1889), p. 143.

†Zeitschrift für Instrumentenkunde, Vol. IV (1884), p. 439.

‡United States Coast Survey Report, 1857, pp. 402, 403; 1876, p. 131.

§Phil. Trans., 1831, pp. 174, 175. United States Coast Survey Report, 1876, p. 131.

|| United States Coast and Geodetic Survey Bulletin No. 12 (1889), pp. 143-146. Baird's Manual for Tidal Observations, pp. 3–6.

¶ Cf. Church, Mechanics of Engineering, p. 736; Trautwine, The Civil Engineer's Pocket-Book, 16th. ed., Art. "Syphon"; Knight, American Mechanical Dictionary, Art., "Siphon."

**United States Coast Survey Report, 1858, pp. 247, 248. This gauge as used at Boston was supplied with glycerine instead of air; it was connected with a self-registering apparatus. See an article entitled "Description of a tide gauge for cold climates," by John M. Batchelder, Am. Jour. Sci. and Arts, Vol. 2 (1871), pp. 67, 68.

††Thomson, Popular Lectures and Addresses, Vol. III, p. 54.

‡‡ United States Coast Survey Report, 1857, pp. 403, 404.

A *tripod or pulley gauge*, sometimes used where observations are made upon ice rising and falling with the tide, has a flexible cord made fast to an anchor on the bottom, and which, passing over a pulley directly above, terminates in a counterpoise. The heights may be indicated by the movement of the counterpoise over a graduated scale; or by the number of revolutions of the pulley*, or by a graduated scale securely fastened to the vertical portion of the rope.

AUTOMATIC OR SELF-REGISTERING TIDE GAUGES.

5. The object of these gauges is to trace a curve, or leave some other record, which will enable one to readily find the height of the sea corresponding to any given instant of time covered by the period of observation.

Many forms have been proposed or constructed from time to time; most of them, however, more or less resemble the one described by Henry R. Palmer in 1831,† which is probably the first self-registering tide gauge ever constructed. The essential parts of any form may be said to be, (1) a float and box similar to those employed in a box gauge, (2) a time piece, and (3) some means of recording the height, either in a continuous manner or at short discrete intervals of time.

Usually the motion of the float as it rises and falls with the tide is communicated to the recording portion of the gauge by means of a flexible cord which passes over a grooved wheel called a *float wheel*.‡ Thence the motion is transferred, but usually on a reduced scale, through some mechanism depending upon the particular kind of gauge, to a pencil which traces a curve upon a moving sheet of paper. The paper is driven or carried along by means of a cylinder connected with a well-regulated clock. The pencil is free to move in a direction perpendicular to the line of motion of the paper. In some gauges the paper used is in the form of a long band, and usually of sufficient length for containing a month's record; it is paid out from one cylinder, passes over a second upon which the tracing pencil rests, and is received upon a third. In others there is but one cylinder and this usually revolves once in twenty-four hours. For gauges of this kind, the sheet of paper is first dampened with a wet sponge or cloth, then wrapped about the cylinder and its edge pasted down; when dry it will so hug the roller as not to slip. In some cases it is made fast with rubber bands. One or several days' record are made upon each sheet; but care must be taken to change the sheet before the record becomes confused by many tracings.

In order to keep the float cord and the bands of paper taut, it is necessary to have some arrangement for counterpoising; either weights or springs may be used for this purpose.§

---

* Smithsonian Contributions to Knowledge, Vol. 13 (1863), pp. 1, 2.

† Phil. Trans., 1831, pp. 209–213.

‡ The word "cord" may be used as a general term including wire, tape, chain, etc. In many forms of gauge a rack-and-pinion takes the place of cord and float wheel.

§ For description of several self-registering gauges, the following references may be consulted:

"Description of the self-registering tide-gauge arranged for the Coast Survey," by Joseph Saxton; United States Coast Survey Report, 1853, pp. 94–96.

"Methods of registering tidal observations," by R. S. Avery; ibid., 1876, pp. 130–142.

"The self-registering tide-gauge;" Baird's Manual for Tidal Observations, pp. 10–14.

(Thomson's gauge); Minutes of Proceedings of the Institution of Civil Engineers (London), Vol. 65 (1881), pp. 2–10.

"Notes relating to self-registering tide-gauges as used by the United States Coast and Geodetic Survey," by J. F. Pratt; Report, 1897, App. No. 7.

"Ueber einen elektrisch registrirenden Fluthmesser der Telegraphen-Bauanstalt von Siemens & Halske"; Zeitschrift für Instrumentenkunde, Vol. IV (1884), pp. 95–99.

"Registrirender Fluthmesser" (F. R. Reitz's); ibid., Vol. V (1885), pp. 165-168.

"Ueber Fluthmesser," by Prof. Eugen Gelcich; ibid., Vol. VI (1886), pp. 86–89.

"Der selbstregistrirende Pegel zu Travenmünde," by Prof. W. Seibt; ibid., Vol. VII (1887), pp. 7–14.

"Der selbstregistrirende Fluthmesser von R. Fuess;" ibid., Vol. VII (1887), pp. 243–246; Engineering News, July 28, 1888.

"Der selbthätige Universalpegel zu Swinemünde, System Seibt-Fuess;" ibid., Vol. XI (1891), pp. 351–365; also ibid., Vol. XIV (1894), pp. 41–45; ibid., Vol. XV (1895), pp. 193–203.

"Neuer Mareograph," by L. Favé; ibid., Vol. XII (1892), pp. 171, 172. This is a self registering pressure gauge described in the Journal de Physique, II, Vol. 10, p. 404.

Other references may be found in the index of Zeitschrift für Instrumentenkunde under the heads "Wasserstandsanzeiger," "Fluthmesser," and "Pegel."

It may be added that several designs for new tide gauges, also for attachments and improvements to older forms, are on file at the office of this Survey.

A *tripod or pulley gauge*, sometimes used where observations are made upon ice rising and falling with the tide, has a flexible cord made fast to an anchor on the bottom, and which, passing over a pulley directly above, terminates in a counterpoise. The heights may be indicated by the movement of the counterpoise over a graduated scale; or by the number of revolutions of the pulley*, or by a graduated scale securely fastened to the vertical portion of the rope.

AUTOMATIC OR SELF-REGISTERING TIDE GAUGES.

5. The object of these gauges is to trace a curve, or leave some other record, which will enable one to readily find the height of the sea corresponding to any given instant of time covered by the period of observation.

Many forms have been proposed or constructed from time to time; most of them, however, more or less resemble the one described by Henry R. Palmer in 1831,† which is probably the first self-registering tide gauge ever constructed. The essential parts of any form may be said to be, (1) a float and box similar to those employed in a box gauge, (2) a time piece, and (3) some means of recording the height, either in a continuous manner or at short discrete intervals of time.

Usually the motion of the float as it rises and falls with the tide is communicated to the recording portion of the gauge by means of a flexible cord which passes over a grooved wheel called a *float wheel.*‡ Thence the motion is transferred, but usually on a reduced scale, through some mechanism depending upon the particular kind of gauge, to a pencil which traces a curve upon a moving sheet of paper. The paper is driven or carried along by means of a cylinder connected with a well-regulated clock. The pencil is free to move in a direction perpendicular to the line of motion of the paper. In some gauges the paper used is in the form of a long band, and usually of sufficient length for containing a month's record; it is paid out from one cylinder, passes over a second upon which the tracing pencil rests, and is received upon a third. In others there is but one cylinder and this usually revolves once in twenty-four hours. For gauges of this kind, the sheet of paper is first dampened with a wet sponge or cloth, then wrapped about the cylinder and its edge pasted down; when dry it will so hug the roller as not to slip. In some cases it is made fast with rubber bands. One or several days' record are made upon each sheet; but care must be taken to change the sheet before the record becomes confused by many tracings.

In order to keep the float cord and the bands of paper taut, it is necessary to have some arrangement for counterpoising; either weights or springs may be used for this purpose.§

---

* Smithsonian Contributions to Knowledge, Vol. 13 (1863), pp. 1, 2.

† Phil. Trans., 1831, pp. 209–213.

‡ The word "cord" may be used as a general term including wire, tape, chain, etc. In many forms of gauge a rack-and-pinion takes the place of cord and float wheel.

§ For description of several self-registering gauges, the following references may be consulted:

"Description of the self-registering tide-gauge arranged for the Coast Survey," by Joseph Saxton; United States Coast Survey Report, 1853, pp. 94–96.

"Methods of registering tidal observations," by R. S. Avery; ibid., 1876, pp. 130–142.

"The self-registering tide-gauge;" Baird's Manual for Tidal Observations, pp. 10–14.

(Thomson's gauge); Minutes of Proceedings of the Institution of Civil Engineers (London), Vol. 65 (1881), pp. 2–10.

"Notes relating to self-registering tide-gauges as used by the United States Coast and Geodetic Survey," by J. F. Pratt; Report, 1897, App. No. 7.

"Ueber einen elektrisch registrirenden Fluthmesser der Telegraphen-Bauanstalt von Siemens & Halske"; Zeitschrift für Instrumentenkunde, Vol. IV (1884), pp. 95–99.

"Registrirender Fluthmesser" (F. R. Reitz's); ibid., Vol. V (1885), pp. 165-168.

"Ueber Fluthmesser," by Prof. Eugen Gelcich; ibid., Vol. VI (1886), pp. 86–89.

"Der selbstregistrirende Pegel zu Travenmünde," by Prof. W. Seibt; ibid., Vol. VII (1887), pp. 7–14.

"Der selbstregistrirende Fluthmesser von R. Fuess;" ibid., Vol. VII (1887), pp. 243–246; Engineering News, July 28, 1888.

"Der selbthätige Universalpegel zu Swinemünde, System Seibt-Fuess;" ibid., Vol. XI (1891), pp. 351–365; also ibid., Vol. XIV (1894), pp. 41–45; ibid., Vol. XV (1895), pp. 193–203.

"Neuer Mareograph," by L. Faué; ibid., Vol. XII (1892), pp. 171, 172. This is a self registering pressure gauge described in the Journal de Physique, II, Vol. 10, p. 404.

Other references may be found in the index of Zeitschrift für Instrumentenkunde under the heads "Wasserstandsanzeiger," "Fluthmesser," and "Pegel."

It may be added that several designs for new tide gauges, also for attachments and improvements to older forms, are on file at the office of this Survey.

6. Fig. 1, taken from the Coast Survey Report for 1853, shows a three-roller gauge designed by Joseph Saxton.

F denotes the float, W the float wheel, P the tracing pencil, and C the clock. The paper is paid out from the roller R, is pulled forward by means of pin points in the cylinder R^2 which is driven by the clock, passes under R^3 which pressing down upon it enables the points to puncture the sheet, and is finally received upon R^4. E, E′, E″, E‴ denote various counterpoises.

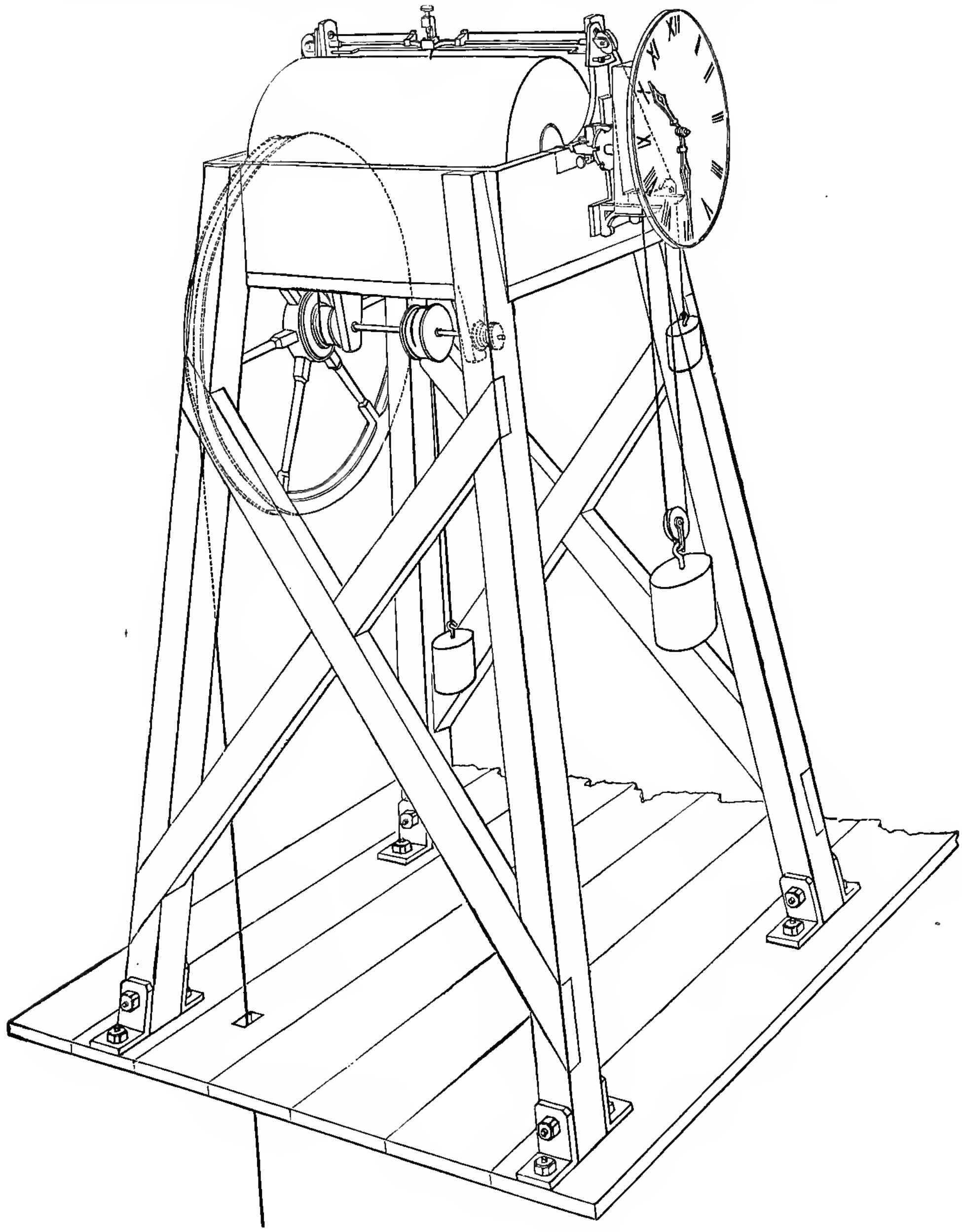


FIG. 2.—Avery's one-roller tide gauge.

This gauge was used by the Survey for many years, the most important change made being the substitution by R. S. Avery of a balance clock for a pendulum clock. This change was necessitated because of the shocks of the waves against the supports of the tide house when in an exposed location.

Several other improvements were made by Avery. One was the putting of band wheels at

the ends of R, R^2, R^4, so that R and R^4 could be moved by R^2 by means of an endless band. This necessitated some means for keeping the paper taut and at the proper tension, which was accomplished by allowing the cores of R and R^4 to revolve with some friction in their cylinders. The friction was produced by friction plates which could be adjusted at will. Another improvement was in the mode of clamping the clock to the cylinder.

Fig. 2 shows a one-roller gauge devised by Avery and described in the Coast Survey Report for 1876.

7. Figs. 3 and 4 show a form of gauge now constructed in the Instrument Division of the Survey.

This is a three-roller gauge capable of receiving one month's record without change of paper. The float wheel has a spiral groove in its periphery so that the float wire, having been made fast at a given point, may be wound around it a number of times without causing any crossing or piling up. A shoulder of the float wheel has a similar spiral groove for the wire or cord of the float counterpoise. The float wheel communicates motion to the recording pencil by means of a coarse screw working in a nut. The float-wheel end of the screw rests upon ball bearings; this secures accuracy as well as ease of working. The recording pencil can be accurately set at any distance from the base line by loosening the float wheel from the coarse screw and rotating the latter, thus driving the nut and pencil.

An attachment is provided for marking the exact hours upon the sheet. This result is accomplished by making use of an additional clock, and adapting its striking apparatus to the sudden movement of the recording pencil each hour.

The paper is kept taut by means of a spring pressing against the roller from which the paper is paid out, and a weight attached to the receiving roller. The distance between the flanges of the rollers, that is, the width of the sheet, is thirteen inches.

8. For special purposes many attachments or additions have been devised. Of these may be mentioned *integrators*, *indicators*, *time-marking* and *printing attachments*. In the more recent gauges, electricity often plays an important part.

An integrator is an arrangement for continuously summing the heights of the sea for the purpose of finding mean sea level. One form of integrator may be described thus:* The cylinder upon which the curve is traced has attached to one end of its axle a smooth disk. A small friction wheel with sharp edge has its plane perpendicular to the plane of the disk, and its axis in the same plane as the axis of the cylinder. This small wheel moves across the face of the disk as the recording pencil moves across the recording sheet. The friction between this disk and the small wheel causes the latter to rotate. When the phase of the tide is at about mean sea level, the edge of the wheel should be set at the center of the disk; then for phases higher than this, it will rotate in one direction; and for lower phases, in the opposite direction. The resulting number of rotations is ascertained by additional wheelwork. This number divided by a quantity proportional to the length of the period covered by the observations, will show how much the observed mean sea level differs from the one assumed.

Another form of integrator† consists essentially of a pendulum whose variable length is dependent upon the height of the sea for the particular instant. A cam connected with the float wheel alters this length in a suitable manner, while a clockwork registers the entire number of beats.

An indicator is either an independent instrument or an attachment to a tide gauge, which shows upon a large dial, or otherwise, the height of the sea at any given instant. The indicator may stand close to the float box and have a mechanical connection therewith; or, it may be located many miles distant and have an electric connection. A broad and clearly-painted tide staff, situated in a conspicuous place, constitutes an indicator of the simplest form.‡

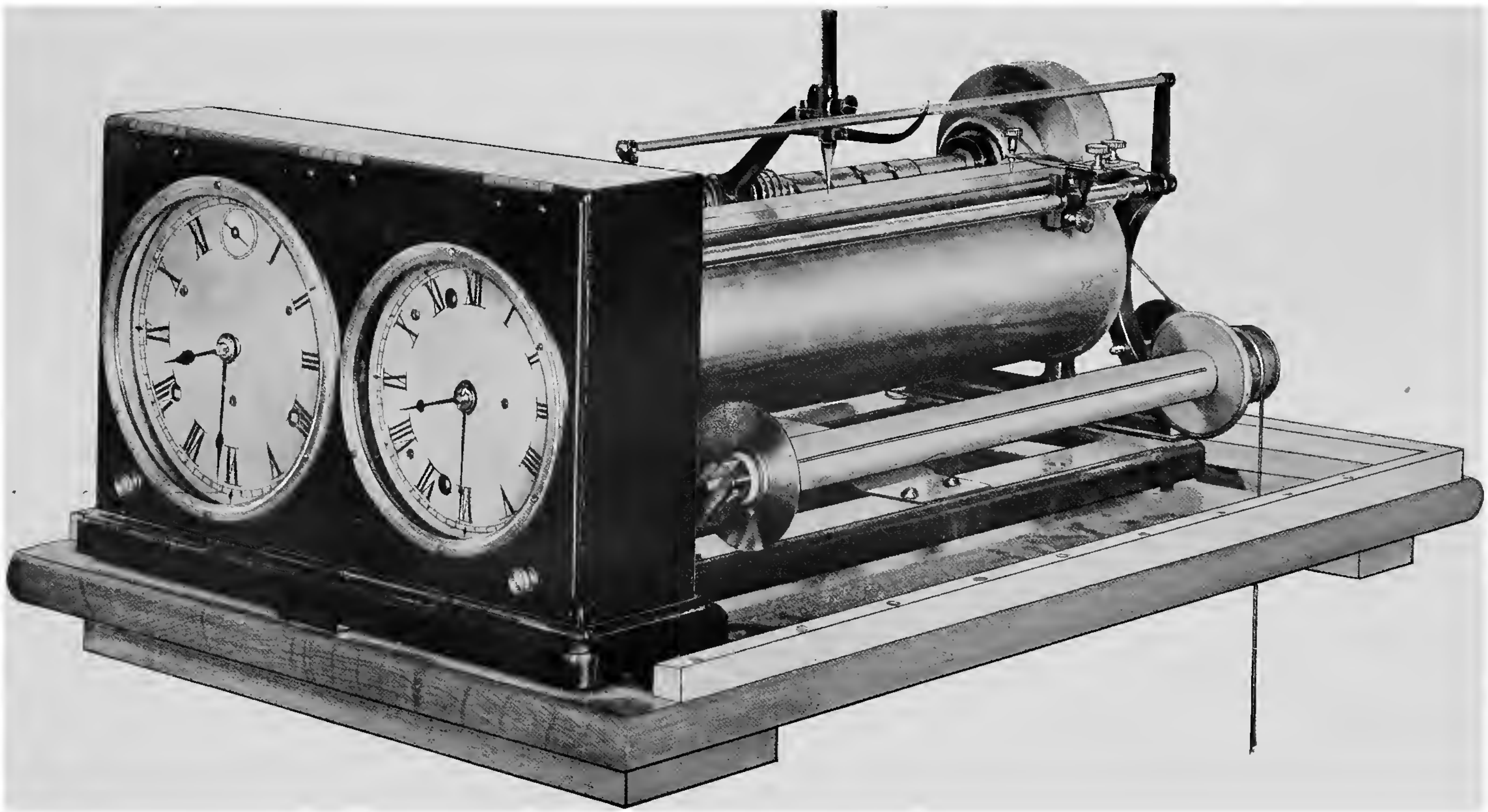
* "Registrirender Fluthmesser" (F. R. Reitz's) Zeitschrift für Instrumentenkunde, Vol. V (1885), pp. 165–168.

† "Der selbthätige Universalpegel zu Swinemünde, System Seibt-Fuess;" Zeitschrift für Instrumentenkunde, Vol. XI (1891), pp. 351–365.

‡ For descriptions and examples of indicators the following references may be consulted:

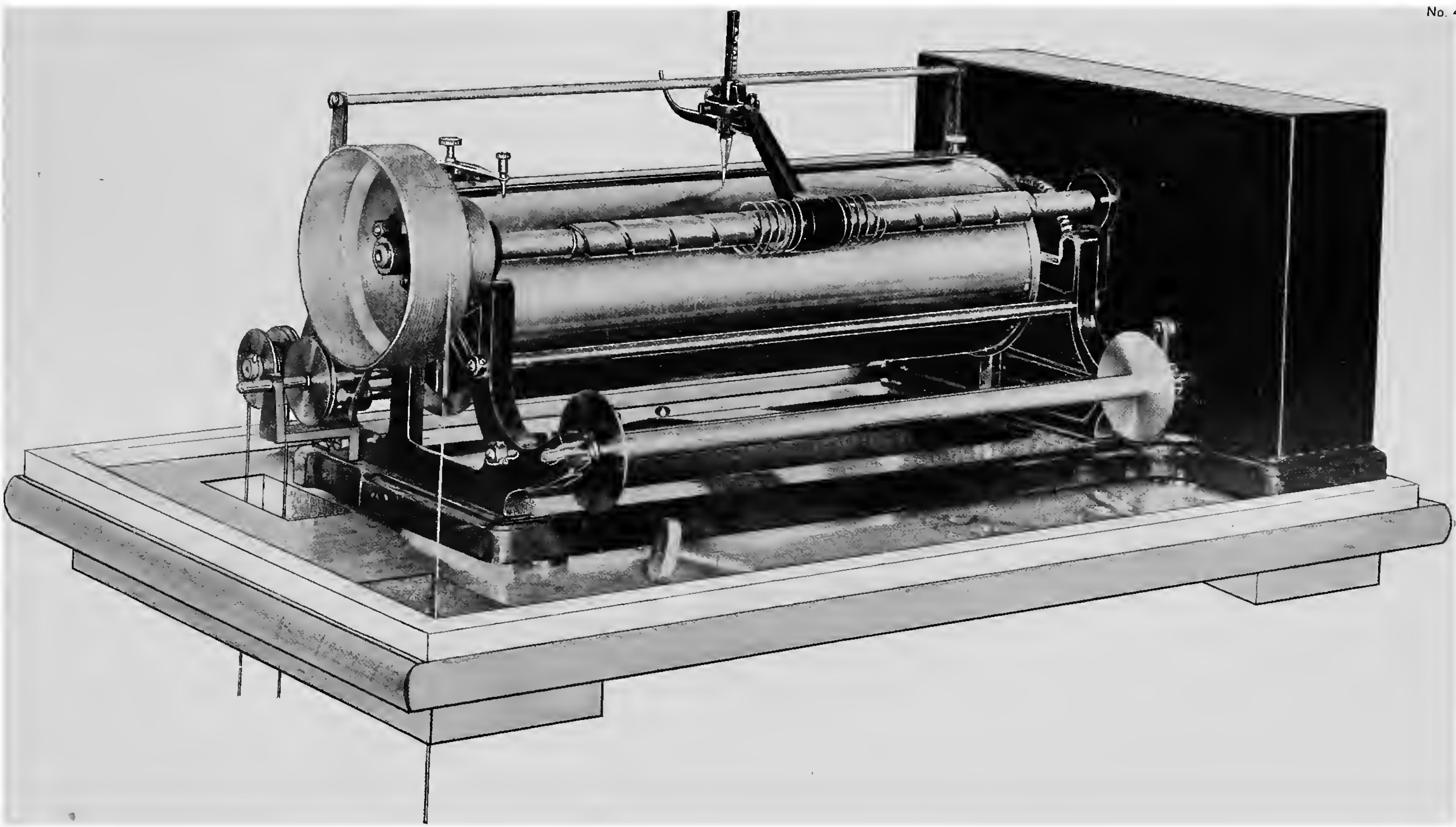
"The tide indicator at Rouen;" Scientific American Supplement, September 23 (1893), p. 14785.

"Tidal indicator, New York Harbor;" Scientific American, March 3 (1894), p. 133. Coast and Geodetic Survey,



STIERLE GAUGE AS MODIFIED BY THE COAST AND GEODETIC SURVEY.

No. 4.



STIERLE GAUGE AS MODIFIED BY THE COAST AND GEODETIC SURVEY.

The Coast and Geodetic Survey has recently erected two tidal indicators, one at Fort Hamilton, N. Y., and one at Reedy Island, Delaware River. The latter, shown in Fig. 5, has a face thirty feet in diameter. It is proposed to erect a similar indicator at Presidio, Cal.

In these indicators the rise and fall of the tide is communicated to a large float wheel carrying a pointer, by means of a flexible wire cord. The float wheel is so counterpoised as to keep this cord constantly taut. The end of the pointer moves along a semicircle whose divisions represent feet and half feet of rise and fall, the zero denoting the position of the pointer when the tide is at the plane of mean low water. The arrow head has its two barbs so hinged as to enable it to point either upward or downward. The upward pointing indicates a rising tide, the downward, a falling.

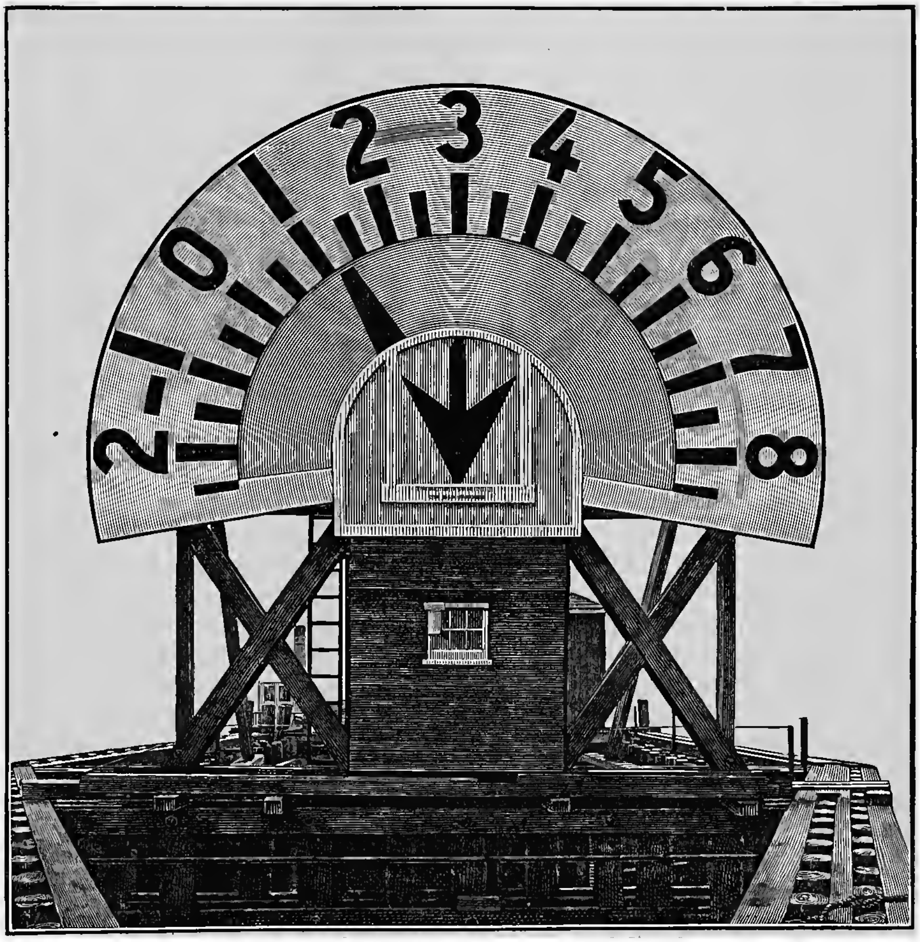


Fig. 5.—Tidal indicator, Delaware River, Delaware.

At the time shown in the figure, the tide is $1\frac{1}{4}$ feet above mean low water and is still falling, as indicated by pointing of the arrow.

Each triangular barb of the arrow head rotates about a point near its obtuse angle when a reversal takes place. The power accomplishing this comes from the float and is communicated to a lever by causing the float cord to pass between a group of three pulleys, thus giving rise to some

"Notice to mariners" No. 177. Harpers Weekly, Vol. 38 (1894), p. 96. United States Coast and Geodetic Survey Report (1893), pp. 27, 28.

A similar indicator is located at Reedy Island, near Philadelphia. "Notice to mariners" No. 202.

"Elektrischer Tiefwasserstandsmesser mit Zifferblatt" (A. Grabié's); Zeitschrift für Instrumentenkunde, Vol. IV (1884), p. 439.

"Elektrischer Wasserstandsanzeiger" (A. Hempel's); ibid., Vol. VIII (1888), p. 224.

"Elektrischer Wasserstandsanzeiger mit Registrirvorrichtung," by W. E. Fein; ibid., Vol. IX (1889), pp. 338–343.

"Der selbthätige Universalpegel zu Swinemünde, System Seibt-Fuess;" ibid., Vol. XI (1891), pp. 351–365.

small amount of friction. The lever actuates a vertical rod in the shaft of the arrow, so to speak, and this in turn actuates an arm or projection attached to each barb. From the figure it is evident that the barbs are of sufficient size to require counterpoising in order that as little work as possible may be required of the float.

The object of time-marking attachments is to indicate upon the record sheet the exact hour as given by a clock, either near or distant, in order to avoid errors which might result from using hour marks made by points fixed upon the cylinder. The gauge of Mr. Palmer, already referred to, marked the positions of the hours by means of a punch actuated by a toothed wheel. The figure was in the form of an arrow whose direction was always that of the wind-vane above the tide house. A cross-mark showed the exact position of the arrow to be taken as the hour mark.

Since the invention of the electric telegraph, the principle of having a clock make (or break) a circuit has been at the foundation of nearly all schemes for the transference of a particular instant of time, whether for astronomical or other purposes. In this way it is easy to have one standard clock mark the exact hours upon several gauges simultaneously.*

The time-marking attachment now used upon the gauges belonging to this survey, and shown in Figs. 3, 4, causes the pencil which traces the tidal curve to suddenly move back and forth, thus dawing a short horizontal mark from the curve at each exact hour. The advantage of horizontal marks over vertical ones is that the hourly heights can be more easily read from the sheet. Some gauges cause hour marks to be made at both edges of the sheet, thus precluding such error as may arise from slanting the height scale when the sheet is being read.

Printing attachments are designed for the purpose of saving time in the reading of a tidal record. One form leaves a record consisting of two rows of printed figures, the one being the hours of the day uniformly distributed, and so easily made; the other, the heights of the sea at the exact hours, or at certain fractions of hours.†

For ascertaining the meteorological effects upon the tides and the condition or density of the water, the following auxiliary instruments may be provided: A self-registering aneroid barometer, a mercurial barometer for checking the aneroid, a self-registering anemometer, a thermometer, and a densimeter or salimeter.

Establishment and care of a self-registering tide gauge; also the reading of the record.

9. A small house or closed shed must be provided for sheltering the registering portion of the gauge. With the most common forms of automatic gauges this house must be located upon a wharf or other staging directly over the float box; but some forms have been devised in which the recording portion of the gauge may be placed wherever convenient. When a special house has to be constructed, it should be large enough to afford room to get at the registering part of the gauge, and be provided with a window for light and ventilation. The site of the gauge should be so selected as to afford as much protection from violent storm waves as possible, while at the same time not so far removed from the port for which the tidal observations are wanted as to introduce any material alteration in the time or range of the tide. The location should be reasonably accessible, and the structure upon which a gauge is placed should be as firm as is practicable. The siphon arrangement already described will be found to be very serviceable where deep water lies far out from the shore line.‡

In setting up the gauge, care must be taken to have it properly leveled in order that all parts may work freely. The float box should be vertical and the float suspended in its center. The statements made in § 3 concerning the size of the openings apply here fairly well. The scale to be used can be decided upon as soon as the range of tide at the place is approximately known.

---

* Some references to time-marking attachments:

"Elektrischer Wasserstandsanzeiger mit Registrirvorrichtung," by W. E. Fein; Zeitschrift für Instrumentenkunde, Vol. IX, (1889), pp. 338-343.

"Der selbthätige Universalpegel zu Swinemünde, System Seibt-Fuess;" ibid., Vol. XI (1891), pp. 351-365.

"Der kurvenzeichnende Kontrolpegel, System Seibt-Fuess," by Wm. Seibt; ibid., Vol. XIV (1894), pp. 41-45.

"Notes relating to self-registering tide gauges as used by the United States Coast and Geodetic Survey," by J. F. Pratt; report (1897), App. No. 7.

† "Ueber einen elektrisch registrirenden Fluthmesser der Telegraphen-Bauanstalt von Siemens & Halske;" loc. cit. ante.

‡ Cf. Baird, Manual for Tidal Observations, pp. 2-4.

The zero of the gauge (i. e., the position of the tracing pencil) can be approximately fixed by noting a reading of the water upon the tide staff. A closer approximation is afterwards made in the manner described beyond.

An automatic tide gauge of any sort is sometimes called a *marigraph*, and the record produced by it a *marigram;* these names can be readily distinguished as to their application by the more common words "telegraph," the instrument, and "telegram," the message sent.

In some forms of marigraphs there is a row of steel pins at each end of a cylinder which make small perforations in the paper, the distance between them indicating hours or half hours of time. After putting the paper on such gauges, and before connecting with the clock, draw a straight line between the pins at opposite ends of the cylinder, place the recording pencil upon this line, and then when it is an exact hour (or half hour, if the pins are close enough to measure that interval of time) connect the driving clock with the apparatus, and note the time along the ruled line.

In other forms of gauges the hour lines and their numbers are previously marked upon the edge of the cylinder, or upon profile paper stretched around it; but care must be taken in starting to make the record and actual time agree.

In the most improved forms of gauge the hours are marked upon the paper by a special clock, or by the driving clock itself, using either electrical or mechanical means. In such gauges there is no need of starting the record at any whole hour or half hour.

On the blank paper, at the beginning and end of each marigram, a note, similar to that below, should be written and filled out:

Station..
Latitude..Longitude...
The time used is..
Tidal record from.......................................to...
Marigram No. ....Marigraph No. ....Scale.....................
Observer...

In connection with every marigraph there should be a fixed tide staff, in order to have a check upon the working of the apparatus; and the readings of this fixed staff, called *staff readings*, together with the times of making them, must be recorded on the marigram. The best time for taking a staff reading is generally at or near the time of high or low water, because the height of the water surface then changes slowly; but it is also desirable that several staff readings be made, at least once a month, when the water is about at its mean level, on both a rising and a falling tide, in order to show that the gauge is working freely and accurately. A general form for making such entries upon the marigram is as follows, the whole being connected, by means of an arrow or other device, with the exact position of the recording pencil of the gauge at the time:

Monday, Dec. 21, 1896, 9^h 35^m A. M.
Gauge clock correct. Staff 6.34 ft.

Such a note should be made at the beginning and end of each sheet or roll of paper, without regard to the phase of the tide.

When there is a time-marking attachment to the gauge, time comparisons should be made at the instant when the hour mark is made. Then a note like the following may be used:

P. M. Wed., Sept. 25, 1895
Correct time, 1:59
Clock set right
At 2:01, staff reads 6.34.

By clock is here meant the clock which makes the hour marks, and the above note implies that this clock made the 2 o'clock hour mark 1 minute too soon.

If the hour marks are made by means of electric connections with a standard timepiece, no such time comparison is necessary.

Whenever anything unusual happens to the record, such as disturbances caused by great storms, or a stoppage of the gauge, an explanatory statement should be written upon the marigram. In fact, a marigram should be a complete record in itself, as notes made elsewhere are liable to become permanently separated from it.

A graduated piece of paper, wood, metal, glass, or other material, called a height scale, is used for reading the marigram. This scale shows the relation between the marigram and nature; for instance, a scale of one-tenth means that a variation of 10 inches in the height of the water surface is indicated by an inch of change on the record; and with such a scale 1·2 inches on the marigram corresponds to a foot on the fixed staff. Gauges can be made to work upon any desired scale, according to the range of tide at the place; but while it is desirable that the scale used should be as near as possible to nature, the average curve produced ought not to occupy much more than one-half of the paper, for unexpected variations of surface level are sure to occur, and these are oftentimes matters of much interest.

Upon starting the gauge care should be taken to so adjust the pencil that half-tide level will fall as nearly as possible in the middle of the record paper. In order to make scale readings it is necessary to have some datum line upon the marigram; the zero of the scale is often placed upon the line of punctures made in the paper by the row of steel pins for marking time, or preferably upon a line traced by a stationary pencil. When the gauge is first started this datum line may be placed anywhere, and arbitrarily called such a division of the scale as will insure positive readings for the curve, but after sufficient record has been obtained to properly determine the relation between such assumed datum line and the fixed staff, it is desirable to so change the datum as to make the scale and staff readings agree as closely as possible.

In the manufacture of marigraphs it frequently happens that mechanical difficulties prevent the obtaining of the exact scale desired, and hence with a new machine one must always find out from the record itself what its working scale really is.

10. *True scale of a marigraph.*

The true or working scale of the tide gauge is ascertained by comparing the staff readings with the readings of the curve corresponding to these times. The sum of the staff readings near high water, less the sum of a like number of staff readings near low water, gives, when divided by this number, a certain range of observed tide. Treating the corresponding scale readings in the same manner, a certain range of recorded tide is obtained. The ratio of these two ranges shows how much the true or working scale of the tide gauge differs from the assumed scale used in reading the curve. For instance, the sum of the three observed ranges in the fourth column in the example below is 9·7 feet, while the sum of the corresponding scale ranges in the seventh column is 9·9.

$\therefore 9{\cdot}7 \div 9{\cdot}9 = 0{\cdot}980$; and, since the assumed scale is 10, the working scale is $0{\cdot}980 \times 10 = 9{\cdot}80$.

Having thus found that scale heights (by assumed scale of 10) must be multiplied by 0·980 in order to give true heights from base line or scale datum, we multiply 26·5 by it, thus obtaining 26·0 feet. Subtracting 26·0 feet from the corresponding staff heights, we obtain $-3{\cdot}3$ feet. Since 6 heights are taken, this must be divided by 6, giving $-0{\cdot}55$ feet, which the staff will read when the tracing pencil crosses the base line. Or, if we construct a scale of 9·8, i. e., a scale such that one unit of the scale $=\frac{1}{9{\cdot}8}$ foot, then it should have its division $-0{\cdot}55$ marked as the one to be applied to the base line of the marigram in making all height readings.

The following method amounts to giving larger individual ranges greater weights in the determination than is given to the smaller ranges, and so is applicable where the diurnal inequality is large or where a more elaborate determination may be desired. The notation employed is of a temporary nature.

Let "*staff*" be a staff reading taken at or near high or low water.

Let "*scale*" be a reading with the assumed scale at the time of a staff reading. The expression for the scale, true or assumed, as here used, is supposed to be greater than unity; that is, if the record has been reduced ten times, it is called a scale of ten, instead of one-tenth. The assumed scale is supposed to be so taken that the ratio R differs little from unity.

Let A, B, C, D, etc., be successive ranges (HW–LW) on staff.

Let A', B', C', D', etc., be successive ranges (HW–LW) on sheet by assumed scale.

If the staff and scale readings involved no error, we should have

$$R=\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}=\frac{D}{D'}=\frac{E}{E'}=\text{etc.}$$

But as an error is likely to occur in each comparison between staff and scale, the precision with which R is determined from the above fractions is in each case proportional to the range. That is, the precisions are as $A:B:C: \ . \ . \ . \ .$, or as $A':B':C': \ . \ . \ . \ .$

If we weight the determinations according to the precisions, we have, as before,

$$R=\frac{\frac{A}{A'}A'+\frac{B}{B'}B'+\frac{C}{C'}C'+\ \ . \ . \ .}{A'\ \ +B'\ \ +C'\ \ +\ \ . \ . \ .}=\frac{A+B+C+\ \ . \ . \ .}{A'+B'+C'+\ \ . \ . \ .}\ . \tag{1}$$

In other words,

$$R=\frac{\text{true scale}}{\text{assumed scale}},$$

$$\therefore \text{ True scale} = R \times \text{assumed scale.}$$

But it is reasonable to suppose that the weights given to different determinations ought to be proportional to some power of the precision greater than unity; that is, to

$$A'^n,\ B'^n,\ C'^n\ \ . \ . \ . \ . \ , \text{ where } n > 1.$$

From the law of accidental errors, it is seen that

$$n=2;^*$$

therefore, giving to $\frac{A}{A'}, \frac{B}{B'}$, etc., the weights A'^2, B'^2, etc., we have

$$R=\frac{\frac{A}{A'}A'^2+\frac{B}{B'}B'^2+\frac{C}{C'}C'^2+\ \ . \ . \ .}{A'^2\ +\ B'^2\ +\ C'^2\ +\ \ . \ . \ .}=\frac{AA'+BB'+CC'+\ \ . \ . \ .}{A'^2+B'^2+C'^2+\ \ . \ . \ .}\ . \tag{2}$$

The reading on staff when the pencil crosses the base line of the sheet is

$$y=\frac{1}{\nu}\Big\{ [\text{staff}] - R\,[\text{scale}] \Big\} \tag{3}$$

where ν is the number of staff or scale readings and the square brackets denote their sums.

Form for computing the true or working scale of a marigraph.

Date.	Staff.			Scale.				Staff range × scale range.
	H W	L W	Range.	H W	L W	Range.	(Range)².	
1896.	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	
Dec. 1	5·0	1·9	3·1	5·6	2·4	3·2	10·24	9·92
2	5·8	2·8	3·0	6·5	3·5	3·0	9·00	9·00
3	5·4	1·8	3·6	6·1	2·4	3·7	13·69	13·32
Sum	16·2	6·5	9·7	18·2	8·3	9·9	32·93	32·24
	6·5			8·3				
	22·7			26·5				

$$R=\frac{32{\cdot}24}{32{\cdot}93}=0{\cdot}979$$

$$\text{True scale} = R \times \text{assumed scale} = R \times 10 = 9{\cdot}79$$

$$y=\frac{1}{6}\Big\{ 22{\cdot}7 - 0{\cdot}979 \times 26{\cdot}5 \Big\} = -\ 0{\cdot}53 \text{ feet.}$$

*Merriman, A Text-Book on the Method of Least Squares, p. 41.

That is, if we construct a true scale we must then raise the base line of the sheet by an amount representing 0·53 feet according to true scale, in order that all readings made thereafter may be reduced to staff.

It is to be noted that R should remain the same as long as the same gauge is employed; but the value of y must change and so necessitate a new determination whenever the relation between scale and staff zeros is altered.

The number of observations necessary to make a good determination of the true or working scale and position of the datum line depends upon the value of this scale or ratio of reduction, as well as upon the smoothness of the water, the cross-section of the float used, and the care taken in indicating the exact place on the tide curve which corresponds to the time of reading the staff; as a general rule, however, about thirty high-water and as many low-water comparative readings will suffice. The time and height of these staff readings should be recorded on the marigram, and afterwards copied into a separate register similar to the above form for computation. The scale readings are made by placing the assumed scale so that it corresponds to the arbitrary datum line upon the marigram, and noting where the curve crosses the edge of the scale, care being taken that the place of crossing is exactly at the point marked as being the position of the recording pencil at the instant of making the staff reading, and that the scale is perpendicular to the datum line.

Having thus found the true scale of the record, it is well to have a paper scale constructed for making subsequent readings of the curve. The arbitrary datum line, or the pencil upon the gauge which traces it, should be changed by the value of y, so that the scale readings may agree with the staff readings as nearly as possible. During the progress of the series it frequently happens that the relation between this datum line and the staff is altered by some adjustment of the marigraph, and hence the value of a new y must be computed whenever there is the least suspicion that any alteration has taken place in the relation between scale and staff readings.

If the observer makes any change in the relation he should record the time and amount of such change upon the marigram. It should be made only after he has carefully determined the relation between the datum line and the zero of the tide staff.

11. *Additional directions to the observer.*

In most localities the time may be obtained from railroad or telegraph stations and carried to the gauge by means of a well-regulated watch which has been compared with this standard not many hours before. The kind of time used is unimportant so long as it is defined upon the marigram and maintained for a considerable period, but it is troublesome to have one kind of time at the beginning and another kind at the end of a series. If for any reason it is found desirable to change the kind of time used, it should be done at some convenient epoch, such as the beginning of a calendar year or the beginning of some month, and ample notes should be made upon the record calling attention to such change.

When distant from telegraph stations, a carefully constructed sundial will be of use in setting the watch, for a good sundial will give the time of apparent noon within one minute, which is sufficiently close for any marigraph. Sun time is converted into standard time by adding (Part III, § 27)

$$L - S + \text{equation of time, Table 30.} \tag{4}$$

The gauge should be visited once or twice every day, especially at about the times of those high and low waters which occur during daylight, so as to make staff readings, time comparisons, and to attend to the various details necessary to keep the gauge running.

In cold weather there is often much annoyance and loss of record caused by the water freezing. At permanent stations a system of pipes passing through the float box and carrying hot water is sometimes provided, thus imparting warmth enough to the confined water to prevent the formation of ice around the float.

In some cases a few gallons of kerosene oil poured into the float box will prevent freezing, but in this case there is a change in the line of flotation which must be allowed for in tabulating the record. Moreover, unless the float box has been constructed with great care to secure tight joints, the oil is sure to soon leak out. If nothing better can be done when the gauge is frozen up, let the observer secure as many readings of the staff as may be practicable, particularly at the

exact hours, for even one or two readings a day will be better than nothing for interpolating the break in the record preparatory to analysis.

12. *Preparation of the marigram for reduction.*

In those forms of gauge without a time-marking attachment it is difficult to secure correct time throughout the different hours of each day, because there is generally some eccentricity in the connection of the driving clock with the axis of the gauge cylinder, thus causing the paper to move irregularly, while the clock may keep correct time. If such a gauge has its axle marked so that whenever connection is made with the driving clock a given hour will always correspond to a fixed portion of the surface of the roller, a scale may be made which will represent quite closely the true length of each hour, no matter how unequal the spaces occupied by them may be. Such scale should be copied upon the marigram.*

If the cylinder is driven at a sensibly uniform rate it is often convenient to subdivide the space between the time notes made by the observer into equal hour spaces. This may be done by first making a paper scale with uniform hour spaces a little longer than the average hour of the marigram, the scale being long enough to reach between successive time notes; and then placing it upon the tide curve, so moving and slanting the scale as to make it exactly agree with two vertical lines drawn through that part of the curve referred to by each of two consecutive time notes. The hours of the time scale, while held in this position, are transferred to the marigram by successive dots, through which vertical lines may be drawn, intersecting the curve at each hour, and subsequently numbered to agree with the notes.

With gauges having steel pins for marking the hours, the perforations in the paper may be used as hour marks, provided the gauge clock is kept always correct and care is taken to start the record exactly on a line joining two steel pins on opposite ends of the roller. As it is practically impossible to maintain correct time without disarranging the relation of the punctures to the clock, and since there is likely to be some eccentricity in the connection between the driving clock and the roller, it will generally be more exact to use one of the methods just described for ascertaining the hours.

When the gauge has a time-marking attachment, it is only necessary to number the marks made by the mechanism.

High and low waters.—It is customary to tabulate the time and height of the high and low waters, but there is often considerable difficulty in fixing upon the proper part of the curve in making these readings. The aim should usually be to select the highest and lowest points of what appears to be the true tidal curve. Some persons select the highest and lowest portions of the curve, regardless of accidental disturbances, as well as of peculiarly shaped high or low waters, but even this latter is likely to introduce considerable irregularity in the times, because the tide curve is generally not symmetrical about its extreme points, and these points are liable to swing back and forth during a lunation. In such cases the usage has sometimes been to imagine a small portion of the curve near high or low waters cut off by a horizontal chord, and to take the point where the perpendicular from the middle of the chord cuts the curve as the point of high or low water.

It is convenient to have a small scale, equal to one hour, subdivided into six parts, so that the number of minutes beyond any hour mark may be easily estimated. The height is read by so placing the scale as to make it agree with the datum line, holding it perpendicular to this line, and noting where the curve at the point selected crosses the edge of the scale.

Hourly heights or ordinates.—It is very important that the height of the sea at each exact hour should be ascertained and recorded. As the marigram has been already subdivided into hours, it is only necessary to see that the scale agrees with the datum line, is perpendicular to that line, and intersects the curve exactly at the hour mark. A form for tabulating hourly heights is given in § 61. A correct datum line upon the marigram, and the relation between staff and scale can be found from the staff readings by aid of § 10. Instead of actually ruling in a new datum line, it is well to mark upon the scale the value by which the assumed datum has been found to differ from the one corresponding to staff; by placing this mark of the scale upon the datum line, the scale readings will then approximately agree with the staff readings.

*Cf. Phil. Trans. 1838, p. 250.

CHAPTER II.

ASTRONOMY; TIDAL COMPONENTS SUGGESTED; ETC.

13. *Mean motions.*

In the harmonic treatment of tides it is important to know the mean sidereal motions of the moon, sun, equinox, lunar perigee, solar perigee, and the moon's node. Upon these depend the periods of the tidal components and tidal inequalities. The values given below are taken from various authorities, as indicated in the right-hand margin.

Mean sidereal motion.

Per Julian year (epoch, Jan. 0, 1900).

Moon	$4812^\circ.6649577 + 0^\circ.0000462\left(\frac{t-1900}{100}\right)$	Hansen, Tables de la Lune, pp. 15, 16, with Newcomb's correction, Researches on the Motion of the Moon, p. 268; or Harkness, Solar Parallax, p. 14, equation (33), omitting term of second power.
Sun	$359.9937311 - 0.0000001\left(\frac{t-1900}{100}\right)$	Newcomb, Tables of the Sun, p. 9. There denoted by n.
Equinox	$-0.0139581 - 0.0000062\left(\frac{t-1900}{100}\right)$ $(50''.2493)$ $(0''.0222)$	Newcomb, Tables of the Sun, p. 9, $\left(n-\frac{dL}{dt}\right)$.
Lunar perigee	$40.6763487 - 0.0002070\left(\frac{t-1900}{100}\right)$	Hansen, Tables de la Lune, pp. 15, 16.
Solar perigee	$0.0032336 + 0.0000029\left(\frac{t-1900}{100}\right)$ $(11''.6410)$ $(0''.0104)$	Newcomb, Tables of the Sun, p. 9, $\left(n-\frac{dg}{dt}\right)$.
Moon's node	$-19.3553827 + 0.0000393\left(\frac{t-1900}{100}\right)$	Hansen, Tables de la Lune, pp. 15, 16, increased by $0''.10$, Newcomb's correction, Researches on the Motion of the Moon, p. 274. [Cf. Harkness, Solar Parallax, pp. 16, 17, 140.]

Mean sidereal motion.

Per mean solar day.

	Temporary symbol.	Numerical value (1900). °
Moon	☽$_*$	13.176358543
Sun	⊙$_*$	0.985609120
Equinox	♈$_*$	− 0.000038215
Lunar perigee	ϖ_*	0.111365773
Solar perigee	π_*	0.000008853
Moon's node	☊$_*$	− 0.052992149
Earth's meridian*	⊕$_*$	360.985609120

Mean motion relative to the equinox, i. e., mean motion in longitude.

	Per mean solar day.		Per mean solar hour.	
	Formula.	Numerical value (1900). °	Numerical value. °	Symbol.
Moon	☽$_*$ − ♈$_*$ = ☽♈	13.176396758	0.5490165316	σ
Sun	⊙$_*$ − ♈$_*$ = ⊙♈	0.985647335	0.0410686390	η
Equinox	♈$_*$ − ♈$_*$ =	0.0	0.0	
Lunar perigee	ϖ_* − ♈$_*$ = ϖ♈	0.111403988	0.0046418328	ϖ
Solar perigee	π_* − ♈$_*$ = π♈	0.000047068	0.0000019612	
Moon's node	☊$_*$ − ♈$_*$ = ☊♈	− 0.052953934	− 0.0022064139	
Earth's meridian	⊕$_*$ − ♈$_*$ = ⊕♈	360.985647335	15.0410686390	γ

* I. e., its motion upon the celestial sphere, as seen from the earth's center, 360° = ⊕$_*$ − ⊙$_*$.

Astronomical periods obtained from mean motions.—If we divide 360° by one of the above sidereal motions, or by a combination of them, the length of some mean astronomical period will be obtained. The following list includes the more important of such periods:

Mean astronomical periods.

Formula.	Name.	Numerical value (1900). d
$360° \div ☽_*$	Sidereal month	27·3216609
" $\div \odot_*$	Sidereal year	365·2563605
" $\div [☽_* - \odot_*]$ or $☽\odot$	Synodical month	29·5305881
" $\div [☽_* - \Upsilon_*]$ or $☽\Upsilon$	Tropical month	27·3215816
" $\div [\odot_* - \Upsilon_*]$ or $\odot\Upsilon$	Tropical year	365·2421989
" $\div [☽_* - \varpi_*]$ or $☽\varpi$	Anomalistic month	27·5545503
" $\div [\odot_* - \pi_*]$ or $\odot\pi$	Anomalistic year	365·2596413
" $\div [☽_* - ☊_*]$ or $☽☊$	Nodical month	27·2122191
" $\div [\odot_* - ☊_*]$ or $\odot☊$	Eclipse year	346·6200271
" $\div [\odot_* - \varpi_*]$ or $\odot\varpi$	Evectional period in moon's parallax	411·7846609
" $\div [☽_* - \odot_* - (\odot_* - \varpi_*)]$ or $[☽\odot - \odot\varpi]$	Moon's evectional period	31·8119389
" $\div \oplus_*$	Sidereal day	0·9972696723
" $\div [\oplus_* - \Upsilon_*]$ or $\oplus\Upsilon$	Tropical day*	0·9972695663
" $\div [\oplus_* - ☽_*]$ or $\oplus☽$	Lunar day	1·0350501012
" $\div [\oplus_* - \odot_*]$ or $\oplus\odot$	Solar day	1·0
" $\div \Upsilon_*$	Revolution of equinox	9420384·666
" $\div \varpi_*$	Revolution of lunar perigee or line of apsides	3232·591040
" $\div \pi_*$	Revolution of solar perigee or line of apsides	40664181·63
" $\div ☊_*$	Revolution of moon's node	6793·45916
" $\div [☊_* - \Upsilon_*]$ or $☊\Upsilon$	Node-equinox period	6798·36171

14. *Principle of forced oscillations.*

This principle, due to Laplace,† is of fundamental importance in the analysis and prediction of tides; it may be stated thus:

The state of any system of bodies in which the primitive conditions of the motion have disappeared through the resistances which the motion encounters, is coperiodic with the forces acting on the system.

If there were but a single strictly periodic force acting upon the given system, the effects of successive periodic actions must eventually become identical, and their periods become that of the force. Now the magnitude of any tide-producing force being very small in comparison with the force of terrestrial gravity, the accelerations imparted to the fluid particles must be very small, and so must be the resulting displacements. Therefore if several such forces act simultaneously, they act as if totally independent of one another and so their effects permit of superposition. This being so, the disturbance may be regarded as made up of terms whose periods are the periods of the several forces.

Here is the clue to what periodic terms ought to be found in the tidal wave; for, there ought to be an oscillation corresponding to each term of the causes producing the tide. Such terms follow from the development of the tide-producing potentials of the moon and sun. Before proceeding to this development, it may be well to consider what periodic terms are suggested from a superficial view of the nature of the tide-producing causes.

WHAT COMPONENTS, OR PERIODIC OSCILLATIONS, SHOULD EXIST IN THE TIDE.

15. Since the lunar tide is due to the difference between the moon's attraction upon the earth as a whole and the enveloping sea, there ought to be set up an oscillation whose period is a half lunar day; and likewise, because of the sun's attraction, an oscillation whose period is a half solar day.

Confining our attention to the case of the moon, it may be observed that the actual lunar day is not of constant length, because the moon's orbit is an ellipse, not a circle; because it is

* Generally, but improperly, called "sidereal day."

† Méc. Cél., Book IV, §§ 16, 17.

inclined to the plane of the earth's equator, and this inclination is not constant; also because the sun disturbs the moon, producing evection and variation. These irregularities in the moon's apparent diurnal motion will, sooner or later, be in certain ways reflected in the lunar tide, which but for them might be assumed to be a wave of uniform period for all places, and of constant amplitude at any given place. Denoting the strictly periodic portion of the tide by M_2, let us inquire, what other strictly periodic oscillations or components of about the same period ought to exist in the lunar tide?

Let m_2 denote the hourly speed of M_2, and Π the period, in hours, of some inequality or irregularity in the moon's motion. If a component have the speed

$$m_2 \pm \frac{360}{\Pi} \tag{5}$$

it will gain or lose on M_2 one period during the time Π, as can be seen by multiplying this speed by Π.* In other words, Π is a synodic period for M_2 and a component with either of the above speeds. The nature of the inequality must be taken into account in order to ascertain which sign to take, and also whether, for the present purpose, Π should include the whole or the half period as usually given. The development of the tide-producing potential shows that the two components, one for each sign, are sometimes required.

If Π = an anomalistic month, then

$$m_2 + \frac{360}{\Pi} = 28{\cdot}9841042 + \frac{360}{661{\cdot}309207} = 29{\cdot}5284789,$$

$$m_2 - \frac{360}{\Pi} = 28{\cdot}9841042 - \frac{360}{661{\cdot}309207} = 28{\cdot}4397295.$$

This suggests that there should exist because of the irregularity in the moon's motion due to its varying parallax, one or both of these components, which may be denoted by L_2 and N_2. Since the moon's apparent diurnal motion is slowest when she is in perigree, the perigean tides should have a longer period than the mean tide; and because of the nearness of the moon to the earth, the range should be increased. The most of the parallax inequality in the tide must be due to that component which, when coinciding with M_2, increases the length of the period; in other words, N_2 is the *larger lunar elliptic component*, and L_2 the *smaller lunar elliptic component.*

If Π = a half tropical month, then

$$m_2 + \frac{360}{\Pi} = 28{\cdot}9841042 + \frac{360}{327{\cdot}858979} = 30{\cdot}0821373,$$

$$m_2 - \frac{360}{\Pi} = 28{\cdot}9841042 - \frac{360}{327{\cdot}858979} = 27{\cdot}8860711.$$

Since the moon's apparent diurnal motion is, *cæteris paribus*, greatest when she is in the equator, and since her tendency to produce tides is then greatest (Principia, Bk. III, Prop. XXIV), the speed of the component causing the declinational inequality in the semidiurnal tide is greater than the speed of M_2. This component is the lunar part of K_2. The other component is not required in connection with this inequality.

If Π = a half synodic month, i. e., the moon's variational period, then

$$m_2 + \frac{360}{\Pi} = 28{\cdot}9841042 + \frac{360}{354{\cdot}367057} = 30{\cdot}0000000,$$

$$m_2 - \frac{360}{\Pi} = 28{\cdot}9841042 - \frac{360}{354{\cdot}367057} = 27{\cdot}9682084.$$

---

* Cf. Laplace, Méc. Cél., Bk. XIII, § 3.

Cœteris paribus, the apparent diurnal motion of the moon is least when she is in the syzygies, and her distance to the earth is then least. (Principia, Bk. I, Prop. LXVI, or § 87, Part I.) This shows that the lunar tide is then greater than usual and of longer period; consequently the component causing the variational inequality in the semidiurnal tide is the one whose speed is less than that of M_2. It is denoted by μ_2. If the other component, whose speed is 30° per hour, exist at all it will unite with S_2.

Let Π = half of the evectional period in the moon's parallax. Every time the line of apsides passes the sun, the eccentricity of the lunar orbit becomes a maximum (Principia, Bk. I, Prop. LXVI). That is, the amplitude of the oscillation (N_2) which mainly accounts for the parallax effect upon the tide, would, if no additional component were introduced, have an inequality of a period of about 206 days. A component which would probably represent such an inequality must have for its speed one of the two values,

$$\begin{aligned} n_2 + \frac{360}{\Pi} &= 28{\cdot}4397295 + \frac{360}{4941{\cdot}415931} = 28{\cdot}5125831, \\ n_2 - \frac{360}{\Pi} &= 28{\cdot}4397295 - \frac{360}{4941{\cdot}415931} = 28{\cdot}3668759. \end{aligned} \tag{6}$$

At such times of greatest eccentricity the progression of the line of apsides becomes a maximum. This increases the anomalistic month, and so, by the above formula for the speed of N_2, must increase the speed of the oscillation representing the most of the parallax effect at the time when its amplitude becomes a maximum. Consequently, the principal component due to the moon's evection should have the speed 28·5125831. This component is designated as ν_2.

16. Whenever the moon is not upon the equator, the two tides of a day will generally differ because the moon's north polar distance at the time of a superior transit is not equal to her south polar distance at the time of an inferior transit. In other words, if we suppose the moon and anti-moon to successively cross the meridian of a place, the one will be a body of north declination and the other a body of south declination. It would be natural to try to represent this inequality by a wave of variable amplitude attaining a maximum when the moon is far from the equator and vanishing at about the time when the moon crosses the equator. The speed of such a wave should be m_1, or the apparent diurnal motion of the moon about the earth. When the amplitude becomes zero the phase of the wave should suddenly change by 180°. To avoid this great variability in amplitude and this sudden change of phase, a component, lunar K_1, is suggested which shall gain 360° on the moon in a tropical month, and another component, O_1, of about equal amplitude, which shall lose the same amount; for, the speed of the resultant wave will evidently be that of the moon or m_1, its amplitude will be a maximum when the moon is far from the equator, either north or south, and zero when she is upon or near the equator.

If Π = a tropical month, then

$$\begin{aligned} m_1 + \frac{360}{\Pi} &= 14{\cdot}4920521 + \frac{360}{655{\cdot}717958} = 15{\cdot}0410686 = k_1, \\ m_1 - \frac{360}{\Pi} &= 14{\cdot}4920521 - \frac{360}{655{\cdot}717958} = 13{\cdot}9430356 = o_1. \end{aligned} \tag{7}$$

The same line of reasoning would lead one to infer that all lunar components might be accompanied by small components* of almost the same speeds as themselves for taking into account the effects of the regression of the moon's node. For most purposes it is preferable to suppose this inequality accounted for by means of slight variations in the amplitude and epochs of the components which do not owe their origin to the movement of the node.

* Styled *lunar nodal* components. See Ferrel's Tidal Researches, p. 43; also United States Coast and Geodetic Survey Report, 1878, pp. 270 et seq. For example, let Π = the node-equinox period; then $m_2 - \frac{360}{\Pi} = 28{\cdot}9818978$ = the speed of Ferrel's $M'_{(1,2)}$. Again, $k_1 + \frac{360}{\Pi} = 15{\cdot}0432750$ = the speed of Ferrel's $M'_{(3,1)}$, and $o_1 - \frac{360}{\Pi} = 13{\cdot}9408292$ = the speed of Ferrel's $M'_{(6,1)}$.

Considering now the portion of the tide due to the sun, and placing $\varPi$ = an anomalistic year, then

$$\begin{aligned} s_2 + \frac{360}{\varPi} &= 30{\cdot}0000000 + \frac{360}{8766{\cdot}231391} = 30{\cdot}0410667, \\ s_2 - \frac{360}{\varPi} &= 30{\cdot}0000000 - \frac{360}{8766{\cdot}231391} = 29{\cdot}9589333, \end{aligned} \tag{8}$$

and we obtain two speeds which are denoted by r_2 and t_2, respectively.

By placing $\varPi$ = a half tropical year, the first speed becomes equivalent to k_2. By placing $\varPi$ = a tropical year, then

$$\begin{aligned} s_1 + \frac{360}{\varPi} &= 15{\cdot}0000000 + \frac{360}{8765{\cdot}812774} = 15{\cdot}0410686, \\ s_1 - \frac{360}{\varPi} &= 15{\cdot}0000000 - \frac{360}{8765{\cdot}812774} = 14{\cdot}9589314, \end{aligned} \tag{9}$$

and we obtain two speeds of which the first is k_1 and the second is denoted by p_1.

There are other oscillations having a truly astronomical origin, but their speeds, as a rule, are less readily obtained from the mean motions of the moon and sun than from the speeds of certain components like O_1, K_1, etc. (See Table 2, already referred to.)

17. *Overtides.*

So far we have been mainly concerned with the periods of the oscillations. That they are simply harmonic is not self evident. In fact, where the water is shallow the crest of the tidal wave must generally move faster than the trough. This, as the wave proceeds, causes the duration of fall to exceed the duration of rise. If upon a simple harmonic oscillation we superpose in a suitable manner a small one of double the speed, we obtain the effect desired. We are therefore led to infer that there probably exist along with M_2, S_2, N_2, etc., the components M_4, S_4, N_4, etc. Again, it seems natural to suppose that, for instance, the principal lunar part of the tide may have other departures than the kind just noted from a simple harmonic form. But whatever its shape may be, it can be represented by a Fourier series of terms whose speeds are simple multiples, like 2, 3, 4, etc., of the speed of the principal component. That is, M_2 may naturally enough in shallow water be accompanied by M_4, M_6, M_8, etc. So for S_2, and other components. These are sometimes called overtides because of their analogy to overtones in musical sounds.

18. *Compound tides.*

In shallow water there may be sensible compound tides; that is, components whose speeds are the sums or differences of the speeds of the principal components. These were suggested by Helmholtz's theory of compound sounds. In fact, we have only to multiply together, in pairs, the principal simple harmonic (cosine) terms which constitute the tide in order to ascertain the theoretical relations between their amplitudes, arguments, and epochs. Ferrel gives quite extended lists of such components in the Report of the United States Coast and Geodetic Survey for 1878, pp. 274–276 (about equivalent to Table 36), and Darwin, a list of the more important cases in the British Association Report for 1883, pp. 74–78.

The velocity of propagation for a high water of, say, the component M_2 is

$$\sqrt{g(h + 3\,M_2)} = (gh)^{\frac{1}{2}} + \tfrac{3}{2}\frac{gM_2}{(gh)^{\frac{1}{2}}}, \tag{10}$$

and of the trough

$$\sqrt{g(h - 3\,M_2)} = (gh)^{\frac{1}{2}} - \tfrac{3}{2}\frac{gM_2}{(gh)^{\frac{1}{2}}}, \tag{11}$$

g being the acceleration of gravity and h the depth at half-tide level. The difference between these two velocities is proportional to

$$\left(\frac{g}{h}\right)^{\frac{1}{2}} M_2. \tag{12}$$

At a given place (since g and h are constant) the time distortion is evidently proportional to M_2; for, to a first approximation, all components, whatever their amplitudes, require about the same time in their propagation from point to point—the velocity being $(gh)^{\frac{1}{2}}$. For convenience, suppose M_2 to be a somewhat varying amplitude, that is, let M_2 not stand for the true amplitude of M_2, but the resultant amplitude when combined with another component of about equal speed. Let its value on two different occasions be denoted by $(M_2)_{\prime}$, $(M_2)_{\prime\prime}$. The question now arises, how do the corresponding amplitudes of M_4 compare with each other? In the first case the duration of fall exceeds one-fourth lunar day by an amount proportional to $(M_2)_{\prime}$, and in the second, to $(M_2)_{\prime\prime}$, according to the equations just obtained But it can be readily seen by combining waves [eqs. (60), (61), Part III] that this excess in the one case is proportional to $\frac{(M_4)_{\prime}}{(M_2)_{\prime}}$ and in the other to $\frac{(M_4)_{\prime\prime}}{(M_2)_{\prime\prime}}$.

$$\therefore \frac{(M_4)_{\prime}}{(M_2)_{\prime}} : \frac{(M_4)_{\prime\prime}}{(M_2)_{\prime\prime}} = (M_2)_{\prime} : (M_2)_{\prime\prime}$$

or

$$(M_4)_{\prime} : (M_4)_{\prime\prime} = (M_2)_{\prime}^2 : (M_2)_{\prime\prime}^2; \tag{13}$$

that is, if the amplitude of a component be different on two occasions, the amplitude of the overtide having double the speed of the fundamental, will vary as the square of the amplitude of the fundamental. (Applying this to the diurnal and semidiurnal waves, one might perhaps surmise that the amplitude of the diurnal wave, and so the diurnal inequalities, vary, from place to place, as the square root of the semidiurnal range.)

Suppose that the cause of the variation in M_2 is the addition of S_2, producing spring tides. At the times of the spring tides, what is the amplitude x of the overtide M_4?

$$M_4 : x = M_2^2 : (M_2+S_2)^2,$$

$$\therefore x = M_4 \left(1+\frac{2\,S_2}{M_2}+\frac{S_2^2}{M_2^2}\right). \tag{14}$$

At the neap tides we have

$$x = M_4 \left(1-\frac{2S_2}{M_2}+\frac{S_2^2}{M_2^2}\right). \tag{15}$$

Thus we see that if the quarter diurnal component is to have an approximately fixed position upon the semidiurnal wave it must have a variable amplitude. But, just as in § 15, we infer the speeds of one or two additional quarter diurnals by putting Π = a half synodic month expressed in hours.

$$\left.\begin{aligned} m_4+\frac{360}{\Pi} &= 57{\cdot}9682084+\frac{360}{354{\cdot}367057} = 58{\cdot}9841044, \\ m_4-\frac{360}{\Pi} &= 57{\cdot}9682084-\frac{360}{354{\cdot}367057} = 56{\cdot}9523124. \end{aligned}\right\} \tag{16}$$

The component defined by the first speed $(=m_2+s_2)$ is MS or $(MS)_4$. The values of x just written show that the amplitude is given by the relation

$$MS = \frac{2S_2}{M_2} \times M_4. \tag{17}$$

The speed of MS being the arithmetical mean between the speeds of M_4 and S_4, the period of MS is, of course, the harmonic mean between the periods of M_4 and S_4.

By putting Π = one-fourth of a synodic month, the speed defining the component S_4 is obtained. The values of x show that

$$S_4 = \frac{S_2^2}{M_2^2} \times M_4, \tag{18}$$

which might have been inferred from (13) by writing S_2 for one of the M_2's.

Similarly, there should be compound tides dependent upon M_2, N_2; M_2, K_2; etc. So for S_2, N_2; etc.

The speed of MS being the speed of M_2 plus the speed of S_2, its phase must of course vary as the sum of the phases of M_2 and S_2 varies, and so it is customary to take its initial argument equal to the sum of the initial equilibrium arguments of M_2 and S_2. At spring tides MS should conspire with M_4, and at neap tides it should interfere. The speed of M_4 is equal to twice the speed of M_2, and it is customary to take for its initial argument twice the initial equilibrium argument of M_2.

$$\therefore \mathrm{ms} = \mathrm{m}_2 + \mathrm{s}_2; \tag{19}$$

$$\arg_0 \mathrm{MS} = \arg_0 \mathrm{M}_2 + \arg_0 \mathrm{S}_2; \tag{20}$$

$$\text{phase MS } (= \text{phase } \mathrm{M}_4, \text{ at springs}) = \text{phase } \mathrm{M}_4 + \text{phase } \mathrm{S}_2 - \text{phase } \mathrm{M}_2; \tag{21}$$

$$\text{phase MS } (= \text{phase } \mathrm{M}_4 \pm 180°, \text{ at neaps}) = \text{phase } \mathrm{M}_4 + \text{phase } \mathrm{S}_2 - \text{phase } \mathrm{M}_2; \tag{22}$$

$$\mathrm{ms}t + \arg_0 \mathrm{MS} - \mathrm{MS}° = \mathrm{m}_4 t + \arg_0 \mathrm{M}_4 - \mathrm{M}_4° + \mathrm{s}_2 t + \arg_0 \mathrm{S}_2 - \mathrm{S}_2° - (\mathrm{m}_2 t + \arg_0 \mathrm{M}_2 - \mathrm{M}_2°); \tag{23}$$

and so

$$\mathrm{MS}° = \mathrm{M}_4° + \mathrm{S}_2° - \mathrm{M}_2°. \tag{24}$$

The more general treatment of this subject is given in § 48.

19. *Meteorological tides.*

The land and sea breezes, and the daily variation in atmospheric pressure, may give rise to a tide whose period is a solar day; and, as the cause is not directly astronomical, it is natural to suppose that overtides would have to accompany the simple harmonic form in order to represent a tide whose origin is so remote.

The change of seasons gives rise to an annual tide, Sa. Such a tide must represent the stages of rivers at river stations. As very high stages are usually of comparatively short duration, it is not reasonable to suppose that a single component can represent the annual changes in river level. In other words, the overtides become comparatively large.

It is hardly necessary to add that while the determination of meteorological tides from long series of observations is valuable for some purposes, their recurrence is not generally certain enough to make them of much value in tidal predictions.

The foregoing has led to, or at least suggested, the most of the components which are to be sought in the process of analyzing the tidal wave. Some of those already brought to notice are, from the nature of their origin, too small to be included in a working schedule.

It may be worth while to here call attention to a practical application of § 18, or rather to empiral rules there hinted at.

20. *Rules for inferring certain nonharmonic quantities from values at a neighboring station.*

Suppose that for a secondary station we know the value of a semidiurnal range of tide, say Mn, and wish to estimate the value of another semidiurnal range, say Sg, or Np from a neighboring principal station where the tide is supposed to be fully known. We naturally put

$$(\mathrm{Sg})_{\prime\prime} = (\mathrm{Sg})_{\prime} \frac{(\mathrm{Mn})_{\prime\prime}}{(\mathrm{Mn})_{\prime}}, \; \therefore \; (\mathrm{Mn})_{\prime\prime} = (\mathrm{Mn})_{\prime} \frac{(\mathrm{Sg})_{\prime\prime}}{(\mathrm{Sg})_{\prime}}; \tag{25}$$

$$(\mathrm{Np})_{\prime\prime} = (\mathrm{Np})_{\prime} \frac{(\mathrm{Mn})_{\prime\prime}}{(\mathrm{Mn})_{\prime}}. \tag{26}$$

Similarly for perigean and apogean ranges. But § 18 suggests the hypothesis which observations have in a measure corroborated, that the amplitude of the diurnal wave and all quantities proportional thereto vary between neighboring stations not as the ratio of the mean ranges of tide varies, but rather as the square root of this ratio. Consequently, putting temporarily k for $\sqrt{\frac{(\mathrm{Mn})_{\prime\prime}}{(\mathrm{Mn})_{\prime}}}$ we have*

$$(\mathrm{D}_1)_{\prime\prime} = (\mathrm{D}_1)_{\prime} k, \tag{27}$$

$$(\mathrm{HWQ})_{\prime\prime} = (\mathrm{HWQ})_{\prime} k, \tag{28}$$

$$(\mathrm{LWQ})_{\prime\prime} = (\mathrm{LWQ})_{\prime} k, \tag{29}$$

$$(\text{depression of mean LLW below MSL})_{\prime\prime} = \tfrac{1}{2}(\mathrm{Mn})_{\prime\prime} + [(\text{ditto})_{\prime} - \tfrac{1}{2}(\mathrm{Mn})_{\prime}]k, \tag{30}$$

* For notation see §§ 2–7, Part I.

(depression of Indian or harmonic tide plane below MSL)$_{,,}$

$$= 0{\cdot}49\ (\mathrm{Sg})_{,,} + [(\mathrm{ditto})_{,} - 0{\cdot}49\ (\mathrm{Sg})_{,}]\ k, \tag{31}$$

$$= 0{\cdot}49\ [(\mathrm{Sg})_{,,} + (\mathrm{D}_1)_{,,}],\ \text{approximately.} \tag{32}$$

Here "ditto" refers to the left-hand member.

The Indian tide plane is $M_2 + S_2 + K_1 + O_1$ below mean sea level (MSL). (33)

(depression of tropic LLW below MSL)$_{,,}$

$$= [(\mathrm{ditto})_{,} - \tfrac{1}{2}(\mathrm{Mn})_{,}]\ \frac{(\mathrm{LWQ})_{,,}}{(\mathrm{LWQ})_{,}}, \tag{34}$$

$$(\mathrm{Gc})_{,,} = (\mathrm{Mn})_{,,} + [(\mathrm{Gc})_{,} - (\mathrm{Mn})_{,}]\ \frac{(\mathrm{HWQ})_{,,} + (\mathrm{LWQ})_{,,}}{(\mathrm{LWQ})_{,} + (\mathrm{LWQ})_{,}} = (\mathrm{Mn})_{,,} + [(\mathrm{Gc})_{,} - (\mathrm{Mn})_{,}]\ k, \tag{35}$$

$$(\text{tropic HHWI})_{,,} = (\mathrm{HWI})_{,,} + [(\text{tropic HHWI})_{,} - (\mathrm{HWI})_{,}]\ \frac{(\mathrm{Mn})_{,}}{(\mathrm{Mn})_{,,}} \cdot \frac{(\mathrm{D}_1)_{,,}}{(\mathrm{D}_1)_{,}}, \tag{36}$$

$$(\text{tropic LLWI})_{,,} = (\mathrm{LWI})_{,} + [(\text{tropic LLWI})_{,} - (\mathrm{LWI})_{,}]\ \frac{(\mathrm{Mn})_{,}}{(\mathrm{Mn})_{,,}} \cdot \frac{(\mathrm{D}_1)_{,,}}{(\mathrm{D}_1)_{,}}. \tag{37}$$

When $(\mathrm{D}_1)_{,,}$ is known from observation, the value of k from (27) should be used.

21. *Mean longitude.*

The *mean longitude* of a body may be defined as the distance along any given great circle moved over by a uniformly moving or "mean" body from the intersection of its orbit and the circle. It is simply the time since the body passed such intersection multiplied by the average angular velocity of the body. For a fixed body or point the mean, as well as the true, longitude is measured from the foot of the perpendicular let fall from the body or point in question upon the circle in which longitude is reckoned.

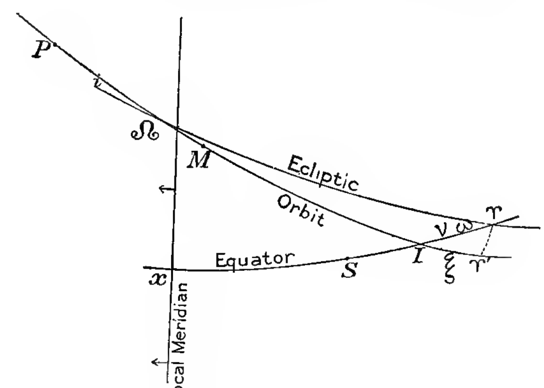


FIG. 6.

If two or more great circles meet in a point, each circle will have a mean body of its own; but all these mean bodies will be at the same distance, at any given instant, from the origin; that is, the real body will have the same mean longitude reckoned in whichever circle. If origins in the several circles be taken a constant distance from the common intersection, the mean longitude in each circle will be altered by the same amount.

The mean longitude of the sun (h) from ♈ is the same in the equator as in the ecliptic. If ν denote the R. A. or mean longitude of I in the equator (see Fig. 6), then $h - \nu$ is the sun's mean longitude from the point I. If we take a point ♈′ upon the moon's orbit such that ☊ ♈′ = ☊ ♈, and denote the distance ♈′ I by ξ, then the moon's mean longitude from I, whether in the orbit or the equator, is $s - \xi$, s being her mean longitude from ♈′ or ♈ in the orbit or the equator.

Fig. 6 supposes the observer to be within the celestial sphere looking outward. The ecliptic, moon's orbit, and the celestial equator are supposed to be fixed, while the projection of the terrestrial meridian upon the celestial sphere moves eastward, as indicated by the arrows, and at the rate of 15° per sidereal hour. S' is a slowly moving point upon the equator, distant 180° from the *mean sun* (S), and so ♈ $S' = h \pm 180°$. When the meridian passes over the point S' it is mean local midnight, where we shall assume t, the mean solar time, to be zero. When the meridian is at x, then t, or 15 t, = $S'x$. The hour angle of ♈ may be temporarily denoted by g. M denotes the

position of the mean moon, and P that of the lunar perigee. The following eqnations are given for convenience of reference; t is the time expressed in degrees instead of hours:

$$
\begin{array}{lll}
\Upsilon I = \nu, & \Upsilon' I = \xi, & Ix = \chi = t + h \pm 180^\circ - \nu, \\
\Upsilon S = h, & \Upsilon' M = s, & IM = \sigma_{,} = s - \xi, \\
\Upsilon S' = h \pm 180^\circ, & \Upsilon' P = p, & IP = \varpi_{,} = p - \xi. \\
\Upsilon x = g = t + h \pm 180^\circ, & \Upsilon' \Omega = N, & \\
\Upsilon \Omega = N, & S'x = t, &
\end{array}
\qquad (38)
$$

Most tables of the moon and sun give mean longitude of the lunar and solar elements reckoned in the orbit or the ecliptic and from the origin Υ. To refer any of them to the equator, using I as origin, it is necessary to know the mean longitude of the moon's node (N); then ν and ξ can be obtained by solving the spherical triangle $\Upsilon I \Omega$. To avoid this labor the values of ν and ξ for any value of N may be taken from Table 7.

In all of the work pertaining to the reduction and prediction of tides, we shall adopt the following (Hansen's) values for s, p, h, p_1, and N, which are those used by Darwin in his report for 1883:

$$
\begin{aligned}
s &= 150^\circ{\cdot}0419 + [13 \times 360^\circ + 132^\circ{\cdot}67900]T + 13^\circ{\cdot}1764D + 0^\circ{\cdot}5490165H, \\
p &= 240^\circ{\cdot}6322 + 40^\circ{\cdot}69035T + 0^\circ{\cdot}1114D + 0^\circ{\cdot}0046418H, \\
h &= 280^\circ{\cdot}5287 + 360^\circ{\cdot}00769T + 0^\circ{\cdot}9856D + 0^\circ{\cdot}0410686H, \\
p_1 &= 280^\circ{\cdot}8748 + 0^\circ{\cdot}01711T + 0^\circ{\cdot}000047D, \\
N &= 285^\circ{\cdot}9569 - 19^\circ{\cdot}34146T - 0^\circ{\cdot}052954D.
\end{aligned}
\qquad (39)
$$

where T is the number of Julian years of $365\frac{1}{4}$ mean solar days each; D the number of mean solar days; H the number of mean solar hours after Greenwich mean noon January 1, 1880. On account of the slowness of the secular changes in the coefficients of T, D, or H the epoch of this table may, for tidal purposes, be regarded as 1900. See Hansen's Tables de la Lune, p. 15, from which these formulæ may be obtained by putting t=80. Newcomb's corrections are not of sufficient magnitude to seriously alter these values.

Affecting the symbols employed by Hansen with a zero subscript, to indicate that Newcomb's corrections have been applied, we have*

$$
\begin{aligned}
s &= 360^\circ - \odot_0 + \omega_0 + \gamma_0 = 335^\circ{\cdot}723766 + (13 \times 360^\circ + 132^\circ{\cdot}67886233)\ (t-1800) + 0^\circ{\cdot}002623\left(\frac{t-1800}{100}\right)^2 + 0^\circ{\cdot}000004\left(\frac{t-1800}{100}\right)^3, \\
p &= 360^\circ - \odot_0 + \omega_0 = 225^\circ{\cdot}398072 + 40^\circ{\cdot}69050683\ (t-1800) - 0^\circ{\cdot}010037\left(\frac{t-1800}{100}\right)^2 - 0^\circ{\cdot}000010\left(\frac{t-1800}{100}\right)^3, \\
h &= 360^\circ - \odot_0 + \omega_0' + \gamma_0' = 279^\circ{\cdot}913791 + 360^\circ{\cdot}00768417\ (t-1800) + 0^\circ{\cdot}000308\left(\frac{t-1800}{100}\right)^2, \\
p_1 &= 360^\circ - \odot_0 + \omega_0' = 279^\circ{\cdot}505952 + 0^\circ{\cdot}01710689\ (t-1800) + 0^\circ{\cdot}000464\left(\frac{t-1800}{100}\right)^2, \\
N &= 360^\circ - \odot_0 = 33^\circ{\cdot}275319 - 19^\circ{\cdot}34147114\ (t-1800) + 0^\circ{\cdot}002275\left(\frac{t-1800}{100}\right)^2 + 0^\circ{\cdot}000002\left(\frac{t-1800}{100}\right)^3,
\end{aligned}
\qquad (40)
$$

where t is the number of Julian year from the epoch 1800·000, or noon December 31, 1799. By making t=100, values for s, p, h, p_1, and N are obtained for noon, January 1, 1900; at that time $s = 283^\circ{\cdot}612626$, $p = 334^\circ{\cdot}438708$, $h = 280{\cdot}682516$, $p_1 = 281^\circ{\cdot}217105$, $N = 259^\circ{\cdot}130482$. Tables 3, 4, and 6, which do not involve Newcomb's corrections, give for the same time, $s = 283^\circ{\cdot}62$, $p = 334^\circ{\cdot}44$, $h = 280^\circ{\cdot}68$, $p_1 = 281^\circ{\cdot}22$, $N = 259^\circ{\cdot}13$.

Newcomb's recent values† for h and p are for the epoch 1900.

22. *Formulæ for computing I, ν, and ξ.*‡

---

* Researches on the Motions of the Moon, pp. 268, 274, Washington Observations, Vol. 22, 1875; Harkness, Solar Parallax, pp. 13, 14, Washington Observations, 1885, Appendix III.

† Tables of the Sun, p. 9, Astronomical Papers, Vol. VI, 1895.

‡ B. A. A. S. Report, 1883, pp. 83, 84.

From Fig. 6

$$\cot (N - \xi) \sin N = \cos N \cos i + \sin i \cot \omega; \tag{41}$$

whence

$$\tan \xi = \frac{\sin N \cos N (\cos i - 1) + \sin N \sin i \cot \omega}{\sin^2 N + \cos^2 N \cos i + \sin i \cot \omega \cos N}, \tag{42}$$

$$= \frac{\sin i \cot \omega \sin N(1 - \tan \frac{1}{2} i \tan \omega \cos N)}{\cos^2 \frac{1}{2} i + \sin i \cot \omega \cos N - \sin^2 \frac{1}{2} i \cos 2 N}. \tag{43}$$

From the figure

$$\cos I = \cos i \cos \omega - \sin i \sin \omega \cos N, \tag{44}$$

$$\sin \nu = \sin i \operatorname{cosec} I \sin N; \tag{45}$$

whence

$$\tan \nu = \frac{\tan i \operatorname{cosec} \omega \sin N}{1 + \tan i \cot \omega \cos N}. \tag{46}$$

Regarding i as small we have

$$\tan \xi = i \cot \omega \sin N - \tfrac{1}{2} i^2 \sin 2 N \frac{1 - \frac{1}{2} \sin^2 \omega}{\sin^2 \omega}, \tag{47}$$

$$\tan \nu = i \operatorname{cosec} \omega \sin N - \tfrac{1}{2} i^2 \sin 2 N \frac{\cos \omega}{\sin^2 \omega}, \tag{48}$$

$$\cos I = (1 - \tfrac{1}{2} i^2) \cos \omega - i \sin \omega \cos N. \tag{49}$$

Table 7, taken from Baird's Manual, was computed upon the assumption that $\omega = 23^\circ\ 27'{\cdot}3$ and $i = 5^\circ\ 8'{\cdot}8$.

23. *Kepler's problem.*

By analytical geometry the vectorial angle (v),* reckoned from the perigee, is connected with the eccentric angle (E) by the equation

$$\cos v = \frac{\cos E - e}{1 - e \cos E}, \tag{50}$$

e being the eccentricity of the orbit. By Kepler's second law equal areas are described in equal times, and so the mean (M) and eccentric anomalies are connected by the equation

$$M = E - e \sin E. \tag{51}$$

The elimination of E from these two equations constitutes the solution of Kepler's problem. For a complete solution see books on mathematical astronomy, such as Dziobek's Mathematical Theories of Planetary Motions, Résal's Mécanique Céleste, etc. For most tidal purposes it is sufficient to carry the solution to the second powers of e, and this may be easily accomplished in the following manner:

If we develop by Taylor's theorem the value of v from the first equation written in the form

$$v = \cos^{-1} (\cos E + h) \tag{52}$$

where

$$h = -\frac{e \sin^2 E}{1 - e \cos E} \tag{53}$$

we obtain

$$v = E + e \sin E + \tfrac{1}{4} e^2 \sin 2 E + \ \ldots \tag{54}$$

The second equation gives

$$M = E - e \sin (M + e \sin M),$$

or

$$E = M + e \sin (M + e \sin M); \tag{55}$$

$$\therefore E = M + e \sin M + \tfrac{1}{2} e^2 \sin 2 M, \text{ to the second power of } e, \tag{56}$$

$$e \sin E = e \sin M + \tfrac{1}{2} e^2 \sin 2 M, \text{ to the second power of } e, \tag{57}$$

---

* The present use of the letters v, E, M, and h will probably be confined to this paragraph.

$$\tfrac{1}{4}\,e^2 \sin 2\,E = \tfrac{1}{4}\,e^2 \sin 2\,M, \text{ to the second power of } e; \tag{58}$$

$$\therefore v = M + 2\,e \sin M + \tfrac{5}{4}\,e^2 \sin 2\,M, \text{ to the second power of } e. \tag{59}$$

24. *Reduction to the equator, and conversely.*

Let A denote the inclination of an orbit to the equator, and B a right angle formed by the equator and the hour circle passing through the body. By spherical trigonometry we have

$$\tan b = \frac{\tan c}{\cos A} = \tan c\,(1 - \sin^2 A)^{-\frac{1}{2}}, \tag{60}$$

$$= \tan c\,(1 + \tfrac{1}{2}\sin^2 A + \tfrac{3}{8}\sin^4 A + \quad . \quad . \quad . \quad); \tag{61}$$

$$\therefore\; b = \tan^{-1}(\tan c + k), \text{ say.} \tag{62}$$

By Taylor's theorem

$$b = c + \tfrac{1}{4}\sin 2c \sin^2 A + (\tfrac{1}{8}\sin 2c + \tfrac{1}{32}\sin 4c)\sin^4 A + \quad . \quad . \quad . \quad . \tag{63}$$

The usual formula for this is, unless the exact solution by trigonometry be preferred,

$$b = c + \tan^2 \tfrac{1}{2}A \sin 2c - \tfrac{1}{2}\tan^4 \tfrac{1}{2}A \sin 4c + \tfrac{1}{3}\tan^6 \tfrac{1}{2}A \sin 6c - \quad . \quad . \quad . \quad ; \tag{64}$$

but for certain purposes it is convenient to expand in powers of $\sin A$. Also,

$$\tan c = \tan b \cos A = \tan b\,(1 - \sin^2 A)^{\frac{1}{2}}, \tag{65}$$

$$= \tan b\,(1 - \tfrac{1}{2}\sin^2 A - \tfrac{1}{8}\sin^4 A); \tag{66}$$

$$\therefore\; c = \tan^{-1}(\tan b + k'), \text{ say,} \tag{67}$$

$$= b - \tfrac{1}{4}\sin 2b \sin^2 A - (\tfrac{1}{8}\sin 2b - \tfrac{1}{32}\sin 4b)\sin^4 A. \tag{68}$$

25. *Approximate expressions for the right ascension of the sun or moon.*

The object of this paragraph is to show that if we displace in time the strictly periodic portion of the equilibrium lunar (solar) tide by the lunar (solar) components, the resulting wave will have its crest beneath the tidal body or 180° therefrom.

The true longitude of the sun is, by Kepler's problem, § 23,

$$l = h + 2e_1 \sin (h - p_1) + \tfrac{5}{4}e_1^2 \sin 2\,(h - p_1), \tag{69}$$

where h denotes the mean longitude of the sun, p_1 that of the solar perigee, and e_1 the eccentricity of the earth's orbit $= 0{\cdot}01679$. The coefficients $2e_1$, $\frac{5}{4}e_1^2$, are converted into degrees by means of the factor 57·3, thus giving

$$l = h + 1^\circ{\cdot}924 \sin (h - p_1) + 0^\circ{\cdot}020 \sin 2\,(h - p_1). \tag{70}$$

The right ascension corresponding to l is, § 24,

$$\alpha = l - \tfrac{1}{4}\sin^2 \omega \sin 2\,l - \sin^4 \omega\,(\tfrac{1}{8}\sin 2\,l - \tfrac{1}{32}\sin 4\,l), \tag{71}$$

$$= h + 1^\circ{\cdot}924 \sin (h - p_1) + 0^\circ{\cdot}020 \sin 2\,(h - p_1) - 2^\circ{\cdot}450 \sin 2\,h + 0^\circ{\cdot}045 \sin 4\,h, \tag{72}$$

ω, the obliquity of the ecliptic being taken as 23° 27′·3.

Now since all solar tides are due to the sun, the crest of the solar wave composed of these partial tides ought, in the case of semidiurnals, to have its phase equal to twice the hour angle of the sun.

Sun's right ascension × 2

$$\begin{aligned} = 2\,h &- \text{acceleration in } S_2 \text{ due to } T_2 \\ &- \quad \text{"} \qquad \text{"}\ S_2 \ \text{"} \ \text{"}\ R_2 \\ &- \quad \text{"} \qquad \text{"}\ S_2 \ \text{"} \ \text{"}\ \text{solar } K_2\text{, Table 15.} \end{aligned} \tag{73}$$

This is approximately equal to

$$2\,h - \frac{T_2}{S_2}\frac{t_2}{s_2}\sin\,(\arg T_2 - \arg S_2, = p_1 - h)$$

$$- \frac{R_2}{S_2}\frac{t_2}{s_2}\sin\,(\arg R_2 - \arg S_2, = h - p_1 + 180^\circ)$$

$$- \frac{K_2}{S_2}\frac{k_2}{s_2}\sin\,(\arg K_2 - \arg S_2, = 2\,h). \tag{74}$$

Here "arg" stands for the equilibrium argument, $V+u$, of Table 1; it is equivalent to the phase of the oscillations, if its epoch (or lag) be zero; K_2 here denotes the solar part of K_2.

Putting in the theoretical values of these component ratios, we have for twice the sun's right ascension *

$$2\,h - 3^\circ{\cdot}32 \sin\,(p_1 - h) + 0^\circ{\cdot}47 \sin\,(h - p_1) - 4^\circ{\cdot}95 \sin 2\,h;$$

or

$$2\,h + 3^\circ{\cdot}79 \sin\,(h - p_1) - 4^\circ{\cdot}95 \sin 2\,h, \tag{75}$$

the half of which is

$$h + 1^\circ{\cdot}90 \sin\,(h - p_1) - 2^\circ{\cdot}48 \sin 2\,h. \tag{76}$$

This value of the sun's right ascension agrees well with value (72). Thus it is seen that T_2, R_2, and solar K_2 account for irregularities in the solar wave corresponding well with the irregularities in the motion of the sun. The quantity $+ 1^\circ{\cdot}90 \sin\,(h - p_1) - 2^\circ{\cdot}48 \sin 2\,h$ is, when converted into time, very nearly the equation of time (to change apparent to mean time), Table 30. For,

Sun's right ascension = right ascension of mean sun + equation of time.

In a precisely similar manner the right ascension of the moon reckoned from I is

$$\sigma_{,} + 6^\circ{\cdot}292 \sin\,(\sigma_{,} - \varpi_{,}) + 0^\circ{\cdot}216 \sin 2\,(\sigma_{,} - \varpi_{,})$$

$$- 14^\circ{\cdot}324 \sin^2 I \sin 2\,\sigma_{,} - \sin^4 I\,(7^\circ{\cdot}162 \sin 2\,\sigma_{,} - 1^\circ{\cdot}790 \sin 4\,\sigma_{,}), \tag{77}$$

$\sigma_{,}$ denoting the mean longitude of the moon from I, $\varpi_{,}$ that of the lunar perigee, the eccentricity of the orbit being 0·05491. A more accurate value from Hansen's tables is, taking into account the higher powers of e, the evection, variation, and the annual inequality,

$$\sigma_{,} + 6^\circ{\cdot}290 \sin\,(\sigma_{,} - \varpi_{,}) + 0^\circ{\cdot}216 \sin 2\,(\sigma_{,} - \varpi_{,})$$

$$- 14^\circ{\cdot}324 \sin^2 I \sin 2\,\sigma_{,} - \sin^4 I\,(7^\circ{\cdot}162 \sin 2\,\sigma_{,} - 1^\circ{\cdot}790 \sin 4\,\sigma_{,})$$

$$+ 1^\circ{\cdot}241 \sin\,(s - 2\,h + p)$$

$$+ 0^\circ{\cdot}596 \sin 2\,(s - h)$$

$$- 0^\circ{\cdot}183 \sin\,(h - p_1). \tag{78}$$

The two terms in $\sin 2\,\sigma_{,}$ may be added together by giving to $\sin I$ a mean value. This expression increased by ν is the moon's right ascension reckoned from ♈.

By § 21 the above expression may be written

$$s - \xi + 6^\circ{\cdot}290 \sin\,(s - p) + 0^\circ{\cdot}216 \sin 2\,(s - p)$$

$$- 15^\circ{\cdot}458 \sin^2 I \sin 2\,(s - \xi) + \sin^4 I\,[1^\circ{\cdot}790 \sin 4\,(s - \xi)]$$

$$+ 1^\circ{\cdot}241 \sin\,(s - 2\,h + p)$$

$$+ 0^\circ{\cdot}596 \sin\,(2\,s - 2\,h)$$

$$- 0^\circ{\cdot}183 \sin\,(h - p_1). \tag{79}$$

* I. e., making use of Table 1 and multiplying by 57·3.

The harmonic components give for twice the right ascension of the moon, likewise reckoned from I,

$$2s - 2\nu - 12^\circ\!\cdot\!50 \sin(s-p) - 1^\circ\!\cdot\!42 \sin 2(p-s) \qquad N_2,\ L_2;\ 2\,N$$

$$- 32^\circ\!\cdot\!47 \sin^2 I \sin 2(s-\xi) \qquad \text{lunar } K_2$$

$$- 2^\circ\!\cdot\!54 \sin(2h-s-p) \qquad \nu_2,\ \lambda_2$$

$$- 1^\circ\!\cdot\!33 \sin 2(h-s). \qquad \mu_2 \qquad (80)$$

Reckoned from ♈, the double right ascension of the moon is this expression with the term 2ν omitted. Since ν is nearly equal to ξ, this agrees well with twice the preceding expression, which shows that the mean lunar tide is so perturbed by additional components as to follow the true moon. Either enables a person to compute the approximate time of the moon's transit by means of Table 3.

In a similar manner we might compare ordinary periodic expressions for the moon's parallax or radius vector with the expression for the amplitude of the semidiurnal wave regarded as the M_2 wave perturbed by N_2, L_2, etc.

26. *Astres fictifs, fictitious moons, etc.*

For aiding the imagination, Laplace* introduced a set of fictitious bodies having certain analogies to the mean sun and mean moon. Such bodies move uniformly in the celestial equator and at a constant distance from the earth. The successive transits of any *astre fictif* across a given terrestrial meridian defines a corresponding day, differing in length but little from twenty-four mean solar hours.

The Tidal Committee of the British Association made use of such fictitious bodies as the harmonic analysis of tides seemed to demand.† By the introduction of suitable bodies, which were not used by Laplace, the parallaxes of the mean sun and mean moon become constant.

The assigned position of a fictitious body has, of course, a direct influence upon the epoch of any component tide determined with reference to the body. The following quotation from a paper by Thomson is found upon page 481 of the British Association Report for 1878:

ε (technically called the epoch) is the angle, reckoned in degrees, which an arm revolving uniformly in the period of the particular tide has to run through till high water of this constituent, from a certain instant or era of reckoning defined for each constituent as follows:—

Definition of ε.‡—To explain the meaning of the values of ε given in the following table of results, it is convenient to use Laplace's "astres fictifs," or ideal stars. Let them be as follows:—

M the "mean moon."
S the "mean sun."
K for diurnal tides, a star whose right ascension is 90°.
K for semi-diurnal tide, the "first point of Aries," or ♈.
O a point moving with angular velocity 2σ, and having 270° of right ascension when M is in ♈.
Q a point moving with angular velocity $2\sigma - \varpi$, and 270° before M in right ascension when the longitude of M is half the longitude of the perigee.
P a point moving with angular velocity 2η, having 270° of right ascension when S is in ♈.
N a point moving with angular velocity, $\frac{3}{2}\sigma - \frac{1}{2}\varpi$, and passing alternately through the perigee and apogee of the moon's orbit when M is in perigee.
L a point moving with angular velocity, $\frac{1}{2}\sigma + \varpi$, and passing alternately through 90° on either side of the perigee of the moon's orbit when M is in perigee.

The value of ε in each case above means the number of 360ths of its period which the corresponding tidal constituent has still to execute till its high-water from the instant when the ideal star crosses the meridian of the place. Thus if n denote the periodic speed of the particular tide in degrees per mean solar hour, its time of high-water is $\frac{\varepsilon}{n}$, reckoned in mean solar hours after the transit of the ideal star.

---

*Méc. Cél., Bk. IV, §§ 17, 19. Cf. expression for the tide in Bk. XIII. See under Laplace.

†B. A. A. S. Reports, 1868, p. 496; 1876, p. 293; 1878, p. 481. Thomson and Tait, Natural Philosophy (Ed. 1883), § 848.

‡"This definition for the several cases of K diurnal, and O, P, Q, and L differs by 90° or 180° or 270° from the definition given in the British Association Report (1876) for a reason obvious on inspection of Tables I. and II., pp. 304 and 305 of that report, which (except in respect to the longitude of perigee and perihelion) show ε as previously reckoned for the several constituents."

It is to be remarked that there are two K bodies always 90° apart in the celestial equator. Darwin discards the *astres fictifs*, and says in his report for 1883:

In the present Report the method of mathematical treatment differs considerably from that of Sir William Thomson. In particular, he has followed, and extended to the diurnal tides, Laplace's method of referring each tide to the motion of an *astre fictif* in the heavens, and he considers that these fictitious satellites are helpful in forming a clear conception of the equilibrium theory of tides. As, however, I have found the fiction rather a hindrance than otherwise, I have ventured to depart from this method, and have connected each tide with an 'argument,'* or an angle increasing uniformly with the time and giving by its hourly increase the 'speed' of the tide. In the method of the *astres fictifs*, the speed is the difference between the earth's angular velocity of rotation and the motion of the fictitious satellite amongst the stars. It is a consequence of the difference in the mode of treatment, and of the fact that the elliptic tides are here developed to a higher degree of approximation, that none of the present Report is quoted from the previous ones.

In case of a diurnal component, the argument is evidently the hour angle of the corresponding *astre fictif;* in case of a semidiurnal, twice the hour angle; etc.

Since these bodies do not really exist in nature, but are created for aiding the imagination, it seems justifiable to carry the fiction one step farther if simplicity can be thereby attained. Imagine a system of bodies to have an apparent and uniform motion around the earth from east to west. Let their periods be equivalent to the periods of the various short-period tidal components. That is, an M_2 moon or body will cross the meridian of a place twice each mean lunar day; an M_4 moon, four times; etc. They are to be so placed in the celestial equator that the equilibrium arguments of the components ($V+u$) are the hour angles of the fictitious bodies. Each body has the property of producing a $\genfrac{}{}{0pt}{}{\text{maximum}}{\text{minimum}}$ of the corresponding component tide a certain number of hours (= epoch expressed in time) after its $\genfrac{}{}{0pt}{}{\text{upper}}{\text{lower}}$ transit. (See Fig. 7, Part I.)

27. *Sum of the series*

$$\cos\theta+\cos 2\theta+\cos 3\theta+\ \ .\ \ .\ \ .\ \ +\cos n\theta. \tag{81}$$

If n denote any positive integer and θ any angle, then

$$2\cos\theta\cos n\theta=\cos(n+1)\theta+\cos(n-1)\theta \tag{82}$$

because the second member of this equation is equal to

$$\cos n\theta\cos\theta-\sin n\theta\sin\theta+\cos n\theta\cos\theta+\sin n\theta\sin\theta.$$

Now giving to n the values 1, 2, 3, . . . n, we have

$$2\cos\theta\sum_{n=1}^{n=n}\cos n\theta=\sum_{n=1}^{n=n}\cos(n+1)\theta+\sum_{n=1}^{n=n}\cos(n-1)\theta. \tag{83}$$

Calling $\sum\limits_{n=1}^{n=n}\cos n\theta$, S, we obviously have

$$2S\cos\theta=2S+1-\cos\theta+\cos(n+1)\theta-\cos n\theta; \tag{84}$$

$$\therefore S=-\tfrac{1}{2}+\frac{\cos n\theta-\cos(n+1)\theta}{2(1-\cos\theta)}, \tag{85}$$

$$=-\tfrac{1}{2}+\tfrac{1}{2}\frac{\sin(2n+1)\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}=-1+\frac{\sin(n+1)\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}\cos\frac{n}{2}\theta. \tag{86}$$

Special case $n\theta=360°$. Either value of S just obtained reduces to zero.

28. *Note on the determination of empirical constants.*

Let there be three empirical constants (x, y, z)† to be determined from n observations upon

---

* This we have generally called "equilibrium argument," denoting it by arg. We then have arg—epoch = phase.

† The notation used in this paragraph is temporary.

the value of a linear function involving them $(a_ix + b_iy + c_iz)$. This function changes its value for different assigned values of the coefficients (a, b, c) which are absolutely known from theory, and which are made to vary in accordance with the circumstances of the observations. Let $-k_i$ denote an observed value under the circumstances whose characteristic is i. If there is no error in this measurement, then

$$a_ix + b_iy + c_iz + k_i = 0. \tag{87}$$

If there were three accurate measurements $(i = 1, 2, 3)$, then x, y, z would be accurately determined; if the three measurements were not accurate, x, y, z would still be definitely, though not accurately, determined. Now when there are more than three $(i = 1, 2, 3, \quad . \quad . \quad . \quad n)$ slightly inaccurate measurements, the values of x, y, z determined from any three observation equations will not exactly satisfy the others. As it is not known which observation is in error, all may be assumed to be slightly in error, and the values of x, y, z might be found by taking the mean values of all determinations. But such a process would not give the most probable values of the unknown quantities.

Instead of actually solving all equations for x, y, z, suppose these latter to have such values as are the most probable under the given set of measurements. Clearly none of the observation equations will now have zero as its right-hand member, but the one whose characteristic is i will have a small residual v_i, say, instead of zero. If x, y, z have their most probable values, it can be shown from the theory of accidental errors that they render the sum of the squares of the residuals $(v_1^2 + v_2^2 + \quad . \quad . \quad . \quad + v_i^2 + \quad . \quad . \quad . \quad + v_n^2)$ of the n observation equations a minimum.* But the minimum of the function

$$\sum_{i=1}^{i=n} (a_ix + b_iy + c_iz + k_i)^2 \tag{88}$$

is found by equating each of the partial derivatives to zero; for, one can show that Lagrange's conditions are satisfied for this function.† The three resulting equations (which are the three normal equations) are

$$\begin{aligned}
&\sum_{i=1}^{i=n} a_i(a_ix + b_iy + c_iz + k_i) = 0, \\
&\sum_{i=1}^{i=n} b_i(a_ix + b_iy + c_iz + k_i) = 0, \\
&\sum_{i=1}^{i=n} c_i(a_ix + b_iy + c_iz + k_i) = 0.
\end{aligned} \tag{89}$$

For any number of unknown quantities the normal equations will have the above form, and their number will be equal to the number of the unknowns.

Special case. If $a_i, b_i, c_i, \quad . \quad . \quad .$ be of the form $\frac{\sin}{\cos}\left(ri\frac{2\pi}{n}\right)$, where r is an integer ranging from 0 to $n-1$, all coefficients in the normal equations will be zero except those of the form Σa_i^2, Σb_i^2, $\Sigma c_i^2 \quad . \quad . \quad .$ which are respectively equal to $\frac{1}{2}$ except for the case $r = 0$, which gives 1 instead of $\frac{1}{2}$. The products $a_i b_i$, $b_i c_i \quad . \quad . \quad .$ must each be zero because they can be written in the form

$$\begin{aligned}
\cos A \cos B &= \tfrac{1}{2} \{ \cos (A - B) + \cos (A + B) \}, \\
\cos A \sin B &= \tfrac{1}{2} \{ \sin (A + B) - \sin (A - B) \}, \\
\sin A \cos B &= \tfrac{1}{2} \{ \sin (A - B) + \sin (A + B) \}, \\
\sin A \sin B &= \tfrac{1}{2} \{ \cos (A - B) - \cos (A + B) \},
\end{aligned} \tag{90}$$

and because the algebraic sum of the sines or cosines of angles which divide a circle into a whole number (n) of equal parts is zero (§ 27).

---

* See any treatise on least squares.

† See any standard treatise on differential calculus.

CHAPTER III.

THE TIDE-PRODUCING POTENTIAL.

29. The attracting force of the moon upon any particle of unit mass whose distance is D from the moon's center is

$$\frac{\mu M}{D^2} \tag{91}$$

where M is the moon's mass and μ the attraction between unit masses unit distance apart.* Now if W be such a function that

$$\frac{\partial W}{\partial D} = -\frac{\mu M}{D^2}, \tag{92}$$

it is, by definition, the *gravitational potential* of the moon at the point where the particle is situated; for, in the direction of D increasing, the force is negative. From this equation it is seen that the moon's potential decreases as the distance of the particle from the moon increases. If

$$W = \frac{\mu M}{D} + \text{constant}, \tag{93}$$

equation (92) is satisfied.† Let

$r =$ the distance of the moon's center from the center of the earth,

$\rho =$ the distance of the disturbed particle from the center of the earth,

$\theta =$ the angle at the earth's center between the disturbed particle and the moon's center. In the plane triangle defined by the earth's center, the moon's center, and the disturbed particle, the lengths of two sides are r and ρ, while the included angle is θ. Consequently the remaining side, whose length is D, has the value

$$\sqrt{r^2 - 2\,r\rho\cos\theta + \rho^2}. \tag{94}$$

Replacing D by this value and making the constant zero, (93) becomes

$$W = \frac{\mu M}{\sqrt{r^2 - 2\,r\rho\cos\theta + \rho^2}} \tag{95}$$

Suppose the earth and the particle P to constitute a system not subject to deformation by the moon. The whole system is urged toward the moon just as if each unit particle of the system had applied to it the force

$$\frac{\mu M}{r^2} \tag{96}$$

acting in a direction parallel to the line joining the centers of the earth and moon. The components of this force are

$$m_1\frac{\mu M}{r^2}, \qquad m_2\frac{\mu M}{r^2}, \qquad m_3\frac{\mu M}{r^2}, \tag{97}$$

where m_1, m_2, m_3 are direction-cosines of the line joining the centers of the earth and moon referring to axes fixed in the earth, the origin being the earth's center. If U denote the potential

---

* $g = \frac{E}{a^2}\mu$; $\therefore\ \mu = g\frac{a^2}{E} = \frac{3}{4}\frac{g}{\pi a\delta_e}$, since $E =$ earth's mass $=$ volume $\times$ density $= \frac{4}{3}\pi a^3\delta_e$.

† It is customary to give the potential a sign such that its value will decrease when we go in the direction indicated by an arrow representing the force; i. e., the partial derivatives are not the forces in the corresponding directions as contemplated in this chapter, but minus such forces.

at P of this force, it must be such a function ot x, y, z, the coördinates of P, that its partial derivations shall be the above component forces. Such a function is

$$U = \frac{\mu M}{r^2}(m_1 x + m_2 y + m_3 z) + \text{constant}. \tag{98}$$

If p_1, p_2, p_3 denote the direction-cosines of P referring to the axes mentioned in connection with m_1, m_2, m_3, we have

$$x = p_1 \rho, \qquad y = p_2 \rho, \qquad z = p_3 \rho\,; \tag{99}$$

$$\therefore \cos\theta = p_1 m_1 + p_2 m_2 + p_3 m_3, \tag{100}$$

$$= \sin\lambda \sin\delta + \cos\lambda \cos\delta \cos\,(\text{moon's hour angle}), \tag{101}$$

where

λ = the latitude of P,
δ = the declination of the moon.

$$\therefore\ U = \frac{\mu M \rho}{r^2}\cos\theta + \text{constant}. \tag{102}$$

Now let the system be subject to deformation; in other words, let there be an opportunity for P to move relatively to the earth's center or to a rigid nucleus which may surround the center. The force causing this movement has for its potential

$$W - U = V, \text{ say}; \tag{103}$$

or

$$\frac{\mu M}{\sqrt{r^2 - 2r\rho\cos\theta + \rho^2}} - \frac{\mu M \rho}{r^2}\cos\theta - \text{constant} = V. \tag{104}$$

At the earth's center the potential of the tide-producing force must be zero because W and U are there equal. Making $\rho = 0$ and $V = 0$, the constant becomes equal to

$$+\frac{\mu M}{r}; \tag{105}$$

$$\therefore V = \frac{\mu M}{\sqrt{r^2 - 2r\rho\cos\theta + \rho^2}} - \frac{\mu M}{r} - \frac{\mu M \rho}{r^2}\cos\theta.* \tag{106}$$

The expression

$$\frac{1}{\sqrt{r^2 - 2r\rho\cos\theta + \rho^2}} \tag{107}$$

may be written

$$\frac{1}{r}\left(1 - 2\frac{\rho}{r}\cos\theta + \frac{\rho^2}{r^2}\right)^{-\frac{1}{2}}. \tag{108}$$

This expanded in powers of $\frac{\rho}{r}$ becomes

$$\frac{1}{r}\left(P_0 + P_1\frac{\rho}{r} + P_2\frac{\rho^2}{r^2} + P_3\frac{\rho^3}{r^3} + \ .\ .\ .\right) \tag{109}$$

* Cf. Laplace Méc. Cél., Bk. III, § 23, and Ferrel, Tidal Researches, p. 25. This expression, without the term $-\mu M/r$, is the disturbing function Ω in the astronomical problem of three bodies.

where

$$\begin{aligned} P_0 &= 1, \\ P_1 &= \cos\theta, \\ P_2 &= \frac{3\cos^2\theta - 1}{2}, \\ P_3 &= \frac{5\cos^2\theta - 3\cos\theta}{2}, \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{110}$$

The P's are functions of θ alone and are called zonal harmonics or Legendre's coefficients. Equation (106) now becomes

$$V = \mu M\left[\frac{\rho^2}{r^3}\left(\frac{3\cos^2\theta - 1}{2}\right) + \frac{\rho^3}{r^4}\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + \ \ldots\ \right], \tag{111}$$

or

$$V = \frac{\mu M}{r}\left[P_2\frac{\rho^2}{r^2} + P_3\frac{\rho^3}{r^3} + \ \ldots\ \right]. \tag{112}$$

30. *Approximate determination of the tide-producing potential.**

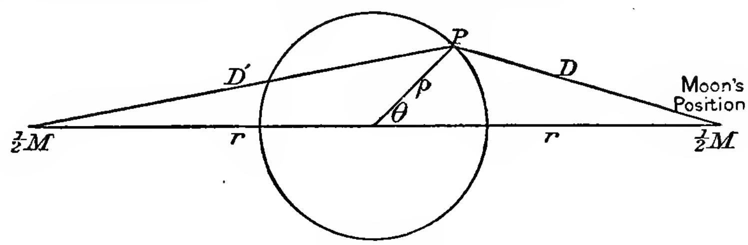


FIG. 7.

Imagine the moon to be divided into two equal parts, the one occupying the moon's position, the other a position diametrically opposite but at an equal distance from the earth's center.

The potential of the entire moon at any point P now becomes

$$W = \tfrac{1}{2}\mu M\left(\frac{1}{D'} + \frac{1}{D}\right) = \frac{\tfrac{1}{2}\mu M}{\sqrt{r^2 + 2r\rho\cos\theta + \rho^2}} + \frac{\tfrac{1}{2}\mu M}{\sqrt{r^2 - 2r\rho\cos\theta + \rho^2}}, \tag{113}$$

$$= \tfrac{1}{2}\mu M\frac{1}{r}\left(P_0 - P_1\frac{\rho}{r} + P_2\frac{\rho^2}{r^2} - \ \ldots\ + P_0 + P_1\frac{\rho}{r} + P_2\frac{\rho^2}{r^2} + \ \ldots\ \right), \tag{114}$$

$$= \frac{\mu M}{r}\left(P_0 + P_2\frac{\rho^2}{r^2} + P_4\frac{\rho^4}{r^4} + \ \ldots\ \right); \tag{115}$$

the P's have been defined in § 29.

$$\therefore\ W = \frac{\mu M}{r} + \mu M\left[\frac{\rho^2}{r^3}\left(\frac{3\cos^3\theta - 1}{2}\right) + \ \ldots\ \right]. \tag{116}$$

Since one-half of the moon is upon one side of the earth's center and the other half upon the other side, and at an equal distance from it, the earth's center, and so the solid portion of the earth, has no tendency to move. That is, the force tending to move the earth's center is zero;

$$\therefore\ \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0; \tag{117}$$

$$\therefore\ U = \text{constant}. \tag{118}$$

* See Thomson and Tait, Natural Philosophy, § 804.

The tide-producing potential is

$$V = W - U = \frac{\mu M}{r} + \mu M \left[\frac{\rho^2}{r^3}\left(\frac{3 \cos^2 \theta - 1}{2}\right) + \quad . \quad . \quad . \quad \right] - \text{constant.} \tag{119}$$

At the center of the earth $V = 0$ and $\rho = 0$;

$$\therefore \text{ constant} = + \frac{\mu M}{r}. \tag{120}$$

The required tide-producing potential is, then,

$$V = \mu M \left[\frac{\rho^2}{r^3}\left(\frac{3 \cos^2 \theta - 1}{2}\right)\right] \tag{121}$$

to the third power of $\frac{1}{r}$, or of the moon's parallax.

31. *Proctor's construction.*

The following is Proctor's construction for showing graphically the disturbing influence of the sun upon the moon; it is here used to show the tide-producing force of the moon M upon a unit particle P.

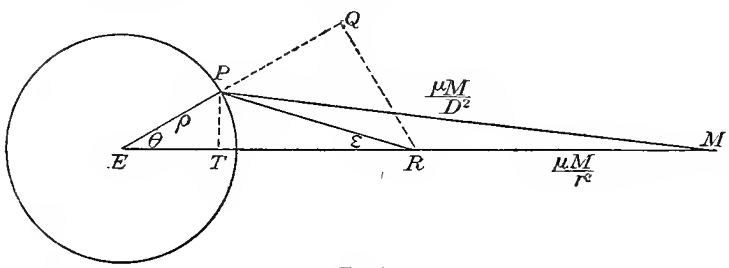


Fig. 8.

It is assumed that the line PM whose length is D, represents a force acting on P toward M, the magnitude of this force really being $\mu M/D^2$; that is, the attraction of the moon on a unit mass situated at P. Upon the same scale RM, where $TR = 2ET$, represents the attraction, in magnitude and direction, of the moon upon a unit mass at E, the earth's center; or, the attraction of the moon upon the entire earth divided by the earth's mass. The line PR represents both in magnitude and direction the disturbing force of M upon P.

For, the attraction at E is

$$PM \times \frac{\overline{PM}^2}{\overline{EM}^2}; \tag{122}$$

or, since PM and TM are very nearly equal and $EM = ET + TM$, it is

$$TM\left(1 - \frac{2ET}{TM}\right),$$

or

$$TM - 2ET. \tag{123}$$

The line PR, completing the triangle of forces, represents the force tending to move P relatively to E.

The potential of this force, since its action is relative to E, can be found by integrating the force PR or its components along the directions of their action. We shall integrate the component PQ along the line EQ.

Upon the same scale as before, the line TR represents, because of the remoteness of M, the force

$$\frac{\mu M}{D^2} - \frac{\mu M}{r^2}. \tag{124}$$

By construction,

$$\tan \varepsilon = \tfrac{1}{2} \tan \theta. \tag{125}$$

The angle RPQ is equal to $\theta + \varepsilon$. The force represented by PQ must be

$$\left(\frac{\mu M}{D^2} - \frac{\mu M}{r^2}\right) \sec \varepsilon \cos (\varepsilon + \theta), \tag{126}$$

$$= \left(\frac{\mu M}{D^2} - \frac{\mu M}{r^2}\right) \frac{3 \cos^2 \theta - 1}{2 \cos \theta}, \tag{127}$$

$$= \mu M \left(\frac{1}{r^2} + \frac{2\rho}{r^2} \cos \theta + \quad . \quad . \quad . \quad - \frac{1}{r^2}\right) \frac{3 \cos^2 \theta - 1}{2 \cos \theta}, \tag{128}$$

$$= \frac{\mu M \rho}{r^3} (3 \cos^2 \theta - 1),^* \tag{129}$$

neglecting higher powers of the moon's parallax.

$$\therefore V = \int \frac{\mu M \rho}{r^3} (3 \cos^2 \theta - 1) \, \partial \rho = \frac{\mu M \rho^2}{r^3} \frac{3 \cos^2 \theta - 1}{2} + \text{constant.} \tag{130}$$

The tangential component at P of the force PR has for its value QR, or

$$\left(\frac{\mu M}{D^2} - \frac{\mu M}{r^2}\right) \sec \varepsilon \sin (\varepsilon + \theta) = 3 \frac{\mu M \rho}{r^3} \sin \theta \cos \theta.^* \tag{131}$$

But this is

$$\frac{-\partial V}{\rho \partial \theta}. \tag{132}$$

Hence the constant in V contains neither ρ nor θ; in other words, it is the same for all points of the earth. At the earth's center it is zero, and must remain so for all positions of P.

$$\therefore V = \frac{\mu M \rho^2}{r^3} \frac{3 \cos^2 \theta - 1}{2}. \tag{133}$$

32. *Spherical harmonic expression for*

$$\cos^2 \theta - \tfrac{1}{3}. \tag{134}$$

By § 29,

$$\cos \theta = p_1 \, m_1 + p_2 \, m_2 + p_3 \, m_3. \tag{135}$$

From analytical geometry we have

$$p_1^2 + p_2^2 + p_3^2 = 1, \qquad m_1^2 + m_2^2 + m_3^2 = 1; \tag{136}$$

and so of course

$$(p_1^2 + p_2^2 + p_3^2) \, (m_1^2 + m_2^2 + m_3^2) = 1. \tag{137}$$

From these relations it follows that

$$\cos^2 \theta - \tfrac{1}{3} = 2 \, p_1 p_2 \, m_1 m_2 + 2 \, \frac{p_1^2 - p_2^2}{2} \, \frac{m_1^2 - m_2^2}{2} + 2 \, p_2 p_3 \, m_2 m_3$$

$$+ 2 \, p_1 p_3 \, m_1 m_3 + \tfrac{3}{2} \, \frac{p_1^2 + p_2^2 - 2 \, p_3^2}{3} \, \frac{m_1^2 + m_2^2 - 2 \, m_3^2}{3}. \tag{138}$$

* Cf. values of $-P$, $-T$, p. 25, Godfray's Lunar Theory; Thomson and Tait, Natural Philosophy, § 812.

The reason why $\cos^2\theta - \frac{1}{3}$ should be expressible in this form is given below, but the verification can be made in the manner just indicated. It may be noted that because of the symmetry in (135), (138) must be symmetric in p_1, p_2, p_3 and m_1, m_2, m_3.

The reason why

$$\rho^2\left(\frac{3\cos\theta^2-1}{2}\right) \tag{139}$$

should be expressible in the terms of

$$x^2-y^2, \quad xy, \quad xz, \quad yz, \quad x^2+y^2-2z^2, \tag{140}$$

$$\rho^2(p_1^2-p_2^2), \quad \rho^2p_1p_2, \quad \rho^2p_1p_3, \quad \rho^2p_2p_3, \quad \rho^2(p_1^2+p_2^2-2p_3^2). \tag{141}$$

Since by (99), (100),

$$\rho\cos\theta = m_1x+m_2y+m_3z, \tag{142}$$

and since

$$\rho^2 = x^2+y^2+z^2, \tag{143}$$

$\rho^2(3\cos^2\theta-1)$ must be a (rational, integral) homogeneous function in x, y, z of the second order.

The property possessed by the functions (140) is that they (and so any function of them of the form $A_0(x^2-y^2)+B_0\,x\,y+C_0\,x\,z+D_0yz+E_0(x^2+y^2-2z^2)$, the coefficients being any constants) satisfy Laplace's equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}=0 \tag{144}$$

where F stands for any one of the quantities x^2-y^2, etc., or any combination of them of the kind just given. Other homogeneous functions analogous to (140), such as

$$x^2-z^2, \quad y^2-z^2, \quad x^2+z^2-2y^2, \quad y^2+6yz-z^2,$$

which are made up from (140) by using suitable coefficients A_0, B_0, etc., are not independent solutions of (144) if (140) be regarded as such.

It can be shown that if

$$\phi=\rho^2\left(\frac{3\cos^2\theta-1}{2}\right) \text{ or } \rho^2P_2\text{; or, more generally, } \rho^nP_n; \tag{145}$$

then

$$\rho\frac{\partial^2(\rho\phi)}{\partial\rho^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)=0.^* \tag{146}$$

But this equation is a particular form of

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0, \tag{147}$$

so that any solution of (146) is also a solution of (144). Since ϕ or $\rho^2\left(\frac{3\cos^2\theta-1}{2}\right)$ is a solution of (146) and, as has just been shown, is a homogeneous function in x, y, z of the second order, ϕ must be some form of F.

In like manner it can be shown that because $\rho^3\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)$ satisfies (146) and is a homogeneous function in x, y, z of the third order, it must satisfy (144).

33. *To show that, by the equilibrium theory, the attraction of the moon produces spherical harmonic deformations of the ocean and also that such deformations are consistent with the equation of continuity.*

---

* Easily proved by direct substitution in (146) for any particular case required in tidal work. For the general proof, see Ferrers, Spherical Harmonics, pp. 4–10.

Since the tide-producing potential represents work, viz., the force of terrestrial gravity multiplied by the height of the tide, the height at various places must be the tide-producing potential divided by g. But g may be regarded as constant, and so the height of the tide will be proportional to V, and so a spherical harmonic deformation.

Suppose everything stationary, and let ρ denote the radius rector of the free surface of the sea; the earth will assume the form of a solid of revolution whose equation is

$$\rho = a + c\left[\frac{\rho^2}{r^3}\left(\frac{3\cos^2\theta - 1}{2}\right) + \frac{\rho^3}{r^4}\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + \quad . \quad . \quad . \right], \tag{148}$$

or

$$\rho = a + c_2\,\rho^2\,\frac{3\cos^2\theta - 1}{2} + c_3\,\rho^3\,\frac{5\cos^3\theta - 3\cos\theta}{2}, \tag{149}$$

where c_1, c_2, c_3 are small quantities.

But this value of ρ satisfies (146), showing, as will be noted in § 35, that the equation of continuity is satisfied, or that no volume is lost or gained by the deformation. That is, the volume generated by revolving the curve

$$\rho = a + c_2\,\rho^2\,\frac{3\cos^2\theta - 1}{2}, \tag{150}$$

or

$$\rho = a + c_3\,\rho^3\,\frac{5\cos^3\theta - 3\cos\theta}{2}, \tag{151}$$

or

$$\rho = a + c_2\,\rho^2\,\frac{3\cos^2\theta - 1}{2} + c_3\,\rho^3\,\frac{5\cos^3\theta - 3\cos\theta}{2}, \tag{152}$$

about the x-axis* is equivalent to the volume of the undisturbed sphere of radius a. This may be shown independently and as follows:

In the first instance, since ρ is nearly equal to a, $\rho = \sqrt{x^2 + y^2} = a + c_2\left(\frac{3}{2}x^2 - \frac{1}{2}a^2\right)$;

$$\therefore\ x^2 + y^2 = a^2 + ac_2\left(3x^2 - a^2\right), \tag{153}$$

or

$$\frac{x^2}{a^2\left(1 + 2ac_2\right)} + \frac{y^2}{a^2\left(1 - ac_2\right)} = 1. \tag{154}$$

The volume of the ellipsoid generated by revolving this ellipse is

$$\tfrac{4}{3}\pi a^3\,\sqrt{1 + 2ac_2}\,\sqrt{1 - ac_2}\,\sqrt{1 - ac_2},$$

$$= \tfrac{4}{3}\pi a^3, \text{ neglecting terms in } c_2^2. \tag{155}$$

In the second instance we have

$$x^2 + y^2 = a^2 + ac_3\left(5x^3 - 3a^2x\right). \tag{156}$$

The required volume is

$$\pi\int y^2 dx \tag{157}$$

taken between the limits

$$\begin{aligned} x &= a + a^3c_3, \\ x &= -a + a^3c_3. \end{aligned} \tag{158}$$

This is, neglecting terms in c_3^2,

$$\tfrac{4}{3}\pi a^3.$$

---

* In this and the next paragraph, the coördinate axes are taken with reference to the instantaneous spheroid, the x-axis passing through the disturbing body, and are not fixed in the earth as is usually the case.

34. *Further illustration.*

A *surface of equilibrium* or a *level surface* is one which has the same potential at all its points. Supposing the earth to be a sphere without rotation, and the moon divided as in § 30; any surface of equilibrium has for its equation

$$\frac{\mu E}{\rho} + \tfrac{1}{2}\,\mu M\left[\frac{1}{D'} + \frac{1}{D}\right] = \text{constant,*} \tag{159}$$

E denoting the earth's mass. Because the water of the sea is incompressible and because the action of the moon is symmetric about a line joining the centers of earth and moon, the surface whose equation is (159) must cut a perfect or undisturbed sphere in two small circles whose planes are perpendicular to the line joining the centers. Let $a =$ the radius vector of the surface (159) at these small circles, $=$ the mean radius of the undisturbed sphere. If now we write

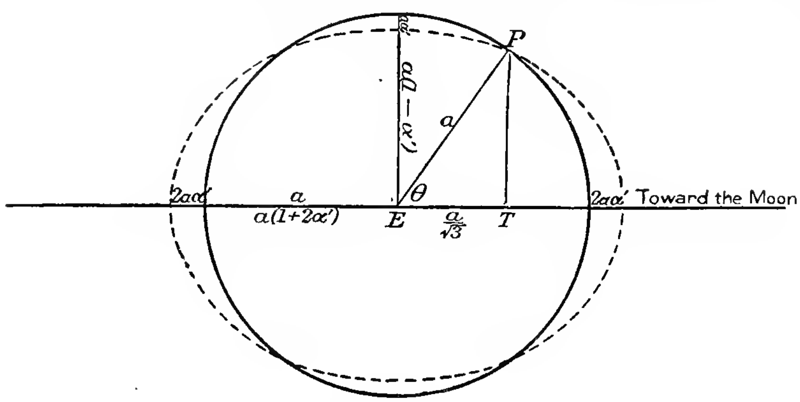


FIG. 9.

$$\rho = a\,(1 + u), \tag{160}$$

au is a very small quantity in comparison with a and represents the variation of ρ from a constant value a. Since (159) is true for all possible values of ρ, it is true when $\rho = a$ or $u = 0$. Developing (159) and putting $\rho = a$ we have, as in (115),

$$\frac{\mu E}{a} + \frac{\mu M}{r}\left[P_0 + P_2\frac{a^2}{r^2} + P_4\frac{a^4}{r^4} + \quad . \quad . \quad . \quad\right] = \text{constant.} \tag{161}$$

Now $2\,P_2 = 3\cos^2\theta - 1, \therefore$ if we make $\cos\theta = \dfrac{1}{\sqrt{3}}$, $P_2 = 0$. Hence, if we omit all terms beyond $P_2\dfrac{a^2}{r^2}$ in the brackets as being comparatively small, (161) becomes

$$\frac{\mu E}{a} + \frac{\mu M}{r} = \text{constant.} \tag{162}$$

Writing this value for the constant and putting $\rho = a\,(1 + u)$, equation (159) gives upon development

$$\frac{E}{a}(1 - u + u^2 - \quad . \quad . \quad . \quad) + \frac{M}{r}\left[P_0 + P_2\frac{a^2\,(1+u)^2}{r^2} + \quad . \quad . \quad . \quad\right] = \frac{E}{a} + \frac{M}{r}. \tag{163}$$

Omitting the second and higher powers of u as being very small, we have from (163)

$$u = \frac{M}{E}\frac{a^3}{r^3}P_2, = 2\,\alpha'\,P_2 = \alpha'\,(3\cos^2\theta - 1), \text{ say.} \tag{164}$$

When

$$\theta = 0,\ u = \frac{M\,a^3}{E\,r^3} = 2\,\alpha';$$

*See Thomson and Tait, Natural Philosophy, §§ 800, 804.

Taking into account the earth's axial rotation, the potential of the centrifugal force is $\frac{1}{2}\,\omega^2 d^2$ where ω is put temporarily for angular velocity and d for the distance of any point from the axis. This should be added to the left side of (159).

when (165)

$$\theta = 90^\circ,\ u = -\tfrac{1}{2}\frac{M}{E}\frac{a^3}{r^3} = -\alpha'.$$

Now au represents the inequality in the radius vector of the surface of equilibrium. If the surface be an ellipsoid of revolution, the section made by a plane passing through the centers of the earth and moon must be an ellipse having semiaxes $a(1+2\alpha')$ and $a(1-\alpha')$, respectively.

For the lunar tide, $a\alpha' = 0.59$ feet; for the solar, 0.27 feet. The equation of such an ellipse is

$$\frac{x^2}{a^2(1+2\alpha')^2} + \frac{y^2}{a^2(1-\alpha')^2} = 1. \tag{166}$$

Now write

$$\begin{aligned} x &= \rho\cos\theta, \\ y &= \rho\sin\theta, \\ \rho &= a\,(1+u), \end{aligned} \tag{167}$$

and (166) becomes

$$\frac{(1+u)^2\cos^2\theta}{(1+2\alpha')^2} + \frac{(1+u)^2\sin^2\theta}{(1-\alpha')^2} = 1; \tag{168}$$

$$\therefore u = \alpha'\,(3\cos^2\theta - 1), \tag{169}$$

which agrees with (164), and shows that the variation in ρ is such that the section made by the plane passing through the centers of the earth and moon is an ellipse.

To show that the condition of continuity is fulfilled, that is, that no volume has been lost or gained by the deforming action of the moon, it is only necessary to remember that the volume of the ellipsoid is

$$\begin{aligned} &\tfrac{4}{3}\pi\,a^3\,(1+2\,\alpha')\,(1-\alpha')\,(1-\alpha'), \\ &= \tfrac{4}{3}\pi\,a^3\,(1+0\,\alpha' - 3\,\alpha'^2 + \ \ldots\ ); \\ &= \tfrac{4}{3}\pi\,a^3 \end{aligned} \tag{170}$$

when α' is so small that all powers beyond the first may be neglected. But this is the volume of a sphere whose radius is a.

In obtaining (163), cos θ was arbitrarily put equal to $\frac{1}{\sqrt{3}}$ in order that P_2 might be equal to zero. The reason for this is that for some value of θ we know that the tide-producing potential must be zero; and to the degree of approximation here assumed, this potential is

$$\frac{\mu M\rho^2}{r^3}\,P_2. \tag{171}$$

35. *Given the displacement of sea level, to show that the displacement potential is of the same form as the tide-producing potential and satisfies the equation of continuity.*

We have seen that the displacement of the sea level in a vertical direction, i. e., along a radius of the earth, is

$$a\alpha'\,(3\cos^2\theta - 1), \tag{172}$$

or

$$\rho\alpha'\,(3\cos^2\theta - 1), \tag{173}$$

since a and ρ are sensibly equal. The volume of water moved in any small displacement of a level surface is its thickness multiplied by its horizontal area; that is, in the case under consideration,

$$\rho\alpha'\,(3\cos^2\theta - 1)\,dS, \tag{174}$$

where dS represents the area of an elementary portion of the surface. The entire amount of outward and inward displacement must be equal to zero, since the volume inclosed by the level surface remains unaltered.

$$\therefore \iint \rho\alpha' \,(3 \cos^2 \theta - 1)\, dS = 0. \tag{175}$$

The *displacement potential** is such a function of the coördinates that the differential coefficient with respect to a coördinate represents the displacement along that coördinate. Denoting this function by ϕ, it must be such a function of the coördinates that

$$\frac{\partial\phi}{\partial\rho} = \text{displacement along radius.} \tag{176}$$

In this case we have

$$\frac{\partial\phi}{\partial\rho} = \rho\alpha' \,(3 \cos^2 \theta - 1), \tag{177}$$

$$\therefore \phi = \frac{\rho^2}{2}\alpha' \,(3 \cos^2 \theta - 1), = \rho^2\alpha' P_2, \tag{178}$$

to which any constant may be added. Thus it is seen that ϕ satisfies (147).

In general, letting n denote the normal to the surface S,

$$\iint \frac{\partial\phi}{\partial n} dS = 0, \tag{179}$$

when the integration extends over the entire closed surface S; that is, there is as much outward as inward displacement when the volume inclosed by S remains unaltered.

To show that (179) is satisfied, first determine an expression for dS. The element of surface generated by revolving the short line of length $\rho\, d\, \theta$ about the x-axis is equal to

$$\rho^2 \sin \theta \, d\, \theta, = dS. \tag{180}$$

The question is, Is

$$\iint \frac{\delta\phi}{\delta\rho}\, dS = 0 \tag{181}$$

when the integral is taken over S? This integral becomes

$$2\, \rho^3\alpha' \int_0^\pi \sin \theta \,(3 \cos^2 \theta - 1)\, d\theta,$$

$$= 2\rho^3\alpha' \left[2 \int_0^\pi \sin \theta \, d\theta - 3 \int_0^\pi \sin^3 \theta \, d\theta \right], \tag{182}$$

$$= 2\, \rho^3\alpha' \Big[-2 \cos \theta \Big]_0^\pi - 2\, \rho^3\alpha' \Big[\tfrac{1}{4} \cos 3\, \theta - \tfrac{9}{4} \cos \theta \Big]_0^\pi = 0, \tag{183}$$

as might have been inferred from the fact that ϕ satisfied (147). (147) is another form of the equation of continuity, as can be shown by considering a small displacement of an elementary volume.

The higher spherical harmonic deformations admit of similar treatment.

36. *Alteration in the tide-producing potential, and so in the height of the tide, caused by the mutual attraction between the fluid particles constituting the tide wave.*

The figures of the heavenly bodies depend on the law of gravity at their surfaces, and as this gravity is the resultant of the attraction of all their particles, it must also depend on their figures; therefore the law of gravity, at the surfaces of the heavenly bodies, and their figures, have a mutual connexion, which renders the knowledge of the one necessary for the determination of the other. In consequence of this, the investigation becomes very difficult, and it seems to require an analysis specially adapted to the subject.—(Laplace, Book III.)

---

* See Ibbetson, Mathematical Theory of Elasticity, p. 57.

So far in this chapter, the mutual attraction of the water particles has been disregarded in the tide-producing potential. Practically nothing is lost by so proceeding, because the continents and the inertia of the water do not permit the tide to assume a spherical harmonic deformation. A complete statement of the equilibrium hypothesis requires this source of disturbance to be noticed, and to that end an application of the special analysis referred to by Laplace will be given. It may be here noted that Newton took account of the mutual attraction of the particles in his theory of the figure of the earth, and in his tidal theory derived therefrom.*

A spherical harmonic distribution of density of attracting matter or, what amounts to the same thing, a thin attracting layer of corresponding thickness, on a spherical surface, produces a similar and similarly placed spherical harmonic distribution of potential over any concentric spherical surface throughout space, external and internal.

Let us ascertain the potential of such a spherical surface at points along the axis of symmetry for the distribution (or thickness), taking the center of the sphere as origin; then by replacing the distance on the axis from the origin by ρ, the distance of any point in space from the origin, and multiplying each term of the developed potential by a surface harmonic of the proper order, the resulting expression of the potential is true for points whether upon the axis or not.

For an internal point we know that the equation (147), often written $\nabla^2 V=0$, must be satisfied because (see § 32) any term of the form $\rho^n P_n$ is a solution; consequently an expression involving the sum of such terms must be a solution; and it is the required solution because when $\theta=0$ it reduces to the expression known to be true along the axis.

The expression applying to external points is a solution of $\nabla^2 V=0$ whose terms are of the form $\dfrac{1}{\rho^{n+1}} P_n$.†

Denoting the equilibrium height of the tide by

$$\tfrac{2}{3}\, a\, \mathfrak{E}\, P_2, \tag{184}$$

the density of the water by σ, then the mass of tidal water in a zone whose width is $a\, d\theta$ is evidently

$$\tfrac{4}{3}\, a^3\, \mathfrak{E}\, \pi\, \sigma \sin\theta\, P_2\, d\,\theta. \tag{185}$$

The distance from this zone to a point in the axis distant z from the origin is

$$\left(\overline{z-a\cos\theta}^{\,2}+\overline{a\sin\theta}^{\,2}\right)^{\frac{1}{2}},$$

or

$$(z^2-2\, az\cos\theta+a^2)^{\frac{1}{2}}. \tag{186}$$

Writing temporarily μ' for $\cos\theta$, the quotient of (185) by (186), i. e., the gravitational potential due to the elementary zone, becomes

$$-\tfrac{4}{3}\frac{a^3\, \mathfrak{E}\, \pi\, \sigma\, P_2\, d\,\mu'}{(z^2-2\, az\, \mu+a^2)^{\frac{1}{2}}}\ddagger, \tag{187}$$

$$=-\tfrac{4}{3}\frac{a^3\, \mathfrak{E}\, \pi\, \sigma\, P_2}{z}\left(P_0+P_1\frac{a}{z}+P_2\frac{a^2}{z^2}+\quad.\quad.\quad.\quad\right)d\,\mu' \tag{188}$$

for an external point on the axis, or

$$-\tfrac{4}{3}\, a^2\, \mathfrak{E}\, \pi\, \sigma\, P_2\left(P_0+P_1\frac{z}{a}+P_2\frac{z^2}{a^2}+\quad.\quad.\quad.\quad\right)d\,\mu' \tag{189}$$

---

* Principia, Bk. I, Prop. 91; Bk. III, Props. 19, 36, and 37.

† Ferrers, Spherical Harmonics, pp. 1, 2.

‡ The quantity μ of § 29 does not, of course, enter into the expression for $\mathfrak{E}$, and so is in this paragraph taken as unity. It may be explicitly introduced as a factor into (192) and (194).

for an internal point. Integrating (188) between the limits $\mu'=+1$ and $\mu'=-1$, we have, since

$$\int_{+1}^{-1} P_n P_m \, d\mu' = 0 \tag{190}$$

provided n and m are unequal positive integers, and since

$$\int_{+1}^{-1} P_n^2 \, d\mu' = -\frac{2}{2n+1},^* \tag{191}$$

$$\tfrac{4}{3}\cdot\tfrac{2}{5}\, a^5 \frac{\mathfrak{x}\, \pi\, \sigma}{z^3}. \tag{192}$$

Now, taking any external point, replace z by ρ, its distance from the origin, and multiply by P_2. Thus we obtain for the potential of the spherical shell

$$\tfrac{8}{15} \frac{a^5\, \mathfrak{x}\, \pi\, \sigma}{\rho^3} P_2. \tag{193}$$

Similarly for the higher harmonic deformations of water. This, added to the tide-producing potential of the moon gives for the entire tide-producing potential, to terms of the third order in $1/r$,

$$\frac{M \rho^2 P_2}{r^3} + \tfrac{8}{15} \frac{a^5}{\rho^3}\, \mathfrak{x}\, \pi\, \sigma\, P_2. \tag{194}$$

This must be equal to $g\, \frac{2}{3}\, a\, \mathfrak{x}\, P_2$ or the work accomplished in elevating the water;

$\therefore$ putting $\rho = a$

$$\mathfrak{x} = \tfrac{3}{2} \frac{a}{r^3} \frac{M}{g\left(1 - \frac{8}{15}\cdot\frac{3}{2}\frac{\pi\, a\, \sigma}{g}\right)}. \tag{195}$$

But

$$g = \frac{E}{a^2} = \tfrac{4}{3}\, \pi\, a\, \delta_e \tag{196}$$

since $E = \frac{4}{3}\, \pi\, a^3 \delta_e$, δ_e being the density of the earth,

$$\therefore \mathfrak{x} = \tfrac{3}{2} \frac{Ma}{r^3\, g} \frac{1}{1 - \frac{3}{5}\, \sigma\, \delta_e},$$

$$= \tfrac{3}{2} \frac{M a^3}{E\, r^3} \frac{1}{1 - \frac{3}{5}\, \sigma/\delta_e}. \tag{197}$$

$\therefore$ The equilibrium height of tide, or

$$\tfrac{2}{3}\, a\, \mathfrak{x}\, P_2, = \frac{M a^3}{E\, r^3} \frac{1}{1 - \frac{3}{5}\, \sigma/\delta_e} P_2. \tag{198}$$

When the density of the fluid (σ) is zero, the equilibrium height of tide or the tide-producing potentials is that found in the preceding paragraphs, or Euler's result. When δ_e is taken equal to σ, the equilibrium height is $\frac{5}{2}$ times as great, or Newton's result. As a matter of fact σ/δ_e is only about 2/11, and so the equilibrium tide would not be greatly increased because of this mutual attraction.† For $\sigma=0$, range of semidiurnal tide at the equator $= a\, \mathfrak{x} = H = 3\, a\alpha'$, §§ 41, 47, Part I.

* Ferrers, Spherical Harmonics, p. 17.

† Cf. Thomson and Tait, Natural Philosophy, §§ 815–817. Harkness, Solar Parallax, p. 139, makes $\delta_e/\sigma = 5{\cdot}576 \pm 0{\cdot}016$.

CHAPTER IV.

DEVELOPMENT OF THE TIDE-PRODUCING POTENTIAL.

37. The tide-producing potential of the moon at any given point depends upon the geographic position of the point and upon the particular time chosen. Of course the direction and distance of the moon enter into the value of this potential, but they are both functions of the time. Consequently the expression of this potential should be in terms of the coördinates of the influenced point and functions of time. Moreover, since the tide-producing causes are periodic in their character, the time functions should be simple harmonic functions; that is, functions consisting of terms of the form $A \cos (at + \alpha)$ or $A \sin (at + \alpha)$ where A, a, and α are constant, while t, the time, varies uniformly.

If c denote the moon's mean distance (i. e., the semiaxis major of the orbit), and e the eccentricity of the orbit, the latus rectum has for its value $c(1-e^2)$. If $\varpi_{\prime}$ denote the longitude of the perigee from the intersection of the orbit with the plane of the equator and l the longitude of the moon from the same origin, both reckoned in the plane of the orbit, $l - \varpi_{\prime}$ is the moon's true anomaly. The polar equation of the ellipse representing the orbit, taking the origin at the focus, is

$$\frac{c(1-e^2)}{r} = 1 + e \cos (l - \varpi_{\prime}). \tag{199}$$

Since the most important part of the tide-producing potential V, § 29, depends upon the third power of $\frac{1}{r}$, we cube both members of this equation and obtain

$$\begin{aligned} \left[\frac{c(1-e^2)}{r}\right]^3 &= 1 + 3e \cos (l - \varpi_{\prime}) + 3e^2 \cos^2 (l - \varpi_{\prime}) + e^3 \cos^3 (l - \varpi_{\prime}) + \quad . \quad . \quad . \quad . \;, \\ &= 1 + 3e \cos (l - \varpi_{\prime}) + 3e^2 \frac{\cos 2 (l - \varpi_{\prime})}{2} + \quad . \quad . \quad . \quad . \;, \end{aligned} \tag{200}$$

where terms having the factor e^3 are omitted.

If $\sigma_{\prime}$ denote the moon's mean longitude measured in her orbit from the intersection, and $\varpi_{\prime}$ the mean, as well as the true, longitude of the perigee measured in the same way, the mean anomaly will be $\sigma_{\prime} - \varpi_{\prime}$. The solution of Kepler's problem, § 23, gives the equation

$$l = \sigma_{\prime} + 2e \sin (\sigma_{\prime} - \varpi_{\prime}) + \tfrac{5}{4} e^2 \sin 2 (\sigma_{\prime} - \varpi_{\prime}) + \quad . \quad . \tag{201}$$

This value of l substituted in (200) gives an expression for

$$\left[\frac{c(1-e^2)}{r}\right]^3 \tag{202}$$

in terms whose angles or arguments vary uniformly with the time.

The next step is to express $\cos^2 \theta - \frac{1}{3}$ in functions whose angles vary uniformly with the time. Its value as given by (138) consists of functions of p_1, p_2, p_3 and m_1, m_2, m_3; but as p_1, p_2, p_3 depend only upon the position of the disturbed particle with respect to axes fixed in the earth, they do not involve the time. We have, then, to express the m-functions, or quantities proportional to them, by means of functions whose angles vary uniformly with the time.

Suppose the moon's orbit to be a fixed circle of the celestial sphere concentric with the earth; then its intersection with the celestial equator is a fixed point to which the rotation of the earth may be referred.* Let the z-axis be the one about which the earth rotates, carrying with it the x-axis and the y-axis which lie in the plane of the equator.

Let χ denote the right ascension of the extremity of the x-axis reckoned from the intersection.

Fig. 10 shows the appearance of the celestial sphere viewed from without. The meridional arcs xz, yz revolve in the direction indicated by the arrows. If M denote the position of the moon, her true longitude from I being l, then

$$m_1 = \cos Mx,\ m_2 = \cos My,\ m_3 = \cos Mz;$$

$$\begin{aligned} \therefore\ m_1 &= \cos l \cos \chi + \sin l \sin \chi \cos I, \\ m_2 &= -\cos l \sin \chi + \sin l \cos \chi \cos I, \\ m_3 &= \sin l \sin I. \end{aligned} \tag{203}$$

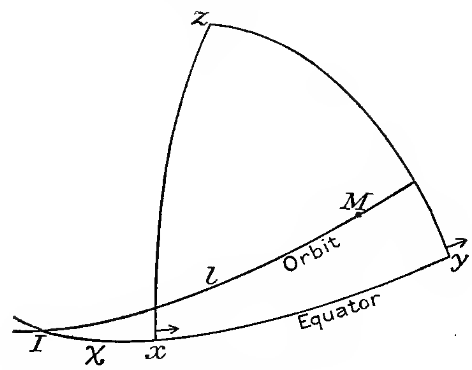


FIG. 10.

We may observe that m_2 is derivable from m_1 by putting $x + \frac{1}{2}\pi$ in place of x.

Now, for brevity, let

$$p = \cos \tfrac{1}{2} I,\ q = \sin \tfrac{1}{2} I; \tag{204}$$

then

$$\begin{aligned} m_1 &= p^2 \cos(\chi - l) + q^2 \cos(\chi + l), \\ m_2 &= -p^2 \sin(\chi - l) - q^2 \sin(\chi + l), \\ m_3 &= 2pq \sin l. \end{aligned} \tag{205}$$

$$\therefore\ m_1^2 - m_2^2 = p^4 \cos 2(\chi - l) + 2p^2q^2 \cos 2\chi + q^4 \cos 2(\chi + l), \tag{206}$$

$$-2m_1 m_2 = \text{the same with sines in place of cosines}, \tag{207}$$

$$m_2 m_3 = -p^3q \cos(\chi - 2l) + pq(p^2 - q^2)\cos\chi + pq^3 \cos(\chi + 2l), \tag{208}$$

$$m_1 m_3 = \text{the same with sines in place of cosines}, \tag{209}$$

$$\tfrac{1}{3}(m_1^2 + m_2^2 - 2m_3^2) = \tfrac{1}{3} - m_3^2 = \tfrac{1}{3}(p^4 - 4p^2q^2 + q^4) + 2p^2q^2 \cos 2l. \tag{210}$$

If for l in these expressions we put its value given by (201), the m-functions, and so $\cos^2\theta - \frac{1}{3}$, will be expressed as functions of angles all of which vary uniformly with the time. Consequently we are now able to express in like manner the entire function, (121) or (133),

$$V = \tfrac{3}{2} \frac{\mu M}{r^3} \rho^2 (\cos^2\theta - \tfrac{1}{3}). \tag{211}$$

For convenience, put

$$X = \left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}} m_1,\ Y = \left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}} m_2,\ Z = \left[\frac{c(1-e^2)}{r}\right]^{\frac{3}{2}} m_3; \tag{212}$$

equations (138) and (211) then give

$$\begin{aligned} V \div \tfrac{3}{2} \frac{\mu M}{c^3(1-e^2)^3} \rho^2 = {} & 2 p_1 p_2 XY + 2 \frac{p_1^2 - p_2^2}{2} \frac{X^2 - Y^2}{2} + 2 p_2 p_3 YZ \\ & + 2 p_1 p_3 XZ + \tfrac{3}{2} \frac{p_1^2 + p_2^2 - 2 p_3^2}{3} \frac{X^2 + Y^2 - 2 Z^2}{3}. \end{aligned} \tag{213}$$

---

* The intersection is here supposed to move only as the equinox moves.

38. Before completely expressing the functions, X^2-Y^2 XY, etc., as simple harmonic functions of the time, it is of interest to examine the special case where the moon's orbit is assumed to be circular instead of elliptical. Since $e=0$, equation (201) gives $l=\sigma_,$;

$$\therefore X^2-Y^2=p^4 \cos 2(\chi-\sigma_,)+2p^2q^2 \cos 2\chi+q^4 \cos 2(\chi+\sigma_,), \tag{214}$$

$$-2XY=p^4 \sin 2(\chi-\sigma_,)+2p^2q^2 \sin 2\chi+q^4 \sin 2(\chi+\sigma_,), \tag{215}$$

$$YZ=-p^3q \cos(\chi-2\sigma_,)+pq(p^2-q^2)\cos\chi+pq^3\cos(\chi+2\sigma_,), \tag{216}$$

$$XZ=-p^3q \sin(\chi-2\sigma_,)+pq(p^2-q^2)\sin\chi+pq^3\sin(\chi+2\sigma_,), \tag{217}$$

$$\tfrac{1}{3}(X^2+Y^2-2Z^2)=\tfrac{1}{3}(p^4-4p^2q^2+q^4)+2p^2q^2\cos 2\sigma_,. \tag{218}$$

From this it appears that there are three classes of tidal causes, and so (§ 14) three classes of tides: *

Semidiurnal tides, period about one-half day.

Diurnal tides, period about one day.

A fortnightly tide, period one-half tropical month.

The constant term in (218) indicates a permanent change in sea level because of the existence of the moon.

From § 13, the hourly variation in χ is $\gamma=15{\cdot}0410686$, and in $\sigma_,$ it is $\sigma=0{\cdot}5490165$; consequently the component tides have for speeds the following values:

$$\begin{aligned} 2(\gamma-\sigma)&=28{\cdot}9841042=\mathrm{m}_2,\\ 2\gamma&=30{\cdot}0821372=\mathrm{k}_2,\\ 2(\gamma+\sigma)&=31{\cdot}1801702=\text{——}\\ \gamma-2\sigma&=13{\cdot}9430356=\mathrm{o}_1,\\ \gamma&=15{\cdot}0410686=\mathrm{k}_1,\\ \gamma+2\sigma&=16{\cdot}1391016=\mathrm{oo},\\ 2\sigma&=\ \ 1{\cdot}0980330=\mathrm{mf}. \end{aligned} \tag{219}$$

The relative size of the components occurring in any particular X-Y-Z function can be roughly determined by putting $p=\cos\frac{1}{2}(23^\circ\ 27'{\cdot}3)=0{\cdot}979$ and $q=\sin\frac{1}{2}(23^\circ\ 27'{\cdot}3)=0{\cdot}203$. Upon inspecting YZ or XZ it is seen that the amplitude of O_1 is a trifle greater than that of lunar K_1.

39. A natural way for developing the function X^2-Y^2, say, is indicated here, the work being carried to terms in e^2:

$$X^2-Y^2=\frac{c^3(1-e^2)}{r^3}[p^4\cos 2(\chi-l)+2p^2q^2\cos 2\chi+q^4\cos 2(\chi+l)], \tag{220}$$

$$\begin{aligned}&=[1+\tfrac{3}{2}e^2+3e\cos(l-\varpi)+\tfrac{9}{2}e^2\cos 2(l-\varpi_,)+\ \ .\ \ .\ \ .\ \ ]\\ &\times[p^4\cos 2(\chi-l)+2p^2q^2\cos 2\chi+q^4\cos 2(\chi+l)].\end{aligned} \tag{221}$$

It is obvious that if l in this product be replaced by its value

$$l=\sigma_,+2e\sin(\sigma_,-\varpi_,)+\tfrac{5}{4}e^2\sin 2(\sigma_,-\varpi_,)+\ \ .\ \ .\ \ .\ \ , \tag{222}$$

X^2-Y^2 will become expressed in terms involving only such angles as vary uniformly with the time. The actual development will show that the terms are all simple harmonic functions of the time. It may be remarked that all angles involved in the above product are of the form $y_1\pm l$, or $y_1\pm 2l$, y_1 having different values for different terms.

$$\therefore \cos(y_1\pm l)=\cos\left[\overline{y_1\pm\sigma_,}\pm\overline{2e\sin(\sigma_,-\varpi_,)+\tfrac{5}{4}e^2\sin 2(\sigma_,-\varpi_,)}\right], \tag{223}$$

* Cf. Thomson and Tait, Natural Philosophy, § 808.

and similarly for cos $(y_1 \pm 2\,l)$. Because e is small we can write

$$2\,e \sin(\sigma_, - \varpi_,) + \tfrac{5}{4} e^2 \sin 2\,(\sigma_, - \varpi_,) \text{ for } \sin\left[2\,e\sin(\sigma_, - \varpi_,) + \tfrac{5}{4}e^2 \sin 2\,(\sigma_, - \varpi_,)\right] \tag{224}$$

and

$$1 - \frac{2^2\,e^2 \sin^2(\sigma_, - \varpi_,)}{2} \text{ for } \cos\left[2\,e\sin(\sigma_, - \varpi_,) + \tfrac{5}{4}e^2 \sin 2\,(\sigma_, - \varpi_,)\right]. \tag{225}$$

Upon following out the work indicated, and reducing by elementary trigonometry, the function $X^2 - Y^2$ becomes expressed in a series of cosine terms, all angles varying uniformly with the time. From the expression for $X^2 - Y^2$ analogous ones for $-2\,XY$, YZ, and XZ may be obtained.

Instead of the above, the following method, used by Darwin, is chosen because of certain symmetries which it puts in evidence:

The m-functions contain trigonometrical functions of the form $\sin(2\,l + \alpha)$ and $\cos(2\,l + \alpha)$. Let

$$R \equiv \left[\frac{c(1-e^2)}{r}\right]^3, \quad \Phi(\alpha) \equiv R\cos(2l + \alpha), \quad \Psi(\alpha) \equiv R\cos\alpha; \tag{226}$$

then

$$X^2 - Y^2 = p^4\,\Phi(-2\,\chi) + 2\,p^2q^2\,\Psi(2\,\chi) + q^4\,\Phi(2\,\chi), \tag{227}$$

$$2\,XY = p^4\,\Phi\left[-2\left(\chi + \frac{\pi}{4}\right)\right] + 2\,p^2q^2\,\Psi\left[2\left(\chi + \frac{\pi}{4}\right)\right] + q^4\,\Phi\left[2\left(\chi + \frac{\pi}{4}\right)\right], \tag{228}$$

$$YZ = -p^3q\,\Phi(-\chi) + pq\,(p^2 - q^2)\,\Psi(\chi) + pq^3\,\Phi(\chi), \tag{229}$$

$$XZ = -p^3q\,\Phi\left(-\chi + \frac{\pi}{2}\right) + pq\,(p^2 - q^2)\,\Psi\left(\chi - \frac{\pi}{2}\right) + pq\,\Phi\left(\chi - \frac{\pi}{2}\right), \tag{230}$$

$$\tfrac{1}{3}(X^2 + Y^2 - 2\,Z^2) = \tfrac{1}{3}(p^4 - 4\,p^2q^2 + q^4)R + 2\,p^2q^2\,\Phi(0, \tag{231}$$

$$\begin{aligned}\Phi(\alpha) = R\cos(2\,l + \alpha) = (1 + \tfrac{3}{2}e^2)&\cos(2\,l + \alpha)\\ &+ \tfrac{3}{2}e\left[\cos(3\,l + \alpha - \varpi_,) + \cos(l + \alpha + \varpi_,)\right]\\ &+ \tfrac{3}{4}e^2\left[\cos(4\,l + \alpha - 2\,\varpi_,) + \cos(\alpha + 2\,\varpi_,)\right]\\ &+ \cdots,\end{aligned} \tag{232}$$

$$\begin{aligned}\Psi(\alpha) = R\cos\alpha = (1 + \tfrac{3}{2}e^2)&\cos\alpha\\ &+ \tfrac{3}{2}e\left[\cos(l + \alpha - \varpi_,) + \cos(l - \alpha - \varpi_,)\right]\\ &+ \tfrac{3}{4}e^2\left[\cos(2\,l + \alpha - 2\,\varpi_,) + \cos(2\,l - \alpha - 2\,\varpi_,)\right]\\ &+ \cdots\end{aligned} \tag{233}$$

Replacing l by its value, we obtain, after suitable reductions,

$$\begin{aligned}\Phi(\alpha) = (1 - \tfrac{11}{2}e^2)\cos(2\,\sigma_, + \alpha) &- \tfrac{1}{2}e\cos(\sigma_, + \alpha + \varpi_,)\\ &+ \tfrac{7}{2}e\cos(3\,\sigma_, + \alpha - \varpi_,) + \tfrac{17}{2}e^2\cos(4\,\sigma_, + \alpha - 2\,\varpi_,)\\ &+ \cdots,\end{aligned} \tag{234}$$

$$\begin{aligned}\Psi(\alpha) = (1 - \tfrac{3}{2}e^2)\cos\alpha &+ \tfrac{3}{2}e\left[\cos(\sigma_, + \alpha - \varpi_,) + \cos(\sigma_, - \alpha - \varpi_,)\right]\\ &+ \tfrac{9}{4}e^2\left[\cos(2\,\sigma_, + \alpha - 2\,\varpi_,) + \cos(2\,\sigma_, - \alpha - 2\,\varpi_,)\right] + \cdots\end{aligned} \tag{235}$$

By § 37,

$$R = 1 - \tfrac{3}{2}e^2 + 3\,e\cos(\sigma_, - \varpi_,) + \tfrac{9}{2}e^2\cos 2\,(\sigma_, - \varpi_,) + \cdots \tag{236}$$

By giving to α the values -2χ, $+2\chi$, etc., which occur in the expressions $X^2 - Y^2$, $2XY$, YZ, XZ, $\frac{1}{3}(X^2 + Y^2 - 2Z^2)$, their developed values are obtained as far as terms in e^2.

$$
\begin{aligned}
X^2 - Y^2 = {} & (1 - \tfrac{11}{2}e^2)\,[p^4 \cos 2(\chi - \sigma_{,}) + q^4 \cos 2(\chi + \sigma_{,})] + (1 - \tfrac{3}{2}e^2)\, 2p^2q^2 \cos 2\chi \\
& + \tfrac{7}{2} e\,[p^4 \cos (2\chi - 3\sigma_{,} + \varpi_{,}) + q^4 \cos (2\chi + 3\sigma_{,} - \varpi_{,})] \\
& - \tfrac{1}{2} e\,[p^4 \cos (2\chi - \sigma_{,} - \varpi_{,}) + q^4 \cos (2\chi + \sigma_{,} + \varpi_{,})] \\
& + \tfrac{3}{2} e\, 2p^2q^2\,[\cos (2\chi + \sigma_{,} - \varpi_{,}) + \cos (2\chi - \sigma_{,} + \varpi_{,})] \\
& + \tfrac{17}{2} e^2\,[p^4 \cos (2\chi - 4\sigma_{,} + 2\varpi_{,}) + q^4 \cos (2\chi + 4\sigma_{,} - 2\varpi_{,})] \\
& + \tfrac{9}{4} e^2\, 2p^2q^2\,[\cos (2\chi + 2\sigma_{,} - 2\varpi_{,}) + \cos (2\chi - 2\sigma_{,} + 2\varpi_{,})]. \qquad (237)
\end{aligned}
$$

From (227) and (228) it may be inferred that the expression for $-2XY$ is deducible from that of $X^2 - Y^2$ by putting sine in the place of cosine. The same inference may be made otherwise. For, $-2m_1m_2 = -\frac{1}{2}\frac{d(m_1^2 - m_2^2)}{d\chi}$; and since R does not contain χ,

$$
\begin{aligned}
-\tfrac{1}{2}\frac{d(X^2 - Y^2)}{d\chi} &= -\tfrac{1}{2}\frac{d[R(m_1^2 - m_2^2)]}{d\chi} = -\tfrac{1}{2}\frac{R\,d(m_1^2 - m_2^2)}{d\chi} = -2Rm_1m_2 \\
&= -2XY. \qquad (238)
\end{aligned}
$$

But $X^2 - Y^2$ consists of cosine terms only; therefore $-2XY$ is the same expression with sines in the place of cosines. It will not be necessary to write out the expression for $-2XY$ because the coördinate axes will be so taken as to render zero the term of V which contains it. To obtain the expression for YZ from that of $X^2 - Y^2$, change 2χ into χ, p^4 into $-p^3q$, $2p^2q^2$ into $pq(p^2 - q^2)$, and q^4 into $p\,q^3$. Because of the choice of axes, the term of V containing YZ will also disappear. The expression for XZ is obtained from that of YZ by writing sines in the place of cosines.

$$
\begin{aligned}
\therefore\ XZ = {} & -(1 - \tfrac{11}{2}e^2)\,[p^3q \sin (\chi - 2\sigma_{,}) - pq^3 \sin (\chi + 2\sigma_{,})] \\
& \qquad + (1 - \tfrac{3}{2}e^2)\, pq(p^2 - q^2) \sin \chi \\
& - \tfrac{7}{2} e\,[p^3q \sin (\chi - 3\sigma_{,} + \varpi_{,}) - pq^3 \sin (\chi + 3\sigma_{,} - \varpi_{,})] \\
& + \tfrac{1}{2} e\,[p^3q \sin (\chi - \sigma_{,} - \varpi_{,}) - pq^3 \sin (\chi + \sigma_{,} + \varpi_{,})] \\
& + \tfrac{3}{2} e\, pq(p^2 - q^2)\,[\sin (\chi + \sigma_{,} - \varpi_{,}) + \sin (\chi - \sigma_{,} + \varpi_{,})] \\
& - \tfrac{17}{2} e^2\,[p^3q \sin (\chi - 4\sigma_{,} + 2\varpi_{,}) - pq^3 \sin (\chi + 4\sigma_{,} - 2\varpi_{,})] \\
& + \tfrac{9}{4} e^2\, pq(p^2 - q^2)\,[\sin (\chi + 2\sigma_{,} - 2\varpi_{,}) + \sin (\chi - 2\sigma_{,} + 2\varpi_{,})]. \qquad (239)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\tfrac{1}{3}(X^2 + Y^2 - 2Z^2) = {} & \tfrac{1}{3}(p^4 - 4p^2q^2 + q^4)\ [(1 - \tfrac{3}{2}e^2) + 3e \cos (\sigma_{,} - \varpi_{,}) + \tfrac{9}{2} e^2 \cos 2(\sigma_{,} - \varpi_{,})] \\
& + 2p^2q^2\,[(1 - \tfrac{11}{2}e^2) \cos 2\sigma_{,} + \tfrac{7}{2} e \cos (3\sigma_{,} - \varpi_{,}) - \tfrac{1}{2} e \cos (\sigma_{,} + \varpi_{,}) + \tfrac{17}{2} e^2 \cos (4\sigma_{,} - 2\varpi_{,})]. \qquad (240)
\end{aligned}
$$

These, then, are the required developments as far as terms in e^2.

40. The obliquity of the ecliptic is 23° 27′·3, and I oscillates between 5° 8′·8 greater and 5° 8′·8 less than that value. The value of q or $\sin \frac{1}{2} I$, when I is 23° 27′·3, is ·203, and its square is ·041, and its cube ·0084. The eccentricity of the lunar orbit $e = ·0549$; hence q^2 is a little smaller than e.

The preceding developments have been carried as far as e^2, principally on account of the terms involving $\frac{17}{2}e^2$, which, as e is about $\frac{1}{18}$, have nearly the same magnitude as if the coefficient had been $\frac{1}{2}e$.

It is proposed, then, to regard q^2 and q^3 as of the same order as e, and to drop all terms of the order e^2, except in the case where the numerical factor is large. This rule will be neglected with regard to one term for a special reason, which appears below; and for another, because the numerical coefficient is just sufficiently large to make it worth retaining.

Adopting this approximation, we may write (237), (239), (240), thus,—

$$X^2 - Y^2 = (1 - \tfrac{11}{2} e^2)\, p^4 \cos 2\,(\chi - \sigma_,) + (1 - \tfrac{3}{2} e^2)\, 2\, p^2 q^2 \cos 2\,\chi$$
$$+ \tfrac{7}{2}\, e\, p^4 \cos (2\,\chi - 3\,\sigma_, + \varpi_,)$$
$$- \tfrac{1}{2}\, e\, p^2\, [p^2 \cos (2\,\chi - \sigma_, - \varpi_,) - 6\, q^2 \cos (2\,\chi - \sigma_, + \varpi_,)]$$
$$+ \tfrac{17}{2}\, e^2 p^4 \cos (2\,\chi - 4\,\sigma_, + 2\,\varpi\,), \tag{241}$$

$$X\,Z = -\,(1 - \tfrac{11}{2} e^2)\, [p^3\, q \sin (\chi - 2\,\sigma_,) - p\, q^3 \sin (\chi + 2\,\sigma\,)]$$
$$+ (1 - \tfrac{3}{2} e^2)\, p\, q\, (p^2 - q^2) \sin \chi - \tfrac{7}{2}\, e\, p^3\, q \sin (\chi - 3\,\sigma + \varpi_,)$$
$$+ \tfrac{1}{2}\, e\, p\, q\, [p^2 \sin (\chi - \sigma_, - \varpi_,) + 3\, (p^2 - q^2) \sin (\chi - \sigma_, + \varpi_,)]$$
$$+ \tfrac{3}{2}\, e\, p\, q\, (p^2 - q^2) \sin (\chi + \sigma - \varpi\,) - \tfrac{17}{2}\, e^2 p^3\, q \sin (\chi - 4\,\sigma_, + 2\,\varpi_,), \tag{242}$$

$$\tfrac{1}{6}\,(X^2 + Y^2 - 2\,Z^2) = \tfrac{1}{6}\,(p^4 - 4\,p^2 q^2 + q^4)\, [(1 - \tfrac{3}{2} e^2) + 3\, e \cos (\sigma_, - \varpi_,)]$$
$$+ 2\, p^2 q^2\, [(1 - \tfrac{11}{2} e^2) \cos 2\,\sigma_, + \tfrac{7}{2}\, e \cos (3\,\sigma_, - \varpi_,)]. \tag{243}$$

The terms which have been retained in violation of the rule of approximation are that in $X^2 - Y^2$ with argument $2\,\chi - \sigma_, + \varpi_,$, and that in $\frac{1}{6}\,(X^2 + Y^2 - 2\,Z^2)$ with argument $3\,\sigma_, - \varpi_,$.

The only other term which could have any importance is

$$\tfrac{3}{2}\, e\, 2\, p^2 q^2 \cos (2\,\chi + \sigma_, - \varpi_,) \text{ in } X^2 - Y^2.^* \tag{244}$$

Since the motion of the lunar perigee is slow, the two terms in $X^2 - Y^2$ whose arguments are $2\,\chi - \sigma_, - \varpi_,$ and $2\,\chi - \sigma_, + \varpi_,$ may be combined into one having a slowly varying amplitude and period. This is done by putting the bracketed portion of

$$-\,\tfrac{1}{2}\, e\, p^2\, [p^2 \cos (2\,\chi - \sigma_, - \varpi_,) - 6\, q^2 \cos (2\,\chi - \sigma_, + \varpi_,)] \tag{245}$$

into the approximate form

$$p\, \sqrt{p^2 - 12\, q^2 \cos 2\,\varpi_,}\, \cos (2\,\chi - \sigma_, - \varpi_, - R) \tag{246}$$

where

$$\tan R = \frac{\sin 2\,\varpi_,}{\frac{1}{6} \cot^2 \frac{1}{2} I - \cos 2\,\varpi_,}. \tag{247}$$

Because

$$2\,\chi - \sigma_, + \varpi_, = (2\,\chi - \sigma_, - \varpi_,) + 2\,\varpi_,,$$

$$p^2 \cos (2\,\chi - \sigma_, - \varpi_,) - 6\, q^2 \cos (2\,\chi - \sigma_, - \varpi_,) \tag{248}$$

may be written

$$(p^2 - 6\, q^2 \cos 2\,\varpi_,) \cos (2\,\chi - \sigma_, - \varpi_,) + 6\, q^2 \sin 2\,\varpi_, \sin (2\,\chi - \sigma_, - \varpi_,). \tag{249}$$

If we assume this equivalent to

$$p\, f' \cos (\overline{2\,\chi - \sigma_, - \varpi_,} - R), \tag{250}$$

then from comparison with expression (247),

$$\tan R = \frac{6\, q^2 \sin 2\,\varpi_,}{p^2 - 6\, q^2 \cos 2\,\varpi_,} = \frac{\sin 2\,\varpi_,}{\frac{1}{6} \cot^2 \frac{1}{2}\, I - \cos 2\,\varpi_,}, \tag{251}$$

$$f'^2 = p^2 - 12\, q^2 \cos 2\,\varpi_, + 36\, \frac{q^4}{p^2}, \tag{252}$$

$$f' = \sqrt{p^2 - 12\, q^2 \cos 2\,\varpi_,}, \text{ approximately.} \tag{253}$$

* B. A. A. S. Report 1883, pp. 57, 58. Small type in the text generally implies direct quotation, as above.

The two terms thus combined may be written

$$-\tfrac{1}{2}\, ep^4 \sqrt{1-12\tan^2 \tfrac{1}{2} I \cos 2\varpi_{\prime}}\, \cos (2\chi - \sigma_{\prime} - \varpi_{\prime} - R); \tag{254}$$

that is, the term having $2\chi - \sigma_{\prime} + \varpi_{\prime}$ for argument simply produces a slow variation in the amplitude and period of the predominating one whose argument is $2\chi - \sigma_{\prime} - \varpi_{\prime}$. Since $\frac{1}{2} I$ is always less than 15°, the denominator of $\tan R$ is always positive and so R must always lie in the first or fourth quadrant.

In $X\,Z$ occur the terms

$$\tfrac{1}{2}\, epq\, [p^2 \sin (\chi - \sigma_{\prime} - \varpi_{\prime}) + 3\,(p^2 - q^2) \sin (\chi - \sigma_{\prime} + \varpi_{\prime})]. \tag{255}$$

These might be combined into one having $\chi - \sigma_{\prime} + \varpi_{\prime}$ as argument, and whose amplitude and period would be subject to slow variations. But as either argument is nearly equal to $\chi - \sigma_{\prime}$, it is convenient, as will afterwards appear, to suppose a component of argument $\chi - \sigma_{\prime}$ having a slowly varying amplitude and period. We are to transform the expression

$$4 \cos \varpi_{\prime} \sin (\chi - \sigma_{\prime}) + 2 \sin \varpi_{\prime} \cos (\chi - \sigma_{\prime}) \tag{256}$$

into the form

$$f'' \sin (\chi - \sigma_{\prime} + Q). \tag{257}$$

Comparing this with (256), we have

$$\tan Q = \tfrac{1}{2} \tan \varpi_{\prime}, \tag{258}$$

$$f''^2 = 16 - 12 \sin^2 \varpi_{\prime}, \tag{259}$$

$$f'' = 2\sqrt{\tfrac{5}{2} + \tfrac{3}{2} \cos 2\varpi_{\prime}}. \tag{260}$$

Consequently the two terms become

$$ep^3q\, \sqrt{\tfrac{5}{2} + \tfrac{3}{2} \cos 2\varpi_{\prime}}\, \sin (\chi - \sigma_{\prime} + Q) \tag{261}$$

where

$$\tan Q = \tfrac{1}{2} \tan \varpi_{\prime}.$$

Since $\tan Q$ passes through zero or infinity with $\frac{1}{2} \tan \varpi_{\prime}$, Q must always lie in the same quadrant as $\varpi_{\prime}$.

The object of the transformations (254), (261), which may seem theoretically undesirable, is as follows:—

The numerical harmonic analysis of the tides is made to extend over one year, and this period is not long enough to distinguish completely a tide whose argument is $2\chi - \sigma_{\prime} - \varpi_{\prime}$ from one whose argument is $2\chi - \sigma_{\prime} + \varpi_{\prime}$, nor one whose argument is $\chi - \sigma_{\prime} - \varpi_{\prime}$ from one whose argument is $\chi - \sigma_{\prime} + \varpi_{\prime}$. In fact, the tide with argument $2\chi - \sigma_1 + \varpi_{\prime}$ (for which no analysis has been as yet carried out) will only produce an irregularity in that of argument $2\chi - \sigma_{\prime} - \varpi_{\prime}$ called the smaller elliptic semidiurnal tide; such irregularity has in fact been noted, but no explanation has previously been given of it.

Again, the pair of terms with arguments $\chi - \sigma_{\prime} \pm \varpi_{\prime}$ will appear in the harmonic analysis with the single argument $\chi - \sigma_{\prime}$, and the resulting numbers will necessarily appear very irregular, unless compared with the theoretical expression (261).

41. *The evection and variation.*

To the first power of e, the inequality in the moon's longitude due to evection is represented by

$$\theta = s + \tfrac{15}{4}\, me \sin (s - 2h + p^*),\dagger \tag{262}$$

and in radius vector

$$\frac{c\,(1-e^2)}{r} = 1 + \tfrac{15}{8}\, me \cos (s - 2h + p^*), \tag{263}$$

where θ is put for the moon's longitude in the ecliptic, and m for the ratio of the sun's to the moon's mean angular motion.

* p, denoting mean longitude, will be marked for the present with an asterisk.

† See Godfray, An Elementary Treatise on the Lunar Theory, §§ 71, 92. For more accurate values of the coefficients of $\sin (s - 2h + p^*)$ and $\cos (s - 2h + p^*)$, see Hansen, Tables de la Lune, § 1.

If we neglect the distinction between longitudes in the orbit and in the ecliptic [which is in effect neglecting a term with coefficient $\sin^2(\frac{1}{2} \times 5^\circ 9')$], we have from (262),

$$l = \sigma_{,} + \tfrac{15}{4} me \sin(s - 2h + p^*); \tag{264}$$

whence

$$\cos(2l + \alpha) = \cos(2\sigma_{,} + \alpha) + \tfrac{15}{4} me \left[\cos(2\sigma_{,} + s - 2h + p^* + \alpha) - \cos(2\sigma_{,} - s + 2h - p^* + \alpha)\right]. \tag{265}$$

And from (263) and the definitions of R, Ψ, Φ in (226),

$$R = \left[\frac{c(1-e^2)}{r}\right]^3 = 1 + \tfrac{45}{8} me \cos(s - 2h + p^*), \tag{266}$$

$$\Psi(\alpha) = \cos\alpha + \tfrac{45}{16} me \left[\cos(s - 2h + p^* + \alpha) + \cos(s - 2h + p^* - \alpha)\right], \tag{267}$$

$$\Phi(\alpha) = \cos(2\sigma_{,} + \alpha) + \tfrac{105}{16} me \cos(2\sigma_{,} + s - 2h + p^* + \alpha) - \tfrac{15}{16} me \cos(2\sigma_{,} - s + 2h - p^* + \alpha). \tag{268}$$

Then substituting from (266), (267), (268) in (227)-(231), and dropping the terms which are merely a reproduction of those already obtained, and neglecting terms in q^2 and q^3, we have

$$X^2 - Y^2 = \tfrac{105}{16} mep^4 \cos(2\chi - 2\sigma_{,} - s + 2h - p^*) - \tfrac{15}{16} mep^4 \cos(2\chi - 2\sigma_{,} + s - 2h + p^*), \tag{269}$$

$$XZ = -\tfrac{105}{16} mep^3 q \sin(\chi - 2\sigma_{,} - s + 2h - p^*) + \tfrac{15}{16} mep^3 q \sin(\chi - 2\sigma_{,} + s - 2h + p^*)$$

$$+ \tfrac{45}{16} mepq(p^2 - q^2)\left[\sin(\chi + s - 2h + p^*) + \sin(\chi - s + 2h - p^*)\right], \tag{270}$$

$$\tfrac{1}{3}(X^2 + Y^2 - 2Z^2) = \tfrac{1}{3}(p^4 - 4p^2q^2 + q^4)\tfrac{45}{8} me \cos(s - 2h + p^*). \tag{271}$$

It must be noticed that $\frac{105}{16}me$ arises by the addition of the coefficient of the Evection in longitude to three halves of that in the reciprocal of the radius vector; that $\frac{15}{16}me$ is the difference of the same two quantities; and that $\frac{45}{8}me$ is three times the coefficient in the reciprocal of radius vector. When the development of the lunar theory is carried to higher orders these coefficients differ considerably from the amounts computed from the first term, which alone occurs in the above analysis. Hence, when these coefficients are computed, the full values of the coefficients in longitude and reciprocal of radius vector must be introduced. According to Professor Adams, the full values of the coefficients are, in longitude ·022233, and in c/r ·010022.

The ratio of the mean motions m is about $\frac{1}{13}$, and is therefore a little greater than e, hence me is somewhat greater than e^2. Thus we may abridge (269)-(271), and write the expression thus:—

$$X^2 - Y^2 = \tfrac{105}{16} mep^4 \cos(2\chi - 2\sigma_{,} - s + 2h - p^*) - \tfrac{15}{16} mep^4 \cos(2\chi - 2\sigma_{,} + s - 2h + p^*), \tag{272}$$

$$XZ = -\tfrac{105}{16} mep^3 q \sin(\chi - 2\sigma_{,} - s + 2h - p^*), \tag{273}$$

$$\tfrac{1}{3}(X^2 + Y^2 - 2Z^2) = \tfrac{1}{3}(p^4 - 4p^2q^2 + q^4)\tfrac{45}{8} me \cos(s - 2h + p^*). \tag{274}$$

The equations (272)-(274) contain the terms to be added to (241)-(243) on account of the Evection.

The Variation.

Treating this inequality in the same way as the Evection, we have

$$l = \sigma_{,} + \tfrac{11}{8} m^2 \sin 2(s - h), \tag{275}$$

$$\frac{c(1-e^2)}{r} = 1 + m^2 \cos 2(s - h), \tag{276}$$

$$R = 1 + 3m^2 \cos 2(s - h), \tag{277}$$

$$\psi(\alpha) = \cos\alpha + \tfrac{3}{2} m^2 \left[\cos(2(s-h) + \alpha) + \cos(2(s-h) - \alpha)\right], \tag{278}$$

$$\phi(\alpha) = \cos(2\sigma_{,} + \alpha) + \tfrac{23}{8} m^2 \cos(2\sigma_{,} + 2s - 2h + \alpha) + \tfrac{1}{8} m^2 \cos(2\sigma_{,} - 2s + 2h + \alpha). \tag{279}$$

Whence we have to a sufficient degree of approximation,

$$X^2 - Y^2 = \tfrac{23}{8} m^2 p^4 \cos(2\chi - 2\sigma_{,} - 2s + 2h), \quad XZ = 0, \qquad (280), (281)$$

$$\tfrac{1}{3}(X^2 + Y^2 - 2Z^2) = \tfrac{1}{3}(p^4 - 4p^2q^2 + q^4)\, 3m^2 \cos(2s - 2h). \tag{282}$$

In this case also the values of the coefficients are actually considerably greater than the amounts as computed from the first terms; and regard must be paid to this, as in the case of the Evection, when the values of the coefficients in the tidal expressions are computed. According to Professor Adams, the full values of the coefficients are, in longitude ·011489, and in c/r ·008249.

42. *The equilibrium height of tide.*

p_1, p_2, p_3 are the direction-cosines of the place of observation; and, if λ denote the latitude of the place, we have

$$p_1 = \cos\lambda,\ p_2 = 0,\ p_3 = \sin\lambda. \tag{283}$$

$$\therefore p_1^2 - p_2^2 = \cos^2\lambda,\ p_1p_2 = 0,\ p_2p_3 = 0,\ 2\,p_1p_3 = \sin 2\,\lambda, \tag{284}$$

$$\tfrac{1}{3}\,(p_1^2 + p_2^2 - 2\,p_3^2) = \tfrac{1}{3} - \sin^2\lambda. \tag{285}$$

Now supposing the place to be at the earth's surface, then

$$\rho = a, \text{ the earth's radius.}$$

$$\therefore V = \tfrac{3}{2}\,\frac{\mu M\,a^2}{c^3(1-e^2)^3}\,[\tfrac{1}{2}\cos^2\lambda\,(X^2 - Y^2) + \sin 2\,\lambda XZ + \tfrac{3}{2}\,(\tfrac{1}{3} - \sin^2\lambda)\,\tfrac{1}{3}\,(X^2 + Y^2 - 2\,Z^2)]. \tag{286}$$

The X-Y-Z functions being simple time harmonics, the principle of forced vibrations (§ 14) allows us to conclude that the forces corresponding to V will generate oscillations in the ocean of the same periods as the terms in V, but of unknown amplitudes and epochs. Now the work represented by V must clearly be equal to hg, where h is the height of the tide from the undisturbed sea level and g the force of gravity.

$$\therefore V = hg, \text{ or } h = \frac{V}{g} = \frac{Va^2}{E\mu}, \tag{287}$$

where E is the mass of the earth.

$$\therefore h = \tfrac{3}{2}\,\frac{M}{E}\left(\frac{a}{c}\right)^3 \frac{a}{(1-e^2)^3}\,[\tfrac{1}{2}\cos^2\lambda\,(X^2 - Y^3) + \sin 2\,\lambda XZ + \tfrac{3}{2}\,(\tfrac{1}{3} - \sin^2\lambda)\,\tfrac{1}{3}\,(X^2 + Y^2 - 2\,Z^2)]. \tag{288}$$

It is convenient to have the quantities

$$\frac{X^2 - Y^2}{(1-e^2)^3},\quad \frac{XZ}{(1-e^2)^3},\quad \frac{\tfrac{1}{3}\,(X^2 + Y^2 - 2Z^2)}{(1-e^2)^3} \tag{289}$$

expressed as a series of cosine terms, each sign being positive.

Since

$$\begin{aligned} -\cos x &= +\cos\,(x + \pi), \\ -\sin x &= +\cos\left(x + \frac{\pi}{2}\right), \\ +\sin x &= +\cos\left(x - \frac{\pi}{2}\right); \end{aligned} \tag{290}$$

and

$$\frac{1}{(1-e^2)^3} = 1 + 3e^2,$$

approximately, we have

$$\begin{aligned} \frac{X^2 - Y^2}{(1-e^2)^3} = {} & (1 - \tfrac{5}{2}e^2)\,p^4 \cos 2\,(\chi - \sigma_{,}) + (1 + \tfrac{3}{2}e^2)\,2\,p^2q^2 \cos 2\,\chi && \mathrm{M_2,\ K_2} \\ & + \tfrac{7}{2}\,ep^4 \cos\,(2\,\chi - 3\,\sigma_{,} + \varpi_{,}) && \mathrm{N_2} \\ & + \tfrac{1}{2}\,ep^4\,\sqrt{\{1 - 12\tan^2\tfrac{1}{2}\,I\cos 2\,\varpi_{,}\}}\,\cos\,(2\,\chi - \sigma_{,} - \varpi_{,} - R + \pi) && \mathrm{L_2} \\ & + \tfrac{17}{2}\,e^2p^4 \cos\,(2\,\chi - 4\,\sigma_{,} + 2\,\varpi_{,}) && \mathrm{2\,N} \\ & + \tfrac{105}{16}\,mep^4 \cos\,(2\,\chi - 2\,\sigma_{,} - s + 2\,h - p^*) && \nu_2 \\ & + \tfrac{15}{16}\,mep^4 \cos\,(2\,\chi - 2\,\sigma_{,} + s - 2\,h + p^* + \pi) && \lambda_2 \\ & + \tfrac{23}{8}\,m^2p^4 \cos\,(2\,\chi - 2\,\sigma_{,} - 2\,s + 2\,h), && \mu_2 \end{aligned}$$

(291)

$$\frac{XZ}{(1-e^2)^3} = (1 - \tfrac{5}{2} e^2)\,[p^3 q \cos(\chi - 2\,\sigma_{,} + \tfrac{1}{2}\pi) + pq^3 \cos(\chi + 2\,\sigma_{,} - \tfrac{1}{2}\pi)] \qquad \mathrm{O_1, OO}$$

$$+ (1 + \tfrac{3}{2} e^2)\, pq\,(p^2 - q^2) \cos(\chi - \tfrac{1}{2}\pi) \qquad \mathrm{K_1}$$

$$+ \tfrac{7}{2}\, ep^3 q \cos(\chi - 3\,\sigma_{,} + \varpi_{,} + \tfrac{1}{2}\pi) \qquad \mathrm{Q_1}$$

$$+ ep^3 q \sqrt{\{\tfrac{5}{2} + \tfrac{3}{2}\cos 2\,\varpi_{,}\}} \cos(\chi - \sigma_{,} + Q - \tfrac{1}{2}\pi) \qquad \mathrm{M_1}$$

$$+ \tfrac{3}{2}\, epq\,(p^2 - q^2) \cos(\chi + \sigma_{,} - \varpi_{,} - \tfrac{1}{2}\pi) \qquad \mathrm{J_1}$$

$$+ \tfrac{17}{2}\, e^2 p^3 q \cos(\chi - 4\,\sigma_{,} + 2\,\varpi_{,} + \tfrac{1}{2}\pi) \qquad 2\,\mathrm{Q}\dagger$$

$$+ \tfrac{105}{16}\, mep^3 q \cos(\chi - 2\,\sigma_{,} - s + 2\,h - p^* + \tfrac{1}{2}\pi), \qquad \rho_1 \ddagger$$

(292)

$$\frac{\tfrac{1}{3}(X^2 + Y^2 - 2\,Z^2)}{(1-e^2)^3} = \tfrac{1}{3}(p^4 - 4\,p^2q^2 + q^4)\,[1 + \tfrac{3}{2} e^2 + 3\,e \cos(\sigma_{,} - \varpi_{,}) \qquad \mathrm{Mm}$$

$$+ \tfrac{45}{8}\, me \cos(s - 2\,h + p^*) + 3\,m^2 \cos(2\,s - 2\,h)] \qquad \mathrm{MSf}$$

$$+ 2\,p^2q^2\,[(1 - \tfrac{5}{2} e^2) \cos 2\,\sigma_{,} + \tfrac{7}{2}\, e \cos(3\,\sigma_{,} - \varpi_{,})]. \qquad \mathrm{Mf}$$

(293)

In these expressions

$$\tan R = \frac{\sin 2\,\varpi_{,}}{\tfrac{1}{6}\cot^2 \tfrac{1}{2} I - \cos 2\,\varpi_{,}}, \quad \tan Q = \tfrac{1}{2} \tan \varpi_{,}. \tag{294}$$

$p, q, \chi, \sigma_{,}, \varpi_{,}$ should now be replaced by their values in §§ 21, 37, thus giving the general expressions for the equilibrium tidal coefficients and arguments of Table 1, Part III. All tides have the universal coefficient $\tfrac{3}{2}\dfrac{M}{E}\left(\dfrac{a}{c}\right)^3 a$, which is about $1\tfrac{3}{4}$ feet in value.§ By (288), the semidiurnals, diurnals, and tides of long period have $\cos^2 \lambda$, $\sin 2\,\lambda$, and $\tfrac{1}{2} - \tfrac{3}{2}\sin^2 \lambda$ as general coefficients.

43. *The solar tides.*

Expressions for the solar components follow, because of symmetry, from those of the lunar. To pass from the latter to the former we have to put

$$s = h,\ p^* = p_1^*,\ \xi = \nu = 0,\ \sigma = \eta,\ I = \omega,\ e = e_1,\ \varpi = \varpi_1. \tag{295}$$

In order that the relative values of the theoretical amplitudes of solar and lunar components may be readily seen, the universal coefficient $\tfrac{3}{2}\dfrac{M}{E}\left(\dfrac{a}{c}\right)^3 a$ will be retained. This involves the introduction of a factor

$$\frac{\tau_1}{\tau} = \frac{\text{mass of sun}}{\text{mass of moon}} \times \left(\frac{\text{mean dist. of moon}}{\text{mean dist. of sun}}\right)^3 = 0{\cdot}46035 = \frac{1}{2{\cdot}17226} \tag{296}$$

where the mass of the moon is assumed to be 1/81·5 that of the earth.

A tide of greater importance than some of those given in (291), (292), and (293) is one whose argument in (237) is $2\,\chi + \sigma_{,} - \varpi_{,}$. The mean value of its coefficient is 0·00323.

There is also the larger variational diurnal tide, which has been omitted: it would have a coefficient 0·00450; also an evectional termensual tide, $\tfrac{105}{16}\, me \tfrac{1}{2} \sin^2 I \cos(3\,s - 2\,h + p^*)$, with coefficient of magnitude 0·00292. All other tides in a complete development as far as the second order of small quantities, without any approximation as to the obliquity of the lunar orbit, would have smaller coefficients than those comprised in the above list. Such a development has been made by Professor J. C. Adams, and the values of all the coefficients computed therefrom, in comparison with the above.

† The symbol 2 Q is here adopted because Q_1 and 2 Q are analogous to N_2 and 2 N.

‡ A Greek letter is here adopted because λ_2 and ν_2 denote other evectional components.

§ If we assume (cf. Harkness, Solar Parallax, pp. 138, 140) $\dfrac{M}{E} = \dfrac{1}{81{\cdot}07}$, $\dfrac{a}{c} = \dfrac{1}{60{\cdot}34}$, $a = 20\,902\,000$ feet, this coefficient becomes 1·760 feet; if $\dfrac{M}{E} = \dfrac{1}{81{\cdot}5}$, it becomes 1·751 feet. This is approximately the theoretical range of the lunar tide at the equator.

44. *Tides depending on the fourth power of the moon's parallax;* M_1, M_3, *etc.*

By equation (111) this portion of the tide-producing potential in $1/r^4$ is

$$V = \frac{\mu M}{r^4}\rho^3 \left(\tfrac{5}{2}\cos^3\theta - \tfrac{3}{2}\cos\theta\right). \tag{297}$$

Neglecting the eccentricity of the lunar orbit, as well as its inclination to the plane of the earth's equator, we obtain

$$\begin{aligned} V \div \frac{\mu M}{c^4}\rho^3 &= \tfrac{5}{8}\left(p_1^3 - 3\, p_1 p_2^2\right)\left(m_1^3 - 3\, m_1 m_2^2\right) \\ &+ \tfrac{5}{8}\left(p_2^3 - 3\, p_1^2\, p_2\right)\left(m_2^3 - 3\, m_1^2\, m_2\right) \\ &+ \tfrac{3}{8}\left(p_1^3 + p_1\, p_2^2 - 4\, p_1\, p_3^2\right)\left(m_1^3 + m_1\, m_2^2\right) \\ &+ \tfrac{3}{8}\left(p_2^3 + p_1^2\, p_2 - 4\, p_2\, p_3^2\right)\left(m_1^2\, m_2 + m_2^3\right). \end{aligned} \tag{298}$$

We have seen in § 33 that a harmonic deformation of the form of V above, represents a possible shape of a sphere covered by water; that is, the equation of continuity is satisfied.

By (206)

$$m_1 = p^2\cos(\chi - l), \quad m_2 = -p^2\sin(\chi - l); \tag{299}$$

and so

$$m_1^3 - 3\, m_1\, m_2^2 = p^6\cos 3\,(\chi - l), \tag{300}$$

$$m_1^3 + m_1\, m_2^2 = p^6\cos(\chi - l). \tag{301}$$

Now put, as before,

$$p_1 = \cos\lambda,\ p_2 = 0,\ p_3 = \sin\lambda,$$

and

$$V = gh;$$

we have

$$h = \tfrac{3}{2}\frac{M}{E}\left(\frac{a}{c}\right)^3\frac{a^2}{c}\left[\tfrac{5}{12}\cos^2\lambda\, p^6\cos 3\,(\chi - l) + \tfrac{1}{12}\cos\lambda\,(1 - 5\sin^2\lambda).p^6\cos(\chi - l)\right] \tag{302}$$

Now, $\cos\lambda\,(5\sin^2\lambda - 1)$ has its maximum value $\dfrac{16}{3\sqrt{15}}$ when $\cos\lambda = \tfrac{2}{15}\sqrt{15}$: that is to say, when $\lambda = 58°\ 54'$;

thus we may write (302)

$$\begin{aligned} h = \tfrac{3}{2}\frac{M}{E}\left(\frac{a}{c}\right)^3 a \Big[&\cos^3\lambda\ \tfrac{5}{12}\left(\frac{a}{c}\right)\cos^6\tfrac{1}{2}I\cos\left[3\,t + 3\,(h - \nu) - 3\,(s - \xi)\right] \\ &+ \tfrac{3}{16}\sqrt{15}\cos\lambda\,(1 - 5\sin^2\lambda)\ \tfrac{4}{135}\sqrt{15}\left(\frac{a}{c}\right)\cos^6\tfrac{1}{2}I\ \cos\left[t + (h - \nu) - (s - \xi)\right]\Big]. \end{aligned} \tag{303}$$

In this expression observe that there is the same 'general* coefficient' outside [ ] as in the previous development; that the spherical harmonics $\cos^3\lambda$, $\tfrac{3}{16}\sqrt{15}\cos\lambda\,(5\sin^2\lambda - 1)$ have the maximum values unity, the first at the equator and the second in latitude 58° 54′. The 'speeds' of these two tides are respectively $3\,(\gamma - \sigma)$ or 43°·4761563 per mean solar hour, and $\gamma - \sigma$, or 14°·4920521 per mean solar hour.

The coefficient of the tide $3\,(\gamma - \sigma)$, which is comparable with those in (288), is

$$\tfrac{5}{12}\left(\frac{a}{c}\right)\cos^6\tfrac{1}{2}\ I, \tag{304}$$

and the mean value of this function multiplied by $\cos 3\,(\nu - \xi)$ is ·00599; also the coefficient of the tide $(\gamma - \sigma)$, likewise comparable with previous coefficient, is

$$\tfrac{4}{135}\sqrt{15}\left(\frac{a}{c}\right)\cos^6\tfrac{1}{2}I, \tag{305}$$

and the mean value of this function multiplied by $\cos(\nu - \xi)$ is ·00165.

---

* I. e. universal.

The expression for the tides is written in the form applicable to the equatorial belt bounded by latitudes 26° 34′ N. and S. (viz. where $\sin l=\frac{1}{5}\sqrt{5}$). Outside of this belt, what may be called high tide, will correspond with low water. The distribution of land on the earth will probably, however, seriously disturb the latitude of evanescent tide.

It must be noticed that the $\gamma-\sigma$ tide is comparatively small in the equatorial belt, having at the equator only $\frac{4}{9}$ of its value in latitude 58° 54′.

Referring to the schedule of theoretical importance,* we see that the ter-diurnal tide M_3 would come in last but four on the list, and the diurnal tide M_1 (with *rigorous* speed $\gamma-\sigma$) would only be about a half of the synodic fortnightly variational tide.

It thus appears that the ter-diurnal tide is smaller than some of the tides not included in our approximation, and that the diurnal tide should certainly be negligeable.

The value of the M_3 tide, however, is found with scarcely any trouble, from the numerical analysis of the tidal observations, and therefore it is proposed that it should still be evaluated.

45. *On the mean values of the coefficients.*

Any of the preceding lunar tides may be written in the form

$$J\cos(T+u) \tag{306}$$

where J is a function of I, and u a function and ν and ξ; this may be seen upon referring to Table 1. Now since I is by equations (44) or (49) a function of ω, i, and N, so also is J. The expression (306) when developed will give a term independent of N, which may be written in the form

$$J_1\cos T \tag{307}$$

wherein J_1 is the mean value of the semirange in question.

It may be proved (see Table 1 and § 22) that in no case does J involve a term with a sine of an odd multiple of N, and it may also be shown that in every term of $\sin u$ there will occur a sine of an odd muitiple of N; whence it follows that $J\sin u$ has mean value zero, and J_1 is the term independent of N in $J\cos u$.

It may also be proved that in no case does $\cos u$ involve a term in $\cos N$, and that the terms in $\cos 2N$ are all of order i^2; also it appears that J always involves a term in $\cos N$, and also terms in $\cos 2N$ of order i^2.

Hence to the degree of approximation adopted, J_1 is equal to $J_0\cos u_0$, where J_0 is the mean value of J, and $\cos u_0$ the mean value of $\cos u$.

In evaluating $\cos u_0$ from the formulæ (47)-(49), we may observe that wherever $\sin^2 N$ occurs it may be replaced by $\frac{1}{2}$; for $\sin^2 N=\frac{1}{2}-\frac{1}{2}\cos 2N$, and the $\cos 2N$ has mean value zero.

The following are the values of $\cos u_0$ thus determined from (43), (46):—

$$\begin{aligned}
&(\alpha)\quad \cos 2(\nu-\xi)_0=1-i^2\left(\frac{1-\cos\omega}{\sin\omega}\right)^2\\
&(\beta)\quad \cos 2\nu_0=1-i^2\frac{1}{\sin^2\omega}\\
&(\gamma)\quad \cos(2\xi-\nu)_0=1-\tfrac{1}{4}i^2\left(\frac{1-2\cos\omega}{\sin\omega}\right)^2\\
&(\delta)\quad \cos(2\xi+\nu)_0=1-\tfrac{1}{4}i^2\left(\frac{1+2\cos\omega}{\sin\omega}\right)^2\\
&(\varepsilon)\quad \cos\nu_0=1-\tfrac{1}{4}i^2\frac{1}{\sin^2\omega}\\
&(\zeta)\quad \cos 2\xi_0=1-i^2\cot^2\omega.
\end{aligned} \tag{308}$$

The suffix $_0$ indicating the mean value.

Similarly the following are the J_0's or mean values of J:—

$$\begin{aligned}
&(\alpha')\quad \cos^4\tfrac{1}{2}I_0=\cos^4\tfrac{1}{2}\omega\left[1+\tfrac{1}{2}i^2\frac{\sin^2\frac{1}{2}\omega-\cos\omega}{\cos^2\frac{1}{2}\omega}\right]\\
&(\beta')\ \&\ (\zeta')\quad \sin^2 I_0=\sin^2\omega\left[1+i^2\frac{1-\frac{3}{2}\sin^2\omega}{\sin^2\omega}\right]\\
&(\gamma')\quad \sin I_0\cos^2\tfrac{1}{2}I_0=\sin\omega\cos^2\tfrac{1}{2}\omega\left[1+\tfrac{1}{4}i^2\left(\frac{\cos 2\omega}{\sin^2\omega}-\frac{2\cos\omega}{\cos^2\frac{1}{2}\omega}\right)\right]\\
&(\delta')\quad \sin I_0\sin^2\tfrac{1}{2}I_0=\sin\omega\sin^2\tfrac{1}{2}\omega\left[1+\tfrac{1}{4}i^2\left(\frac{\cos 2\omega}{\sin^2\omega}+\frac{2\cos\omega}{\sin^2\frac{1}{2}\omega}\right)\right]\\
&(\varepsilon')\quad \sin I_0\cos I_0=\sin\omega\cos\omega\,[1+\tfrac{1}{4}i^2(\cot^2\omega-5)].
\end{aligned} \tag{309}$$

* This embraces the astronomical tides given in Table 1; also a termensual and an evictional monthly.

On referring to schedules [B],* it appears that (α) multiplied by (α') is the mean value of the $\cos^4 \frac{1}{2} I \cos 2(\nu - \xi)$ which occurs in the semidiurnal terms; and so on with the other letters, two and two. Performing these multiplications, and putting $1 - \frac{1}{4} i^2$ in the results as equal to $\cos^4 \frac{1}{2} i$, and $1 - \frac{3}{2} i^2$ as equal to $1 - \frac{3}{2} \sin^2 i$, we find that the mean values are all unity for the following functions, viz.:

$$\frac{\cos^4 \frac{1}{2} I \cos 2(\nu - \xi)}{\cos^4 \frac{1}{2} \omega \cos^4 \frac{1}{2} i}, \quad \frac{\sin^2 I \cos 2\nu}{\sin^2 \omega (1 - \frac{3}{2} \sin^2 i)}, \quad \frac{\sin I \cos^2 \frac{1}{2} I \cos (2\xi - \nu)}{\sin \omega \cos^2 \frac{1}{2} \omega \cos^4 \frac{1}{2} i},$$
$$\frac{\sin I \sin^2 \frac{1}{2} I \cos (2\xi + \nu)}{\sin \omega \sin^2 \frac{1}{2} \omega \cos^4 \frac{1}{2} i}, \quad \frac{\sin I \cos I \cos \nu}{\sin \omega \cos \omega (1 - \frac{3}{2} \sin^2 i)}, \quad \frac{\sin^2 I \cos 2\xi}{\sin^2 \omega \cos^4 \frac{1}{2} i}. \qquad (310)$$

Lastly, it is easy to show rigorously that the mean value of

$$\frac{1 - \frac{3}{2} \sin^2 I}{(1 - \frac{3}{2} \sin^2 \omega)(1 - \frac{3}{2} \sin^2 i)} \qquad (311)$$

is also unity.

If we write

$$\varpi = \cos \tfrac{1}{2} \omega \cos \tfrac{1}{2} i - \sin \tfrac{1}{2} \omega \sin \tfrac{1}{2} i e^{N\iota} \qquad (312)$$

$$\varkappa = \sin \tfrac{1}{2} \omega \cos \tfrac{1}{2} i + \cos \tfrac{1}{2} \omega \sin \tfrac{1}{2} i e^{N\iota} \qquad (313)$$

where ι stands for $\sqrt{-1}$; and let $\varpi_1, \varkappa_1$ denote the same functions with the sign of N changed, then it may be proved rigorously that

$$\cos^4 \tfrac{1}{2} I \cos 2(\nu - \xi) = \tfrac{1}{2}(\varpi^4 + \varpi_1^4) \qquad (314)$$

$$\sin^2 I \cos 2\nu = 2(\varpi^2 \varkappa_1^2 + \varpi_1^2 \varkappa^2) \qquad (315)$$

$$\sin I \cos^2 \tfrac{1}{2} I \cos (2\xi - \nu) = \varpi^3 \varkappa + \varpi_1^3 \varkappa_1 \qquad (316)$$

$$\sin I \sin^2 \tfrac{1}{2} I \cos (2\xi + \nu) = \varpi \varkappa^3 + \varpi_1 \varkappa_1^3 \qquad (317)$$

$$\sin I \cos I \cos \nu = (\varpi \varkappa_1 + \varpi_1 \varkappa)(\varpi \varpi_1 - \varkappa \varkappa_1) \qquad (318)$$

$$\sin^2 I \cos 2\xi = 2(\varpi^2 \varkappa^2 + \varpi_1^2 \varkappa_1^2) \qquad (319)$$

$$1 - \tfrac{3}{2} \sin^2 I = \varpi^2 \varpi_1^2 - 4 \varpi \varpi_1 \varkappa \varkappa_1 + \varkappa^2 \varkappa_1^2. \qquad (320)$$

The proof of these formulæ, and the subsequent development of the functions of the ϖ's and $\varkappa$'s, constitute the rigorous proof of the formulæ, of which the approximate proof has been indicated above. The analogy between the ϖ's and $\varkappa$'s, and the p, q of the earlier developments of this Report, is that if i vanishes $\varpi = \varpi_1 = p$, $\varkappa = \varkappa_1 = q$. [See a paper in the *Phil. Trans.* R. S. Part II. 1880, p. 713.]

Mean sea level varies slightly on account of the regression of the lunar node. The mean value of the coefficient of change in mean level due to the existence of the moon (cf. § 38) is

$$\tfrac{1}{3}(1 + \tfrac{3}{2} e^2)(1 - \tfrac{3}{2} \sin^2 i)(1 - \tfrac{3}{2} \sin^2 \omega) = 0{\cdot}25224,$$

and the variable part is, approximately,

$$-(1 + \tfrac{3}{2} e^2) \sin i \cos i \sin \omega \cos \omega \cos N, = -0{\cdot}0328 \cos N,$$

N being the longitude of the ascending node, which decreases at the rate of 19°·34 per annum or 0°·0529539 per day (§ 13).

46. *The factor f.*

Since ν, ξ are always small, the mean values of the expressions

$$\frac{\cos^4 \frac{1}{2} I}{\cos^4 \frac{1}{2} \omega \cos^4 \frac{1}{2} i} = \frac{\cos^4 \frac{1}{2} I}{0{\cdot}91538} \qquad (321)$$

$$\frac{\sin^2 I}{\sin^2 \omega (1 - \frac{3}{2} \sin^2 i)} = \frac{\sin^2 I}{0{\cdot}15652} \qquad (322)$$

$$\frac{\sin I \cos^2 \frac{1}{2} I}{\sin \omega \cos^2 \frac{1}{2} \omega \cos^4 \frac{1}{2} i} = \frac{\sin I \cos^2 \frac{1}{2} I}{0{\cdot}38005} \qquad (323)$$

$$\frac{\sin I \sin^2 \frac{1}{2} I}{\sin \omega \sin^2 \frac{1}{2} \omega \cos^4 \frac{1}{2} i} = \frac{\sin I \sin^2 \frac{1}{2} I}{0{\cdot}01638} \qquad (324)$$

* B. A. A. S. Report, 1883, p. 66, or Table 1, this manual.

$$\frac{\sin I \cos I}{\sin \omega \cos \omega \,(1 - \frac{3}{2} \sin^2 i)} = \frac{\sin 2\,I}{0{\cdot}72147} \tag{325}$$

$$\frac{\sin^2 I}{\sin^2 \omega \cos^4 \frac{1}{2} i} = \frac{\sin^2 I}{0{\cdot}15779} \tag{326}$$

are always near unity,

while the mean value of

$$\frac{1 - \frac{3}{2} \sin^2 I}{(1 - \frac{3}{2} \sin^2 \omega)\,(1 - \frac{3}{2} \sin^2 i)} = \frac{1 - \frac{3}{2} \sin^2 I}{0{\cdot}75316} \tag{327}$$

is exactly unity. But these expressions are functions of I proportional to those functions of I which are labeled "coefficients" in Table 1. Therefore they may be taken as factors f by which the mean values of the coefficients are to be multiplied in order to produce a value for a particular time.

The luni-solar tides.—In combining two waves, A and B, of equal speeds, the resultant amplitude is, § 4, Part III,

$$\sqrt{A^2 + 2\,AB \cos(\text{phase } A \sim \text{phase } B) + B^2}, \tag{328}$$

and the displacement or alteration in the phase of A due to B is an angle whose tangent is

$$\frac{\sin(\text{phase } B - \text{phase } A)}{\cos(\text{phase } B - \text{phase } A) + \dfrac{A}{B}}; \tag{329}$$

and this is so regardless of the relative sizes of A and B.

Denoting, for the moment, lunar K_1 by $[K_1]$ and solar K_1 by $\{K_1\}$, and letting the accent signify that the lougitude of the lunar node is involved, these two waves may be written

$$\{K_1\} \cos(t + h + \tfrac{1}{2}\pi - K_1{}^\circ), \tag{330}$$

$$[K_1'] \cos(t + h + \tfrac{1}{2}\pi - \nu - K_1{}^\circ). \tag{331}$$

The first is displaced by the second by an angle $-\nu'$, where

$$\tan \nu' = \frac{\sin \nu}{\cos \nu + \{K_1\}/[K_1']} \tag{332}$$

and the resultant amplitude is

$$K_1' = \sqrt{[K_1']^2 + 2\,[K_1']\,\{K_1\} \cos \nu + \{K_1\}^2}. \tag{333}$$

The phase of the resultant wave is

$$t + h + \tfrac{1}{2}\pi - K_1{}^\circ - \nu'. \tag{334}$$

Now t varies 15° per mean solar hour and h, 0·0410686, and so the hourly variation in $t + h$ is k_1; ∴ the resultant oscillation is, reckoning from $t = 0$ on a day when $h = h_0$,

$$K_1' \cos(k_1 t - \zeta) \tag{335}$$

where

$$\zeta = \kappa - \tfrac{1}{2}\pi - h_0 + \nu'. \tag{336}$$

But

$$\zeta = \kappa - (V_0 + u) \tag{337}$$

and

$$V_0 + u = \tfrac{1}{2}\pi + h_0 - \nu'. \tag{338}$$

The context shows whether t is expressed in degrees or hours. From (325),

$$\frac{[K_1']}{[K_1]} = \frac{\sin I \cos I}{\sin \omega \cos \omega \,(1 - \frac{3}{2} \sin^2 i)} = f \text{ of lunar } K_1 = f\,([K_1]). \tag{339}$$

$$\frac{[K_1']}{\{K_1\}} = \frac{\tau(1 + \frac{3}{2}\, e^2) \sin I \cos I}{\tau_1(1 + \frac{3}{2}\, e_1^2) \sin \omega \cos \omega}; \tag{340}$$

$\therefore f$ of lunisolar K_1 or $f\,(K_1)$

$$= \frac{K_1'}{\{K_1\} + [K_1]} = \frac{\sqrt{1 + 2\dfrac{\{K_1\}}{[K_1']} \cos \nu + \dfrac{\{K_1\}^2}{[K_1']^2}}}{1 + \dfrac{\{K_1\}}{[K_1]}} \times \frac{[K_1']}{[K_1]} \tag{341}$$

where

$$\frac{\{K_1\}}{[K_1]} = \frac{\tau_1\,(1 + \frac{3}{2}\, e_1^2)}{\tau\,(1 + \frac{3}{2}\, e^2)} \frac{1}{1 - \frac{3}{2} \sin^2 i} = 0{\cdot}46407, \tag{342}$$

$$\frac{\{K_1\}}{[K_1']} = \frac{\{K_1\}}{[K_1]} \frac{\sin \omega \cos \omega \,(1 - \frac{3}{2} \sin^2 i)}{\sin I \cos I}. \tag{343}$$

Similarly for K_2

$$\tan 2\,\nu'' = \frac{\sin 2\,\nu}{\cos 2\,\nu + \{K_2\}/[K_2']} \tag{344}$$

factor f for lunisolar K_2

$$= \frac{\sqrt{1 + 2\dfrac{\{K_2\}}{[K_2']} \cos 2\,\nu + \dfrac{\{K_2\}^2}{[K_2']^2}}}{1 + \dfrac{\{K_2\}}{[K_2]}} \frac{[K_2']}{[K_2]} \tag{345}$$

where

$$\frac{\{K_2\}}{[K_2]} = \frac{\tau_1\,(1 + \frac{3}{2}\, e_1^2)}{\tau\ \,(1 + \frac{3}{2}\, e^2)} \frac{1}{1 - \frac{3}{2} \sin^2 i} = 0{\cdot}46407, \tag{346}$$

$$\frac{\{K_2\}}{[K_2']} = \frac{\{K_2\}}{[K_2]} \frac{\sin^2 \omega\,(1 - \frac{3}{2} \sin^2 i)}{\sin^2 I}. \tag{347}$$

The tides L_2 *and* M_1.—By § 42 the L_2 tide is proportional to

$$\cos^4 \tfrac{1}{2} I \sqrt{1 - 12 \tan^2 \tfrac{1}{2} I \cos 2\,(p - \xi)} \times \cos\,[2\,t + 2\,(h - \nu) - 2\,(s - \xi) + (s - p) - R + \pi] \tag{348}$$

where

$$\tan R = \frac{\sin 2\,(p - \xi)}{\frac{1}{6} \cot^2 \frac{1}{2} I - \cos 2\,(p - \xi)}. \tag{349}$$

Let P denote the value of $p - \xi$ at the middle of the series considered, and suppose R to be computed for this same time. The approximate value of the f of L_2 is

$$\frac{\cos^4 \frac{1}{2} I}{\cos^4 \frac{1}{2}\,\omega \cos^4 \frac{1}{2}\, i} \sqrt{1 - 12 \tan^2 \tfrac{1}{2} I \cos 2\,P}. \tag{350}$$

By § 42 the M_1 tide is proportional to

$$\tfrac{1}{2}\, e \sin I \cos^2 \tfrac{1}{2} I \sqrt{\tfrac{5}{2} + \tfrac{3}{2} \cos 2\,(p - \xi)} \times \cos\,[t + (h - \nu) - (s - \xi) + Q + \tfrac{1}{2}\pi] \tag{351}$$

where

$$\tan Q = \tfrac{1}{2} \tan\,(p - \xi). \tag{352}$$

If P denote the value of $p - \xi$ at the middle of the series

$$\tan Q = \tfrac{1}{2} \tan P. \tag{353}$$

Since $\tan \frac{1}{2} P$ and $\tan Q$ pass through zero or infinity simultaneously, it follows that they always lie in the same quadrant.

The f of M_1 may be taken as

$$\frac{\sin I \cos^2 \frac{1}{2} I}{\sin \omega \cos^2 \frac{1}{2} \omega \cos^4 \frac{1}{2} i} \sqrt{\tfrac{5}{2} + \tfrac{3}{2} \cos 2 P}. \tag{354}$$

The average value of this expression is not very near to unity as is the average value of most of the f's of the other components. It is, in fact, about 1·5505, as is shown at the end of Table 10.

For

$$\begin{aligned}
\cos^4 \tfrac{1}{2} I, &\quad f = 1{\cdot}0003 - 0{\cdot}0373 \cos N + 0{\cdot}0002 \cos 2 N; \\
\sin I \cos^2 \tfrac{1}{2} I, &\quad f = 1{\cdot}0088 + 0{\cdot}1886 \cos N - 0{\cdot}0146 \cos 2 N; \\
K_2, &\quad f = 1{\cdot}0243 + 0{\cdot}2847 \cos N + 0{\cdot}0080 \cos 2 N; \\
K_1, &\quad f = 1{\cdot}0060 + 0{\cdot}1156 \cos N - 0{\cdot}0088 \cos 2 N; \\
\sin^2 I, &\quad f = 1{\cdot}0429 + 0{\cdot}4135 \cos N - 0{\cdot}0040 \cos 2 N; \\
1 - \tfrac{3}{2} \sin^2 I, &\quad f = 1{\cdot}0000 - 0{\cdot}1299 \cos N + 0{\cdot}0013 \cos 2 N; \\
L_2, &\quad f = 0{\cdot}9780 + \text{terms in } N \text{ and } P; \\
M_1, &\quad f = 1{\cdot}5505 + \text{terms in } N \text{ and } P.
\end{aligned} \tag{355}$$

47. Table 37 shows the equilibrium amplitudes of several components (disregarding, as usual, the mutual attraction of the fluid) for various latitudes. In order to see what type of tide may belong to a particular latitude, draw the M_2, K_1, and O_1 waves, with the amplitudes given in the table, upon separate pieces of paper. K_1 and O_1 generally conspire for extreme declinations of the moon and interfere when she is near the equator. The resultant K_1O_1 wave combined with the M_2 wave will show the diurnal inequality peculiar to the latitude selected.

Meteorological tides.—As already stated, there must generally be a tidal component S_1 whose period is a mean solar day; for, the daily variation of the barometer is a well-established fact. At some places the land and sea breezes may also give a component of this speed which, of course, combines with the one answering to the variation of the barometer.

In regard to the annual component Sa, it may be said that even if it repeat itself reasonably well at a given place, there is no reason for supposing its curve to be nearly harmonic. Consequently we should expect terms higher than the first to appear in the Fourier series representing it. The semiannual Ssa (partly astronomical and partly meteorological) is the only harmonic usually worked for.

The (equilibrium) argument of Sa is h or the mean longitude of the sun and of Ssa, it is $2 h$. These arguments become zero at the time of the vernal equinox; arguments of Sa, Ssa might be so taken as to become zero at the beginning, say, of the calendar year.

48. *Overtides or shallow-water components.*

Let the height of the tide, exclusive of shallow water components, be denoted by y'; let the total height be, as usual, denoted by y. Then y should be some function of y' such that

$$y = K_1 y' + K_2 y'^2 + K_3 y'^3 + \quad . \quad . \quad . \tag{356}$$

where K_1, K_2, K_3 are the numerical coefficients of the powers of y'. Now we know that where y' is small,

$$y = y'; \text{ that is, } K_1 = 1.$$

Therefore we shall write

$$y = A \cos (\arg A - A^\circ) + B \cos (\arg B - B^\circ) + \quad . \quad . \quad . \quad + K_2 y'^2 + K_3 y_1'^3, \tag{357}$$

wherein A, B, . . . are not shallow-water components. For the shallow-water tides constituting $K_2 y'^2$, we are not concerned with the absolute magnitude of the coefficient K_2, but

rather with the relative values of the coefficients of the constituent terms. Squaring y', taken equal to y, we have, besides the constant terms A^2, B^2, ... ,

$$\tfrac{1}{2} A^2 \cos (\arg 2 A - 2 A^\circ) + \tfrac{1}{2} B^2 \cos (\arg 2 B - 2 B^\circ).$$

$$+ AB \cos (\arg A + \arg B - A^\circ - B^\circ) + AB \cos (\arg A - \arg B - A^\circ + B^\circ)$$

$$+ \text{ similar terms whose coefficients are } \tfrac{1}{2} C^2,\ AC,\ BC,\ \ldots \qquad (358)$$

Now the shallow-water component whose argument is $\arg A \pm \arg B$, must have as its speed $a \pm b$. In this manner the following tables of speeds, arguments, and what may be called *primitive* amplitudes and epochs have been obtained. Having obtained the principal terms of y'^2, one can then proceed to the terms of y'^3 or $y' \times y'^2$. (See Table 36* for a list of the principal shallow-water components.

Considering a group of shallow-water tides whose speeds are approximately equal, let us assume that each theoretical epoch differs from each primitive epoch by a quantity E_0 which is constant for the group. Then

$$A^\circ + B^\circ - E_0 = (AB)^\circ \qquad (359)$$

where $(AB)^\circ$ is the theoretical epoch of a component whose speed is $a+b$. In like manner

$$A^\circ + A^\circ - E_0 = (AA)^\circ \qquad (360)$$

$$B^\circ + B^\circ - E_0 = (BB)^\circ. \qquad (361)$$

Suppose A to be larger than B, and suppose that $(AA)^\circ$ and of course A°, B°, have been determined from observation. Then it is possible to infer $(AB)^\circ$ and $(BB)^\circ$. In fact $E_0, = 2 A^\circ - (AA)^\circ$, substituted in the expressions for $(AB)^\circ$, $(BB)^\circ$ gives

$$(AB)^\circ = (AA)^\circ + B^\circ - A^\circ, \qquad (362)$$

$$(BB)^\circ = (AA)^\circ + 2 B^\circ - 2 A^\circ. \qquad (363)$$

Let it likewise be assumed that in each group the primitive range must be multiplied by the same constant C_0;

$$\therefore \tfrac{1}{2} C_0 \cdot A \cdot A = (AA), \qquad (364)$$

$$C_0 \cdot A \cdot B = (AB), \qquad (365)$$

$$\tfrac{1}{2} C_0 \cdot B \cdot B = (BB), \qquad (366)$$

$$C_0 \cdot A \cdot C = (AC), \qquad (367)$$

where (AA), (AB) . . . denote the theoretical amplitudes of components whose speeds are $a+a$ or $2a$, and $a \pm b$, . . . If we happen to know (AA) (and of course A, B, . . .) from observation, then (AB), (BB) . . . can be inferred. In fact

$$(AB) = 2 (AA) \cdot \frac{B}{A}, \qquad (368)$$

$$(BB) = (AA) \cdot \frac{B^2}{A^2}, \qquad (369)$$

. . . .

Applying these rules to $(MS)_4$ and S_4 the values of their coefficients given in Table 1 may be obtained. This agrees with the inferences made in § 18. For applications to nature, see Ferrel, in the Survey Report for 1882, pp. 442, 443, 445, 447.

---

* Adapted from Ferrel, United States Coast and Geodetic Survey Report, 1878, pp. 273–276.

The "corrected" equilibrium theory.

49. It has been already noticed (§ 40, Part I) that small bodies of water may obey the "corrected" equilibrium theory. That is, their surfaces may be everywhere perpendicular to the force of gravity as perturbed by the moon. It remains to develop the disturbing force into a series of harmonic terms.

The value of the potential of the tidal forces is

$$V = \mu M \left[\frac{\rho^2}{r^3}\left(\frac{3\cos^2\theta - 1}{2}\right)\right]$$

where

$$\mu = g\,\frac{a^2}{E}.$$

$$\therefore \frac{V}{g} = h = \tfrac{3}{4}\,\frac{M}{E}\,\frac{a^4}{r^3}\Big[\cos^2\lambda\cos^2\delta\cos 2(\psi - l)$$
$$+ 2\sin\lambda\cos\lambda\sin 2\delta\cos(\psi - l)$$
$$+ \tfrac{1}{3}(3\sin^2\lambda - 1)(3\sin^2\delta - 1)\Big]. \qquad (370)$$

Here l is used to denote the west longitude of the point, its former significations (§§ 25, 37) being no longer necessary.

The slopes of the disturbed spherical layer to the surface of the undisturbed sphere are

$$\frac{\partial h}{a\,\partial\lambda},\quad \frac{\partial h}{a\cos\lambda\,\partial l}, \qquad (371)$$

the former being the slope (elevation) in the south-to-north direction, the latter in the east-to-west. These slopes are the deviations of the plumb line from the vertical, or they are the forces causing its deviation where g is the vertical force.*

$$-\frac{\partial h}{a\cos\lambda\,\partial l} = \text{eastward component} = \tfrac{3}{2}\,\frac{M}{E}\,\frac{a^3}{r^3}\Big[-\cos\lambda\cos^2\delta\sin 2(\psi - l)$$
$$- \sin\lambda\sin 2\delta\sin(\psi - l)\Big]. \qquad (372)$$

$$-\frac{\partial h}{a\,\partial\lambda} = \text{southward component} = \tfrac{3}{4}\,\frac{M}{E}\,\frac{a^3}{r^3}\Big[\sin 2\lambda\cos^2\delta\cos 2(\psi - l)$$
$$- 2\cos 2\lambda\sin 2\delta\cos(\psi - l)$$
$$+ \sin 2\lambda(1 - 3\sin^2\delta)\Big]. \qquad (373)$$

Let e denote the easting of a given point from the no-tide point and s the southing, expressed in feet; then at any instant the height of the tide (H) is $e \times$ eastward component $+ s \times$ southward component. Let the height due to semidiurnal eastward component be denoted by H_{2e}, and similarly for southward component by H_{2s}, we have

$$\frac{H_{2e}}{h_2} = -\frac{2e}{a}\,\frac{\tan 2(\psi - l)}{\cos\lambda}, \qquad (374)$$

$$\frac{H_{2s}}{h_2} = +\frac{2s}{a}\tan\lambda. \qquad (375)$$

But from § 42 we have for the height of the (uncorrected) equilibrium semidiurnal lunar tide

$$h_2 = \tfrac{3}{2}\,\frac{M}{E}\left(\frac{a}{c}\right)^3 a\cos^2\lambda \times$$

$$\begin{cases} \tfrac{1}{2}(1 - \tfrac{5}{2}e^2)\cos^4\tfrac{1}{2}I\cos[2t + 2(h - \nu) - 2(s - \xi)] \\ + \tfrac{1}{2}(1 + \tfrac{3}{2}e^2)\tfrac{1}{2}\sin^2 I\cos[2t + 2(h - \nu)] \\ + \tfrac{1}{2}\cdot\tfrac{7}{2}e\cos^4\tfrac{1}{2}I\cos[2t + 2(h - \nu) - 2(s - \xi) - (s - p)] \\ + \;\ldots\ldots \end{cases} \qquad (376)$$

---

* The diurnal term of Eq. 23^{iv}, § 812, Thomson and Tait, should be multiplied by 2; and in Eq. 23^{v}, the diurnal term should have its sign changed.

In this equation e denotes the eccentricity of the lunar orbit.

The time of the maximum of a single periodic disturbance of level can be found as follows:

It is obvious that H_{2_s} has its maximum simultaneous with the maximum or minimum of h_2; H_{2_e} has its maximum three hours, or 90°, later or earlier. The resultant disturbance has its maximum between these two times. The hour angle ($c_2 t$) reckoned from the transit of the fictitious body or the maximum of h_2, is

$$\tan^{-1} \frac{H_{2_e}}{H_{2_s}}. \tag{377}$$

The corresponding height is

$$H_{2_s} \cos c_2 t + H_{2_e} \sin c_2 t. \tag{378}$$

For M_2,

$$\frac{H_{2_s}}{M_2} = \frac{2\,s}{a} \tan \lambda\,; \therefore H_{2_s} = 0{\cdot}800 \frac{s}{a} \sin 2\,\lambda, \tag{379}$$

$$H_{2_e} = -\,0{\cdot}800 \frac{2\,e}{a} \cos \lambda\,; \tag{380}$$

a, the earth's mean radius, is 20 902 000 feet, and $0{\cdot}800 \cos^2 \lambda$ the equilibrium value of M_2.

Example.—From a map of Lake Superior we see that the no-tide point (center of gravity of the surface) is 6 miles north of Keweenaw Point (Lat. 47° 32′, Lon. 87°). The line joining this 43′
point to Duluth is 210 miles in length and bears S. 76½° W. The no-tide point is 4° 19′ (or 17^m) E. of Duluth. Required the amplitude and epoch of M_2 at Duluth.

Here

$$\begin{aligned} s &= 49 \times 5280 \text{ feet,} \\ e &= -\,204 \times 5280 \quad \text{``} \end{aligned}$$

$$\begin{aligned} H_{2_s} &= 0{\cdot}0099, \\ H_{2_e} &= 0{\cdot}0557; \end{aligned}$$

$$\therefore\; c_2 t = 80°,$$

$$H_{2_s} \cos 80° + H_{2_e} \sin 80° = 0{\cdot}0566 = M_2,$$

$\dfrac{80°}{28{\cdot}984} = 2^h\,46^m =$ time of HW at Duluth in no-tide point time. $\therefore\; 2^h\,46^m - 17^m = 2^h\,29^m =$ HWI for Duluth $M_2° = 80° - 8° = 72°$.

Observation gives* $M_2 = 0{\cdot}063$ feet, $M_2° = 81°$.

50. The "corrected" equilibrium tide can be obtained from the uncorrected in the following manner whether the sea be small or large:

In the first place make two stereographic projections, one of the northern and one of the southern hemispheres, the pole in each case being at the center. Upon these mark the outlines of the sea in question. Upon a partially transparent sheet, using the same kind and size of projection, let the equilibrium heights of a given component, say of M_2, be written in their proper places. Let the center of this sheet be placed upon the center of either hemisphere, and place the radiating line of greatest height upon a given terrestrial meridian. The surface of the sea is divided by the meridians and parallels into rectangles whose areas are proportional to the cosines of their latitudes. The volume of the uncorrected equilibrium tide is found by multiplying the elementary areas into their respective thicknesses at the given time. The transparent sheet shows the thickness for the assumed time. The volume divided by the area shows how much the (uncorrected) equilibrium spheroid lies above the "corrected" equilibrium spheroid at the assumed component hour. At another hour it will have another value. These values tabulated with opposite signs will define a curve drawn once for all for the sea in question, which when added to the (uncorrected) equilibrium curve at any place will give the "corrected" equilibrium tide at that place.

* Obtained by analyzing the heavy curve shown in Plate III, App. BB, Report of the Survey of the Northern and Northwestern Lakes (1873).

So proceed with each of the important components. We may not completely separate the height into components if we, for a given declination of the moon, construct a stereographic projection with the moon distant δ from the bounding circle and the (uncorrected) equilibrium heights written upon it. But this process is less convenient than the former.

The same theory is expressed analytically after Thomson and Tait in the following manner:

If h or au denote the (uncorrected) equilibrium height of the tide, then

$$au - \alpha \tag{381}$$

will denote the "corrected" height, wherein

$$\alpha = \frac{\text{tidal volume over sea}}{\text{area of sea}}, \tag{382}$$

$$= \frac{a \iint u\, d\sigma}{\Omega} \tag{383}$$

where Ω = area of sea and the elementary area $d\sigma = \cos\lambda\, d\lambda\, dl$, and

$$u = \frac{3}{2}\frac{M}{E}\frac{a^3}{r^3}(\cos^2\theta - \tfrac{1}{3}). \tag{384}$$

This gives for the "corrected" height

$$\begin{aligned} au - \alpha = {} & \tfrac{3}{4}\frac{M}{E}\frac{a^4}{r^3}[(\cos^2\lambda\cos 2l - \mathfrak{A})\cos 2\psi + (\cos^2\lambda\sin 2l - \mathfrak{B})\sin 2\psi]\cos^2\delta \\ & + 3\frac{M}{E}\frac{a^4}{r^3}[(\sin\lambda\cos\lambda\cos l - \mathfrak{C})\cos\psi + (\sin\lambda\cos\lambda\sin l - \mathfrak{D})\sin\psi]\sin\delta\cos\delta \\ & + \tfrac{1}{4}\frac{M}{E}\frac{a^4}{r^3}(3\sin^2\lambda - 1 - \mathfrak{E})(3\sin^2\delta - 1), \end{aligned} \tag{385}$$

where

$$\mathfrak{A} = \frac{1}{\Omega}\iint \cos^2\lambda\cos 2l\, d\sigma, \qquad \mathfrak{B} = \frac{1}{\Omega}\iint \cos^2\lambda\sin 2l\, d\sigma,$$

$$\mathfrak{C} = \frac{1}{\Omega}\iint \sin\lambda\cos\lambda\cos l\, d\sigma, \qquad \mathfrak{D} = \frac{1}{\Omega}\iint \sin\lambda\cos\lambda\sin l\, d\sigma,$$

$$\mathfrak{E} = \frac{1}{\Omega}\iint (3\sin^2\lambda - 1)\, d\sigma.$$

In the integrations or quadratures for α, ψ and δ are regarded as constants, and so are taken from beneath the integration signs. This height may be written in the form

$$R_0[3\sin^2\delta - 1] + R_1\sin 2\delta\cos[\psi - l - \varepsilon_1] + R_2\cos^2\delta\cos[2(\psi - l) - \varepsilon_2]. \tag{386}$$

CHAPTER V.

THE HARMONIC ANALYSIS OF TIDAL OBSERVATIONS.

ON THE SUMMATION OF HOURLY ORDINATES.

51. In the harmonic analysis it is convenient to consider *component days*, whose lengths are the periods of the various diurnal components or twice the periods of the semidiurnals.* Such days are divided into twenty-four equal parts called *component hours*. If the tidal curve be read at the component hours and sums made by combining for each hour all readings belonging to it, twenty-four sums will be obtained. These sums are then analyzed in the manner described in § 58. To avoid tabulating the curve for each kind of component time, the tabulation in mean solar time is made to serve for all. This is done by distributing the (solar) hourly heights among the component hours as nearly as possible. The speeds or periods of the components determine where the various component hours fall upon the solar hours. The manner being always the same for a given component, tables of such correspondences between component and solar hours may be prepared as follows: If we put $s_1 = 15°$ for the hourly speed of the mean sun or diurnal solar component, and c_1 for the speed of any other diurnal component, then

$$\frac{c_1}{s_1}=\frac{c_1}{15} \tag{387}$$

will represent the portion of any component hour corresponding to a solar hour. While this comparison of component and solar hours is only required to the nearest whole component hour, in order to secure even this degree of approximation throughout the hours of a whole year, it is desirable to carry the value of $\frac{c_1}{15}$ out to about eight decimal places. For all components, zero hour is always taken to coincide with zero hour of the first day of the series. By successive additions of $\frac{c_1}{15}$ the component hour corresponding to any solar hour may be found; the first solar hour of the first day of series corresponds to the $\frac{c_1}{15}$ component hour, the second solar hour to the $\frac{2\,c_1}{15}$ component hour, and the nth solar hour from the beginning to the $\frac{nc_1}{15}$ component hour. A half component hour will be lost or gained according as c_1 is less or greater than s, when

$$\tfrac{1}{2} \times \frac{s_1}{s_1 \sim c_1} = \frac{15}{30 \sim 2\,c_1} \tag{388}$$

solar hours shall have elapsed from the beginning; that is, the solar and component hours agree from the beginning of the series until this number of solar hours has been reckoned, when the component hour taken to the nearest whole hour will differ by one from the solar hour. At subsequent regular intervals of

$$\frac{s_1}{s_1 \sim c_1} = \frac{15}{15 \sim c_1} \tag{389}$$

solar hours, a whole component hour will be lost or gained, that is, the difference between component and solar hours will increase one at each such time. If $c_1 < s_1$, as is usually the case, two adjacent solar hours at one of these times fall upon the same component hour, i. e., within a half com-

* The length of a component day in solar hours may be found by dividing 360° by the speed per hour (c_1), Table 1.

ponent hour of the time aimed at; but if $c_1 > s_1$, a component hour will be skipped, because no solar hour occurs within a half component hour of it. In either case all the solar hours are represented by component hours, the maximum divergence being a half component hour. If the maximum divergence allowed be assumed to be a half solar hour, then all solar hours are not represented by component hours when $c_1 < s_1$; and when $c_1 > s_1$, a solar hour may occasionally be taken to represent each of two consecutive component hours.

The times when the differences between solar and component hours change, are given in Table 42, designated "Component hours derived from solar hours." In this table, the values on the left hand of *each column* denote mean solar hours, and those on the right hand show how much the numbering of these hours must be altered in order to obtain the corresponding numbering of the component hours. The component and solar hours, as we have seen, start together with zero hour at the beginning of the series, and, reckoned to the nearest whole component hour, they continue to agree until the first tabular value is reached, when the component hour is the sum or the difference (according to whether a plus or minus sign is used) of the two given values. After this the component hours continue to differ from the solar hours by the first right-hand value until the next tabular value is reached, when the difference becomes that right-hand value, which is to be used until the next given value, and so on. Whenever the value to be added to the solar hour is such that the sum is equal to or exceeds 24, the sum should be diminished by 24; and whenever the solar hour is less than the value to be subtracted from it, increase the solar hour by 24 before making the subtraction; it is unnecessary to keep track of the component days.

For the long period components, where the change during a solar hour is very small, it is proposed to use the daily sums of the hourly heights as the quantities to be operated upon, and Table 43 shows what component hour each one of the daily sums corresponds to, reckoned to the nearest whole component hour.

52. As an example of the use of Table 42, find the M hours on the fourth day of series corresponding to the first five hours of solar time. Entering the table for the fourth day of series and in the column M, one sees that for the second solar hour the difference between solar and M hours is −3, which difference will continue from that time until the next tabular value; but for that portion of the day preceding the second solar hour on the fourth day of series we must look to the tabular value next above, which is −2 at the twenty-first hour of the second day of series, and as this difference continues until the next tabular value, it is the difference for the first and second M hours required. The resulting M hours are therefore as shown above. It will be noticed that the first and second solar hours both correspond to the twenty-third M hour; this is due to using whole M hours, as may be seen by multiplying 0·9661368, which is the portion of an M hour corresponding to a solar hour, by the number of solar hours from the beginning of the series up to these hours, showing that on the fourth day the first solar hour corresponds to 22·53 M hours, while the second solar hour corresponds to 23·49 M hours.

Fourth day of series.	
Solar hours.	M hours.
0	22
1	23
2	23
3	0
4	1

Whenever the difference between the solar and component hour is such that it changes in a period less than a solar day, the table gives two or more columns to the component; but the above example will suffice to explain the use of the table even in such cases.

Having tabulated the component hours which most closely correspond to the solar hours, the hourly heights of any series to be analyzed may be distributed in accordance with this relation. This, however, is a laborious process, for it not only requires as many copies of the hourly heights as there are components sought, but each copy must have its heights arranged differently, according to the relation existing between the hours of the component being worked for and the solar hours, as shown by the table.

Instead of using a table, blank forms may be made out once for all indicating where the hourly heights are to be written. Darwin in his report for 1883 gives a sample of such form. He has since prepared and published a set of blank forms for eighteen kinds of summation. Before making a particular summation the hourly heights of the series to be analyzed are to be copied into the form in the way indicated by certain marks thereon. This method requires as many copies of the hourly heights as there are components to be worked for, but does away with the inconvenience of following a table to find the order of arrangement.

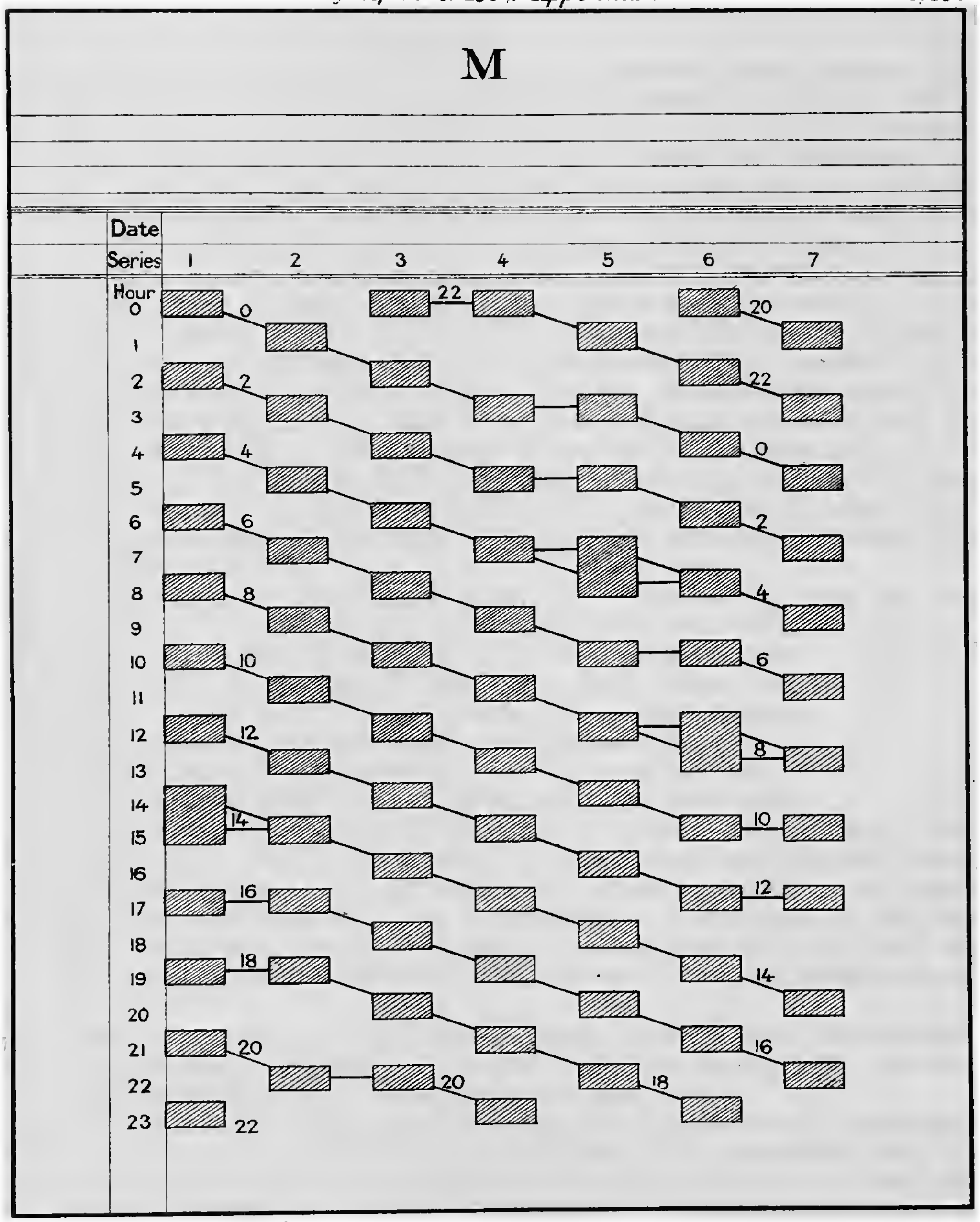


Fig. 11. Stencil for the M summation, sheet for the even hours.

Darwin* has recently devised a set of movable scales for saving labor in distributing the hourly heights for each component; but in matters of simplicity, convenience, cheapness, and rapidity of use, the apparatus does not compare favorably with the stencils described below.

53. *Stencils.*†

This term has been applied to a series of sheets so perforated as to mechanically indicate what observed hourly heights belong to the various component hours. Their use does away with the necessity of copying or rearranging the tabulated hourly heights.

Directions for constructing stencils for any given component.—Select blank forms similar to those upon which the hourly height to be summed have been tabulated. These forms should be so contrived as to cause the heights to stand sufficiently far apart from one another that no two of them can ever be seen through the same opening when a stencil is applied; the mere leaving of a large space in which to write each height will not answer, for the heights must be always found in a definite place, which may be designated by light ruled lines. In addition to the usual way of denoting the date by the day of the month, it is desirable to use a series of consecutive numerals, known as "days of series," which always begin with 1 on the first day of the record used and end so that its last value indicates the number of days taken. The stencil sheets, being intended for use upon any series, have merely the "days of series" upon them as dates. For the sake of clearness in using the stencils it has been found desirable to separate the component hours into even and odd, thus making two stencil sheets for each page of tabulation, and the sheets which thus constitute a pair must have the same days of series. Having thus prepared blank forms with a duplicate set of days of series, and having written the symbol of the component for which it is designed at the head of each page, turn to the table designated "Component hours derived from solar hours" and in the manner already explained proceed to make the blank forms into a table showing what component hour corresponds to each solar hour throughout the series, entering the even component hours on one blank and the odd hours on the other. Join those spaces containing the same component hours by a broken line, and write upon this line, at suitable distances, the number of the component hour it represents. The spaces where the component hours fall are then stamped with a steel punch which makes openings of sufficient size for showing the tabulated heights.

The accompanying figure shows the first seven days of the even hour of the M stencil. The portions inclosed with lines represent openings which are cut through the sheet so as to show the tabulated hourly heights to be summed, when the stencil is placed over them. The size of these sheets is governed by the size used in tabulating the hourly heights. The marginal arguments are the solar days of the series and the solar hours, reckoned from midnight. The broken lines joining openings show that the heights appearing through all openings so connected are to be added together, and each such sum belongs to the M hour written upon the line.

It is generally desirable, particularly in summing for smaller components, to so omit or repeat certain hourly heights that each component hour of the period covered receive one, and but one, hourly height; in other words, to make the maximum divergence between the two kinds of time a half solar, instead of a half component, hour. To construct stencils suitable for this purpose use the same table as before, omitting the unmarked hours when $c<s_1$, but repeating the marked hours when $c_1>s_1$. By marked values are meant those pointed out by the arrow, Table 42.

Directions for using the stencils.—The stencils are to be applied one sheet at a time to the tabulated hourly heights. Care must be taken to see that the proper sheet is applied in each instance, which is done by making the days of series upon the stencils agree with the corresponding days upon the sheets of hourly heights. The stencil must be placed carefully so as to accurately coincide with the tabulation beneath it, using paper weights to hold it in position while making the summations. If a broken line for any component hour runs out at the top or bottom of the stencil, there will in general be another portion of the same hour on the opposite edge, which should be included in taking the sum. As the hourly heights are summed through the stencil openings for each component hour, the sums are set down in a suitable form having the 24 component hours and pages of the tabulated heights as arguments. After all the stencils have been applied,

* B. A. A. S. Report, 1892, pp. 345–389.

† See United States Coast and Geodetic Survey Report, 1893, I, p. 108. Also this manual, Part I, § 145.

the sums for each component hour on the summation form, are combined into a single sum for each of the 24 component hours throughout the period of observation used. The divisors for these final sums are obtained by counting the number of openings in the stencils for each component hour; and as a convenience these divisors may be written, once for all, on the left margin of the stencils, or given in a table. The twenty-four means thus obtained may then be converted into residuals by subtracting from each the mean of all, and these residuals are analyzed in the way explained under harmonic analysis; or, the twenty-four means may be analyzed directly.

Sum checks.—Each page of the hourly heights of the sea should be summed horizontally and vertically before any of the stencils are applied. Any stencil sum for the whole page (adding together the sums belonging to the odd and even hours) should be the same as the sum of the vertical columns or horizontal lines, provided all hourly heights are used once and but once. But when stencils are constructed with reference to the marked values of Table 42 an additional or third stencil sheet should accompany each pair which will point out the hourly heights omitted or used twice according as $c_1 \lessgtr s$. The sum obtained by aid of this sheet must be added to or subtracted from the total sum obtained from the even and odd hour sheets in order to check the work.

For a component like K, P, R, or T, whose speed differs little from that of S, lines joining the openings will frequently become horizontal. When this happens openings should be made in the right-hand margin of the stencil sheets, so that the horizontal sums already made may be simply copied upon the proper component hours. In this connection see § 66.

54. *Adding machines.*

Several varieties of adding machines are used by the Survey in making these and other summations, viz., the "Comptometer" and the "Comptograph," manufactured by the Felt & Tarrant Manufacturing Company, Chicago, Ill.; the "Burroughs Registering Accountant," by the American Arithmometer Company, St. Louis, Mo., and a computing machine made by A. Burkhardt, Glasshütte, Germany. The machine last mentioned is designed more especially for multiplication and division.

55. *A proposed machine for obtaining component sums.*

Having seen that the stencils mechanically point out where the hourly values must go in making up the partial sums, the idea naturally suggests itself of having the equivalent of stencils so control a registering apparatus as to simultaneously give all the required summations.

Let there be as many cylinders—each, say, 26 inches long and 10 inches in diameter—as there are independent summations to be made. Each cylinder will represent the stencil of a single component for, say, 370 days. The circumference of each cylinder should be divided into 370 equal parts, each division fixing a line or element which represents a day. All cylinders are supposed to have a common movement in the direction of their axes an inch or so in extent for bringing the holes about to be mentioned, into their proper positions for the various hours of the day. Each day line contains 24 holes in the surface of the cylinder, determined by the correspondence between solar and component hours for the day in question, the small movement along the axis having been taken into account. The recording or adding apparatus for each kind of summation consists of two series of toothed wheels, all wheels of a series being upon a common shaft. The number of teeth upon each wheel of the first series may be taken as 300, and of the second series 299. The number of wheels in each series is 25, one for each component hour and one for those few hourly heights which are used only in checking the sums of the 24 partial sums. The number of revolutions made by the 300-tooth wheels can be found by subtracting the readings of the 300 tooth wheels from the readings of the 299-tooth wheels. The parts of revolutions are, of course, the direct readings.

The cylinders are placed side by side in a horizontal frame, all axes being parallel. This frame is supported by a table or framework and is capable of the small amount of motion already referred to. All cylinders are made to rotate together by means of a rack and spur wheels.

Above these cylinders, or above the intervening spaces, the two sets of toothed wheels serving as counters are mounted. The shafts bearing the 300-tooth wheels can all be made to rotate the same amount by means of parallel rods and cranks. The operator imparts motion to the mechanism by means of a crank at one end of the framework. This carries a pointer which,

moving over a graduated dial, indicates the amount of its rotation and of the 300-tooth wheels, which are not held fast by the levers about to be mentioned; that is, it indicates what number or hourly height is being entered. The crank can be released and returned to its initial position without causing any of the shafts to rotate.

The cylinders control the 300-tooth wheels by means of levers, one for each wheel. The 25 levers for each component are upon a common axis parallel to the axis of the cylinder. At one end of each lever is a needle-like projection for entering the perforations in the surface of the cylinder, while at the other end is a sharp edge, extending upward, for engaging the wheel above, thereby preventing its rotation. Since the preventing of a wheel from revolving with its shaft must give rise to friction and wear, it seems best to stop but one out of each 25 rather than to stop 24 of them. This involves no extra work on the part of the operator except the subtracting of the final machine readings from a constant number—the grand total of all hourly heights.

For each succeeding day, the cylinders are all turned forward one notch; and for each succeeding hour of the day, they are all carried forward automatically a small but constant amount along the line of their axes. As already intimated, the hourly heights when entered once are to be simultaneously summed in all the kinds of summations required in analysis, thus enabling a person to sum for all components almost as quickly as he now sums for one upon an ordinary adding machine. This machine is designed to take the place of an harmonic analyzer in tidal work. Its merits are its positive workings, the great number of components which can be included, and its simplicity of construction, in that hundreds of its parts are exactly alike.

56. *The Thomson harmonic analyzer.*

The immediate object of the harmonic analyzer is to determine the coefficients H_0, $\bar{A}$, $\bar{\bar{A}}$, $\bar{B}$, $\bar{\bar{B}}$, $\bar{C}$, $\bar{\bar{C}}$, . . . from the observed tidal curve, whose equation may be written

$$\begin{aligned} y = H_0 + \bar{A} \cos at + \bar{B} \cos bt + \bar{C} \cos ct + \quad . \ . \ . \\ + \bar{\bar{A}} \sin at + \bar{\bar{B}} \sin bt + \bar{\bar{C}} \sin ct + \quad . \ . \ . \end{aligned} \tag{390}$$

The average value of y is

$$\frac{1}{t}\int_{t=o}^{t=t} y dt. \tag{391}$$

Replacing y by its value given above the result is readily integrated, giving

$$\begin{aligned} \frac{1}{t}\Big[H_0 t + \frac{\bar{A}}{a} \sin at + \frac{\bar{B}}{b} \sin bt + \frac{\bar{C}}{c} \sin ct + \quad . \ . \ . \\ - \frac{\bar{\bar{A}}}{a} \cos at - \frac{\bar{\bar{B}}}{b} \cos bt - \frac{\bar{\bar{C}}}{c} \cos ct - \quad . \ . \ . \Big]. \end{aligned} \tag{392}$$

When t is large, the average value of y approaches H_0. Any planimeter which enables one to find the area of the curve, and so the average value of y, can be used for finding the value of H_0. For instance, if a disk rotate uniformly with t, and has upon its face a small friction wheel whose axis intersects the axis of the disk perpendicularly, and if this friction wheel be moved inward and outward according to the value of y at each instant, the number of rotations of the friction wheel will be proportional to the area of the curve.

Suppose that the rotation or angular velocity of the disk be a more complicated function of the time than t multiplied by a constant, say $\int_0^t \phi(t)\, dt$. Let the equation of the ordinate of the curve be $y = \psi(t)$. Now, if the rotation of the disk be proportional to the ordinate of a curve whose equation is

$$y' = \int_0^t \phi(t)\, dt, \tag{393}$$

the number of revolutions of the friction wheel will be proportional to

$$\int_0^t \psi(t)\, \phi(t)\, dt.* \tag{394}$$

* Thomson and Tait's Natural Philosophy, Part I, pp. 493–495, and Proc. Roy. Soc., Vol. 24 (1876), pp. 266–268.

For, in the place of $k t$ we now have $\int_0^t \phi(t)\, d t$, k being a constant, and so in the place of $k\, d\, t$, $\phi\,(t)\, d\, t$. In the harmonic analysis of the tide curve $y = \psi(t)$, the function ϕ, as will be presently explained, is of the form

$$\phi(t) = \frac{\sin}{\cos}(nt), \tag{395}$$

and so $y' = \int_0^t \phi\,(t)\, d\, t$ is of the form

$$y' = -\frac{1}{n}\cos(nt) \text{ or } \frac{1}{n}\sin(nt). \tag{396}$$

That is, the rotation of the disk is to have a simple harmonic motion instead of a uniform rotary motion. In this case the reading obtained will be, when multiplied by a proper factor, the values

$$\int_0^t y \cos at\, dt, \text{ or } \int_0^t y \sin at\, dt. \tag{397}$$

Writing for y its value (390) and integrating, the connection between the values of these integrals and $\bar{A}$, $\bar{\bar{A}}$, respectively, becomes known. For a large value of t, the number of revolutions of the friction wheel are proportional to $\bar{A}$ or $\bar{\bar{A}}$. In like manner for determining $\bar{B}$ or $\bar{\bar{B}}$ we have to mechanically evaluate the integral

$$\int_0^t y \cos bt\, dt, \text{ or } \int_0^t y \sin bt\, dt. \tag{398}$$

So on for all the other components. Since the speed ratios a, b, c . . . are constant, and since y is the same in all integrals, it becomes possible to evaluate all integrals simultaneously by having an integrator for each coefficient H_0, $\bar{A}$, $\bar{\bar{A}}$, $\bar{B}$, $\bar{\bar{B}}$, $\bar{C}$, $\bar{\bar{C}}$, etc. In fact, the friction wheel in each will be displaced from the center of the disk the same amount at any given instant of time, and suitable gears can be provided for imparting angular velocities proportional to a, b, c, etc., while the harmonic or reciprocating motion can in each case be obtained by means of a pin working in a slot perpendicular to the required motion. The rectilinear harmonic motion is converted into circular harmonic by means of a rack and toothed sector.

The disk, globe, and cylinder integrator, the invention of Prof. James Thomson, has as its peculiar merit, the avoiding of the sliding motion of the friction wheel along the diameter of the disk.* A sphere replaces the friction wheel. It is moved outward and inward along a diameter of the disk by means of a forked guide. The motion to be recorded is that which takes place perpendicularly to the radii of the disk; it is indicated by the number of revolutions of a cylinder turned by the sphere. The disk is inclined to the horizontal at an angle of about 45°; the recording cylinder turns freely upon its axis which is parallel to the plane of the disk. The sphere by its own weight crowds against the disk and cylinder. As it rolls along a diameter of the disk and an element of the cylinder, its center describes a straight line parallel to the axis of the cylinder. The ordinate of the curve shows how far the ball is to be moved.

A series of these integrators properly connected constitute the Thomson harmonic analyzer. They are arranged in a horizontal row. The point which follows the tide curve as the marigram is passed over a cylinder is fixed on a long horizontal rod. The rod has as many fork-like projections for moving the balls as there are integrators in the machine.

A working model of the first analyzer was exhibited by Sir William Thomson before the Royal Society on the 9th of May, 1878. This consisted of five disk, globe, and cylinder integrators. It served for finding H_0, $\bar{A}$, $\bar{\bar{A}}$, $\bar{B}$, $\bar{\bar{B}}$, where $b = 2\,a$. This is described in the Proceedings of the Society, Volume 27 (1878), pages 371–373. It was soon turned over to the Meteorological Office.

* Thomson and Tait's Natural Philosophy, Part I, pp. 488–492, 505–508.

The first analyzer for actual service was constructed, probably in 1879 and 1880, upon the recommendation of the Meteorological Council. A description of the machine may be found in Engineering for December 17, 1880; also in the Proceedings of the Royal Society, Volume 40 (1886), pages 382–392, where will be found tests of its working. There are seven disk, globe, and cylinder integrators for finding H_0, $\bar{A}$, $\bar{\bar{A}}$, $\bar{B}$, $\bar{\bar{B}}$, $\bar{C}$, $\bar{\bar{C}}$, where $b=2a$, $c=3a$.

The second harmonic analyzer was designed for analyzing tides. It is provided with eleven disk, globe, and cylinder integrators, and serves to determine the coefficients of five principal tidal components. Here $a=\mathrm{m}_2$, $b=\mathrm{s}_2$, $c=\mathrm{k}_1$, $d=\mathrm{o}_1$, $e=\mathrm{p}_1$.

A description of this machine is given in Volume 65 of the minutes of the Proceedings of the Institution of Civil Engineers, and in Popular Lectures and Addresses, by Sir William Thomson, Volume III, pages 177–183.

57. *Augmenting factors.*

If the observations used in finding any particular ordinate of a component do not fall exactly upon it, but constitute a group scattered (uniformly) over some distance on either side, the resulting mean value will be a trifle smaller (numerically) than the true ordinate.

Let any component C be represented by the curve

$$y = C \cos (ct + \gamma). \tag{399}$$

The mean value of y between the times $t - \frac{\tau}{2}$ and $t + \frac{\tau}{2}$ (i. e., over a group τ solar hours in length) is

$$\frac{2 \sin \frac{c\tau}{2}}{c\tau} C \cos (ct + \gamma), \tag{400}$$

and so the augmenting factor is

$$\frac{\pi}{180} \frac{c\tau}{2 \sin \frac{c\tau}{2}}, = \frac{\text{arc } c\tau}{\text{chord } c\tau}. \tag{401}$$

Since this factor applies to any ordinate of the component curve, it applies to the amplitude (C) or to the components of the amplitude ($\bar{C}$, $\bar{\bar{C}}$).

If in the summation of hourly ordinates for the short period tides, the extent of each group is a component hour (the S series excepted), then

$$\tau = 1 \text{ solar hour} \times \frac{15^\circ}{c_1}. \tag{402}$$

when $c = c_1$, the factor becomes

$$\frac{\text{arc } 15^\circ}{2 \sin 7^\circ 30'}; \tag{403}$$

when $c = c_2 = 2\, c_1$, the factor becomes

$$\frac{\text{arc } 30^\circ}{2 \sin 15^\circ}; \tag{404}$$

and so on.

The following table applies to all short period components, excepting the S series where no augmentation is required, if the sums be so taken that each hourly height of the original tabulation is used once and once only. The value of τ is one component hour.

Subscript.	Augmenting factor.	Logarithm.
1	1·00286	0·0012403
2	1·01152	0·0049745
3	1·02617	0·0112193
4	1·04720	0·0200296
5	1·07513	0·0314610
6	1·11072	0·0456046
7	1·15497	0·0625707
8	1·20920	0·0824980

If τ be a solar hour, as is the case when each component hour of the observation period receives one and but one hourly height, the augmenting factors for the various components differ somewhat from one another, the difference depending upon the difference of speed.

Their values are given in Table 38.

If 48 half-hourly heights be tabulated each day, and if the 24 component hours be scattered among these 48 times as nearly as possible in constructing the stencils or making the summation, then $\tau = \frac{1}{2}$ solar hour in the expression (401) for the augmenting factor.

THE ANALYSIS.

58. As already stated in § 49, Part I, the most probable values of the unknown quantities in the equation

$$h = H_0 + \bar{A}_1 \cos at + \bar{\bar{A}}_1 \sin at + \bar{A}_2 \cos 2\,at + \bar{\bar{A}}_2 \sin 2\,at + \quad . \quad . \quad . \tag{405}$$

where $at = 0^\circ, 15^\circ, 30^\circ, \quad . \quad . \quad . \quad 345^\circ$, are

$$H_0 = \tfrac{1}{24}\,\Sigma h, \quad \bar{A}_1 = \tfrac{1}{12}\,\Sigma h \cos at, \quad \bar{\bar{A}}_1 = \tfrac{1}{12}\,\Sigma h \sin at,$$

$$\bar{A}_2 = \tfrac{1}{12}\,\Sigma h \cos 2\,at, \quad \bar{\bar{A}}_2 = \tfrac{1}{12}\,\Sigma h \sin 2\,at,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ;^{*} \tag{406}$$

It may be well to here verify this statement.

There are 24 values of h, viz., $h_0, h_1, h_2, \quad . \quad . \quad . \quad h_{23}$, given by the 24 partial or hourly sums, corresponding to $at = 0^\circ, 15^\circ, 30^\circ, \quad . \quad . \quad . \quad 345^\circ$, i. e., to the 24 component hours. By writing equation (405) in these 24 forms, one under the other, beginning with $at = 0^\circ$, we have

$$1 \cdot H_0 + \cos 0^\circ \bar{A}_1 + \sin 0^\circ \bar{\bar{A}}_1 + \cos 2\,(0^\circ)\,\bar{A}_2 + \sin 2\,(0^\circ)\,\bar{\bar{A}}_2 + \quad . \quad . \quad . \quad - \quad h_0 = 0,$$

$$1 \cdot H_0 + \cos 15^\circ \bar{A}_1 + \sin 15^\circ \bar{\bar{A}}_1 + \cos 30^\circ \bar{A}_2 + \sin 30^\circ \bar{\bar{A}}_2 + \quad . \quad . \quad . \quad - \quad h_1 = 0,$$

$$. \quad . \tag{410}$$

The normal equations are linear equations in the unknown quantities $H_0, \bar{A}_1, \bar{\bar{A}}_1, \quad . \quad . \quad . \quad .$, having for coefficients the sums of the products of the coefficients of (410) properly taken in pairs

* Since the period of h is divided into twenty-four equal parts, $\bar{A}_i = \frac{1}{12}\,\Sigma h \cos iat$, is twice the average value of the function $h \cos iat$. When the period is supposed to be divided into a large number of parts this value becomes

$$\bar{A}_i = \frac{1}{\pi}\int_{at=0}^{at=360} h \cos iat\; d(at) = \frac{1}{\pi}\int_{at=-180}^{at=180} h \cos iat\; d(at). \tag{407}$$

Similarly

$$\bar{\bar{A}}_i = \frac{1}{\pi}\int_{at=0}^{at=360} h \sin iat\; d\,(at) = \frac{1}{\pi}\int_{at=-180}^{at=180} h \sin iat\; d(at). \tag{408}$$

Fourier's series. In general, if at be the independent variable of an arbitrary function f (not necessarily periodic as is h), we have

$$f\,(at) = \frac{A_0}{2} + \bar{A}_1 \cos at + \bar{A}_2 \cos 2\,at + \quad . \quad . \quad .$$

$$+ \bar{\bar{A}}_1 \sin at + \bar{\bar{A}}_2 \sin 2\,at + \quad . \quad . \quad . \tag{409}$$

where

$$\bar{A}_i = \frac{1}{\pi}\int_{at=-\pi}^{at=\pi} f(at) \cos iat\; d(at),$$

$$\bar{\bar{A}}_i = \frac{1}{\pi}\int_{at=-\pi}^{at=\pi} f(at) \sin iat\; d(at),$$

provided

$$-\pi < at < \pi.$$

according to the usual rule.* Then (§ 27) because the average value of the sine or cosine of an angle successively falling on all parts of the circumference is zero, and that of $(\sin)^2$ or $(\cos)^2$ equal to $\frac{1}{2}$, the normal equations reduce to equations (406) above.

The form "Harmonic analysis of tides," § 61, is for facilitating the work indicated by these normal equations. In the form, c_1, s_1 are written instead of $\bar{A}_1$, $\bar{\bar{A}}_1$; c_2, s_2 instead of $\bar{A}_2$, $\bar{\bar{A}}_2$; and so on, because upon that sheet it is not necessary to have the symbol designate the component in any way, and so the same form answers for all components.

The following symbols without subscripts apply to any component. But when it is desired to distinguish between diurnals and semidiurnals, for instance, the symbols take the subscripts 1 and 2.

n denotes the speed of any component.

$V + u$ are together the whole argument of the partial tide according to the equilibrium theory; i. e., $V + u$ is the phase of such tide provided its interval $\varkappa/n$ happens to be zero.

V is the portion of $V + u$ varying uniformly with the time.

$-\zeta$ is the initial phase of the component tide; i. e., the phase when $t = 0$.

ζ/n is the time which must elapse between the beginning of the series and the first high water of the partial tide.

$\varkappa/n$ is the interval or time between the action of the assumed cause of the partial tide and the occurrence of its high water.

$\varkappa$ is the interval expressed in degrees; it denotes the hour angle of the fictitious moon, § 24, at the time of high water of the partial tide.

H is the mean value of the amplitude of any partial tide.

R is the amplitude of any partial tide during the period analyzed.

$$c = R \cos \zeta,$$
$$s = R \sin \zeta.$$

f is the factor by which the constant H must be multiplied, chiefly on account of the variability of the lunar orbit to the plane of the equator, in order to give the amplitude R.

F is the reciprocal of f.

If it is desired to specify the several components, then the above should generally be replaced by other symbols

$$\begin{aligned}
n &= a,\ b,\ c,\ \ldots\ ; \\
V + u &= \arg A,\ \arg B,\ \arg C,\ \ldots\ ; \\
\zeta &= \zeta(A),\ \zeta(B),\ \zeta(C),\ \ldots\ ,\ \text{or} \\
-\zeta &= \alpha,\ \beta,\ \gamma,\ \ldots\ ; \\
\varkappa &= A^\circ,\ B^\circ,\ C^\circ,\ \ldots\ ; \\
H &= A,\ B,\ C,\ \ldots\ ; \\
R &= A',\ B',\ C',\ \ldots\dagger\ ,\ \text{or} \\
&= R(A),\ R(B),\ R(C)\ \ldots\ ; \\
f &= f(A),\ f(B),\ f(C)\ \ldots\ ; \\
c &= \bar{A}',\ \bar{B}',\ \bar{C}',\ \ldots\ ; \\
s &= \bar{\bar{A}}',\ \bar{\bar{B}}',\ \bar{\bar{C}}',\ \ldots\ .
\end{aligned} \tag{411}$$

59. *Ferrel's method of eliminating the effects of components other than the one sought.*

In his "Discussion of tides in Penobscot Bay," Report for 1878, Ferrel takes into account the fact that wherever the series is cut off in summing for any particular component, the hourly

* See § 28, or any text-book on least squares.

† Cf. Ferrel, Tidal Researches, § 97.

sums, and so the resulting phase and amplitude, are affected by the presence of other components. It is assumed that the diurnals when analyzed are free from the semidiurnals, quarter diurnals, etc.; similarly, that the semidiurnals are free from diurnals, quarter diurnals, etc. The amount of disturbance in a diurnal component due to another diurnal, is supposed to depend upon the length of the series and the difference in the initial phases of the two waves; similarly for any two semidiurnals. If the initial phases were each taken into account, instead of the difference merely, an additional argument would be required in the tabulation of the corrections; but it is only necessary to make analyses of hourly sums cut off at various places, in order to convince one's self that the additional argument is wholly unnecessary for a series a month or more in length.

The height of the tide, or surface of the sea, at any time may be written

$$H_0 + A\cos(at+\alpha) + B\cos(b\,t+\beta) + C\cos(ct+\gamma) + \quad . \quad . \quad . \tag{412}$$

or

$$H_0 + \bar{A}_c \cos at + \bar{\bar{A}}_c \sin at \tag{413}$$

where

$$\bar{A}_c = A\cos\alpha, + B\cos(\overline{b-a}t+\beta) + C\cos(\overline{c-a}t+\gamma) + \quad . \quad . \quad . \quad . \;, \tag{414}$$

$$\bar{\bar{A}}_c = -A\sin\alpha, - B\sin(\overline{b-a}t+\beta) - C\cos(\overline{c-a}t+\gamma) + \quad . \quad . \quad . \quad . \tag{415}$$

Subscript c indicates that quantities relating to the component A have not been purified of any of the usually small disturbances due to B, C, ...

If we put

$$\bar{A} = A\cos\alpha\,, \qquad \bar{\bar{A}} = -A\sin\alpha, \tag{416}$$

then from (414) and (415),

$$\bar{A}_c = \bar{A} - \delta\bar{A}\,, \;\therefore \bar{A} = \bar{A}_c + \delta\bar{A}\,; \tag{417}$$

$$\bar{\bar{A}}_c = \bar{\bar{A}} - \delta\bar{\bar{A}}\,, \;\therefore \bar{\bar{A}} = \bar{\bar{A}}_c + \delta\bar{\bar{A}}. \tag{418}$$

Let

$$\alpha = \alpha_c + \delta\alpha \tag{419}$$

then

$$\tan\alpha = \tan(\alpha_c + \delta\alpha) = -\frac{\bar{\bar{A}}}{\bar{A}} = -\frac{\bar{\bar{A}}_c + \delta\bar{\bar{A}}}{\bar{A}_c + \delta\bar{A}}. \tag{420}$$

The next step is to find the values of $\delta\bar{A}$, $\delta\bar{\bar{A}}$ for a series τ hours in length. Since the heights are read at all times from $t=0$ to $t=\tau$, $\bar{A}_c$ and $\bar{\bar{A}}_c$ will be increased or decreased because of the components B, C, ... , according to the length of the series. The average value of $B\cos(\overline{b-a}t+\beta)$ between $t=0$ and $t=\tau$ is

$$B\frac{1}{(b-a)\tau}[\sin(\overline{b-a}\,\tau+\beta) - \sin\beta],^* \tag{421}$$

which may be written

$$\frac{\sin\frac{1}{2}(b-a)\tau}{\frac{1}{2}(b-a)\tau}B\cos[\tfrac{1}{2}(b-a)\tau+\beta]; \tag{422}$$

$$\therefore -\delta\bar{A} = \frac{\sin\frac{1}{2}(b-a)\tau}{\frac{1}{2}(b-a)\tau}B\cos[\tfrac{1}{2}(b-a)\tau+\beta] + \frac{\sin\frac{1}{2}(c-a)\tau}{\frac{1}{2}(c-a)\tau}C\cos[\tfrac{1}{2}(c-a)\tau+\gamma] + \quad . \quad . \quad . \quad . \tag{423}$$

In like manner the average value of $B\sin(\overline{b-a}t+\beta)$ is

$$-\frac{B}{(b-a)\tau}[\cos(\overline{b-a}\,\tau+\beta) - \cos\beta]\,, \tag{424}$$

* This holds for any value of τ because the height of the tide wave is of necessity a single-valued function of t.

which is equal to

$$\frac{\sin\frac{1}{2}(b-a)\tau}{\frac{1}{2}(b-a)\tau}B\sin\left[\tfrac{1}{2}(b-a)\tau+\beta\right]. \tag{425}$$

$$\therefore +\delta\overline{\overline{A}}=\frac{\sin\frac{1}{2}(b-a)\tau}{\frac{1}{2}(b-a)\tau}B\sin\left[\tfrac{1}{2}(b-a)\tau+\beta\right]$$

$$+\frac{\sin\frac{1}{2}(c-a)\tau}{\frac{1}{2}(c-a)\tau}C\sin\left[\tfrac{1}{2}(c-a)\tau+\gamma\right]+\quad .\ .\ .\ . \tag{426}$$

In finding the effects of B, C, . . . upon the amplitude and phase of A we are concerned only with the length of the series; let us therefore suppose the initial phase of A_c, that is α_c, to be zero; then $\overline{\overline{A}}_c=0$, $\overline{A}_c=A_c$.

$$\therefore \tan\delta\alpha=-\frac{\delta\overline{\overline{A}}}{A_c+\delta\overline{A}};\ \text{or,}\ \tan\delta\zeta=\frac{\delta\overline{\overline{A}}}{R_c(A)+\delta\overline{A}} \tag{427}$$

where

$$\zeta_c(A)+\delta\zeta=\zeta(A). \tag{428}$$

$$A=\frac{A_c+\delta\overline{A}}{\cos\delta\alpha};\ \text{or,}\ R(A)=\frac{R_c(A)+\delta\overline{A}}{\cos\delta\zeta} \tag{429}$$

The required values of $\delta\overline{A}$, $\delta\overline{\overline{A}}$ are easily determined. At a time when $\alpha_c=0$, β becomes $\beta-\alpha_c$ and γ, $\gamma-\alpha_c$. $\therefore -\delta\overline{A}$ and $\delta\overline{\overline{A}}$ may be written

$$-\delta\overline{A}=+\frac{\sin\frac{1}{2}(b-a)\tau}{\frac{1}{2}(b-a)\tau}B\cos\left[\tfrac{1}{2}(b-a)\tau+\zeta_c(A)-\zeta(B)\right]$$

$$+\frac{\sin\frac{1}{2}(c-a)\tau}{\frac{1}{2}(c-a)\tau}C\cos\left[\tfrac{1}{2}(c-a)\tau+\zeta_c(A)-\zeta(C)\right]+\quad .\ .\ .\ . , \tag{430}$$

$$\delta\overline{\overline{A}}=\frac{\sin\frac{1}{2}(b-a)\tau}{\frac{1}{2}(b-a)\tau}B\sin\left[\tfrac{1}{2}(b-a)\tau+\zeta_c(A)-\zeta(B)\right]$$

$$+\frac{\sin\frac{1}{2}(c-a)\tau}{\frac{1}{2}(c-a)\tau}C\cos\left[\tfrac{1}{2}(c-a)\tau+\zeta_c(A)-\zeta(C)\right]+\quad .\ .\ .\ . \tag{431}$$

Special case.—Suppose that we are concerned with two waves, A and B, whose speeds are exactly equal. Then formulæ (430) and (431) give

$$-\delta\overline{A}=B\cos\left[\zeta_c(A)-\zeta(B)\right], \tag{432}$$

$$\delta\overline{\overline{A}}=B\sin\left[\zeta_c(A)-\zeta(B)\right]. \tag{433}$$

These values substituted in (427) and (429) clear the A of the effects of B.

Table 41 is given for the purpose of showing how the disturbing effects may be tabulated once for all for an observation period of fixed length. Somewhat smaller effects could have been obtained by selecting lengths suitable for the several components, but covering nearly the same period. The length best adapted to the determination of a particular component would generally be a synodic period of that component with one or more of the largest components of its class, i. e., with diurnals or semidiurnals according as the component is diurnal or semidiurnal.

Each tabular value consists of two parts, the first is the amplitude, or the numerical value of

$$\frac{180}{\pi}\frac{\sin\frac{1}{2}(b-a)\tau}{\frac{1}{2}(b-a)\tau},\ \text{or}\ 57{\cdot}29578\frac{\sin(b-a)\frac{\tau}{2}}{(b-a)\frac{\tau}{2}}; \tag{434}$$

the second is the angle

$$\tfrac{1}{2}(b-a)\tau \tag{435}$$

but with multiples of 180° rejected or added so as to leave the angle between 0 and +360° and,

when substituted in the numerator, to render the above amplitude positive; when π is written underneath it indicates that an odd multiple of 180° has been rejected or added.

It will be noticed that the amplitude (A_c) and phase (α_c) of the component sought are taken as they come out from the analysis; while for the components (B, C, . . .) whose effects are to be eliminated, the best amplitudes and phases obtainable are to be used. In the above work A, B, C, . . . denote the R's of A, B, C, . . . instead of the H's, and this fact may be signified by accenting the A, B, C, ...

60. *The effect of a short-period component upon daily mean sea level.*

By "daily mean sea level" we shall generally imply that the 24 hourly heights corresponding to 0^h, 1^h, 2^h, . . . 23^h are simply added together and the mean taken. The value will obviously pertain to $11^h\ 30^m$ a. m. instead of noon. The sum or mean could be made to pertain to noon by including the 0^h value of the next day, in which case half weight should be given to this value and half to the 0^h value of the day in question.

Let the equation of the short-period wave be

$$y = A \cos (at + \alpha) \tag{436}$$

in which the time is supposed to be reckoned from 0^h a. m. of the first day of the series as usual.

The average height of the surface of the sea for any day (rth day of series) is, so far as dependent upon A,

$$y_r = \frac{A}{24a} 2 \cos \left[(24\,\overline{r-1}\,a) + 11\tfrac{1}{2}\,a + \alpha \right] \sin 12a \tag{437}$$

since t is taken between $24\,(r-1) - \frac{1}{2}$ and $24\,(r-1) + 23\frac{1}{2}$ hours. This is rendered a maximum or minimum according as $\sin 12a$ is positive or negative by putting

$$\alpha = -24a\,(r-1) - 11\tfrac{1}{2}a; \tag{438}$$

this gives for the maximum elevation or depression

$$y_r = \frac{A}{24a} 2 \sin 12a.$$

Assuming that $24a$ is not far from some multiple of 360°, equation (437) may be represented by a curve whose abscissæ are proportional to $24\,(r-1) + 11\frac{1}{2}$. The amplitude of this long-period wave, which we may for the present call L, is

$$\frac{A}{24a} | 2 \sin 12a |; \tag{439}$$

the speed is

$$s \sim a;$$

the phase is

$$\lambda = \alpha \text{ [and so } \zeta\,(L) = \zeta\,(A)\text{]}, \tag{440}$$

when $\sin 12\,a$ is positive, and

$$\lambda = \alpha \pm 180° \text{ [and so } \zeta\,(L) = \zeta\,(A) \pm 180°\text{]} \tag{441}$$

when $\sin 12\,a$ is negative.

Again, suppose that α' represents the phase of A at *any* given midnight. The mean of the 24 hourly heights for the following day is, § 27,

$$\tfrac{1}{24} \sum_{t=0}^{t=23} A \cos (at + \alpha') = \frac{A \sin 12a}{24 \sin \frac{1}{2}a} \cos (11\,\tfrac{1}{2}a + \alpha'); \tag{442}$$

that is, discrete intervals increase the value of L.*

The following mechanical means of determining the average height of the sea for any given day has been suggested by Prof. J. C. Adams: †

* See Laska, Sammlung von Formeln, pp. 409, 417. † Report B. A. A. S., 1883, p. 104. Or §§ 68, 69, below.

The value of y, the average daily height for midnight preceding the rth day, dependent upon several short period components $A, B, \ldots$ is

$$y = \frac{A \sin 12a}{24 \sin \frac{1}{2}a} \cos [24\,\overline{r-1}\,a + \alpha] + \frac{B \sin 12b}{24 \sin \frac{1}{2}b} \cos [24\,\overline{r-1}\,b + \beta] + \ldots, \quad (443)$$

or if i be written for $r - 1$

$$y = \frac{A \sin 12a}{24 \sin \frac{1}{2}a} \cos [24ia + \alpha] + \frac{B \sin 12b}{24 \sin \frac{1}{2}b} \cos [24ib + \beta] + \ldots \quad (444)$$

while the expression for the height of the tide at any time is

$$y = A \cos (at + \alpha) + B \cos (bt + \beta) + \ldots. \quad (445)$$

If a tide predictor which mechanically sums (445) when the amplitudes introduced into it are $A, B, \ldots$, be set with amplitudes

$$\frac{A \sin 12a}{24 \sin \frac{1}{2}a}, \frac{B \sin 12b}{24 \sin \frac{1}{2}b}, \ldots \quad (446)$$

and with phases

$$\alpha = \arg_{\circ} A - A^{\circ}, \beta = \arg_{\circ} B - B^{\circ}, \ldots, \quad (447)$$

(to any of which 180° should be applied when the amplitude (446) is negative) it will give at 11^{h} 30^{m} of each day the average value of the 24 hourly heights of that day so far as these heights depend upon $A, B, \ldots$.

61. *Example showing the application of the harmonic analysis.*

Hourly tidal ordinates.

Station, Sitka, Alaska.
Observer, Fremont Morse.
Tabulator, A. F. Z.
Kind of time used, mean local civil.

Lat. 57° 4′ N.; long. 135° 20′ W.
Date, 1893.
Tide gauge No. 34; scale, $\frac{1}{16}$.
Readings are reduced to staff.

Day of month.	July 1	2	3	4	5	6	7	Horizontal sums.
Day of series.	1	2	3	4	5	6	7	
h. m.	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	
0 00	13·9	13·1	11·6	10·0	8·4	7·3	6·8	71·1
1 00	14·5	14·1	13·0	11·4	9·6	8·0	6·7	77·3
2 00	14·2	14·4	13·8	12·5	10·9	9·2	7·3	82·3
3 00	13·0	13·8	13·8	13·2	12·0	10·4	8·5	84·7
4 00	10·9	12·2	12·9	13·0	12·4	11·5	9·7	82·6
5 00	8·5	10·1	11·2	11·9	12·3	12·0	10·8	76·8
6 00	6·0	7·5	8·9	10·1	11·1	11·7	11·3	66·6
7 00	4·3	5·5	6·8	8·0	9·5	10·8	11·2	56·1
8 00	3·5	4·1	4·9	6·1	7·6	9·3	10·5	46·0
9 00	3·9	3·9	4·1	4·8	6·0	7·8	9·3	39·8
10 00	5·3	4·7	4·2	4·3	5·1	6·5	8·1	38·2
11 00	7·5	6·5	5·3	4·9	5·1	5·9	7·1	42·3
Noon.	9·5	8·4	7·1	6·2	5·7	6·0	6·7	49·6
13 00	11·4	10·5	9·2	8·2	7·2	6·9	6·9	60·3
14 00	12·5	12·1	11·2	10·2	9·2	8·4	7·9	71·5
15 00	12·8	12·9	12·4	11·7	10·9	10·2	9·3	80·2
16 00	12·3	12·8	12·8	12·7	12·4	11·9	11·0	85·9
17 00	11·1	12·0	12·4	12·8	13·1	13·1	12·6	87·1
18 00	9·8	10·7	11·3	12·2	13·0	13·6	13·7	84·3
19 00	8·8	9·4	10·0	10·9	12·1	13·2	14·0	78·4
20 00	8·4	8·5	8·8	9·5	10·7	12·1	13·5	71·5
21 00	8·8	8·3	7·9	8·3	9·1	10·5	12·1	65·0
22 00	10·0	8·9	8·0	7·6	8·0	8·8	10·4	61·7
23 00	11·5	10·2	8·7	7·7	7·3	7·4	8·5	61·3
Sums.	232·4	234·6	230·3	228·2	228·7	232·5	233·9	1 620·6

Hourly tidal ordinates.

Station, Sitka, Alaska.
Observer, Fremont Morse.
Tabulator, A. F. Z.
Kind of time used, mean local civil.

Lat. 57° 4′ N.; long. 135° 20′ W.
Date, 1893.
Tide gauge No. 34; scale, $\frac{1}{16}$.
Readings are reduced to staff.

Day of month.	July 8	9	10	11	12	13	14	Horizontal sums.
Day of series.	8	9	10	11	12	13	14	
h. m.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	
0 00	7.0	8.3	10.0	12.5	14.4	16.0	16.2	84.4
1 00	6.1	6.4	7.6	10.0	12.3	14.8	16.1	73.3
2 00	5.8	5.3	5.6	7.3	9.6	12.3	14.6	60.5
3 00	6.5	5.0	4.2	5.0	6.6	9.3	12.1	48.7
4 00	7.5	5.5	3.9	3.5	4.1	6.1	8.9	39.5
5 00	8.9	6.8	4.5	3.0	2.5	3.6	5.7	35.0
6 00	10.2	8.4	6.0	3.8	2.2	2.0	3.2	35.8
7 00	11.0	9.9	7.9	5.5	3.3	2.0	1.9	41.5
8 00	11.2	10.9	9.7	7.6	5.2	3.2	2.1	49.9
9 00	10.7	11.5	11.1	9.7	7.8	5.6	3.8	60.2
10 00	9.9	11.3	11.8	11.5	10.2	8.4	6.4	69.5
11 00	8.9	10.6	11.8	12.4	12.0	11.0	9.3	76.0
Noon.	8.0	9.6	11.2	12.4	12.9	12.6	11.7	78.4
13 00	7.6	8.9	10.3	11.6	12.8	13.5	13.3	78.0
14 00	7.9	8.4	9.2	10.5	11.9	13.2	13.9	75.0
15 00	8.8	8.5	8.6	9.2	10.5	12.1	13.4	71.1
16 00	10.2	9.3	8.6	8.5	9.1	10.5	11.9	68.1
17 00	11.8	10.7	9.5	8.4	8.2	8.9	10.1	67.6
18 00	13.4	12.4	10.8	9.4	8.2	7.9	8.5	70.6
19 00	14.4	13.9	12.6	11.0	9.2	8.1	7.7	76.9
20 00	14.5	14.8	14.3	12.8	10.9	9.1	7.9	84.3
21 00	13.8	14.9	15.3	14.5	13.0	11.1	9.1	91.7
22 00	12.3	14.0	15.4	15.6	14.8	13.3	11.2	96.6
23 00	10.2	12.2	14.4	15.5	16.0	15.2	13.5	97.0
Sums.	236.6	237.5	234.3	231.2	227.7	229.8	232.5	1 629.6

Hourly tidal ordinates.

Station, Sitka, Alaska.
Observer, Fremont Morse.
Tabulator, A. F. Z.
Kind of time used, mean local civil.

Lat. 57° 4′ N., long. 135° 20′ W.
Date, 1893.
Tide gauge No. 34; scale, $\frac{1}{16}$.
Readings are reduced to staff.

Day of month.	July 15	16	17	18	19	20	21	Horizontal sums.
Day of series.	15	16	17	18	19	20	21	
h. m.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	
0 00	15.2	13.9	11.8	9.3	7.6	6.8	7.2	71.8
1 00	16.2	15.6	13.8	11.0	8.8	7.3	7.0	79.7
2 00	15.9	16.3	15.2	12.7	10.3	8.3	7.3	86.0
3 00	14.3	15.9	15.6	13.7	11.7	9.6	8.1	88.9
4 00	11.6	14.0	14.8	13.9	12.5	10.7	9.2	86.7
5 00	8.5	11.4	13.2	13.2	12.7	11.5	10.2	80.7
6 00	5.4	8.3	10.5	11.4	11.9	11.6	11.0	70.1
7 00	3.1	5.6	7.7	9.1	10.4	11.0	11.2	58.1
8 00	2.4	4.1	5.6	7.1	8.7	9.9	10.9	48.7
9 00	3.1	3.7	4.4	5.6	7.1	8.8	10.2	42.9
10 00	4.9	4.7	4.5	5.0	6.1	7.7	9.4	42.3
11 00	7.8	6.8	5.7	5.5	5.9	7.1	8.7	47.5
Noon.	10.5	9.4	7.9	6.8	6.5	7.1	8.3	56.5
13 00	12.8	12.0	10.2	8.8	7.8	7.8	8.5	67.9
14 00	14.3	13.9	12.6	11.0	9.7	9.1	9.1	79.7
15 00	14.5	14.9	14.1	12.8	11.5	10.6	10.2	88.6
16 00	13.7	14.9	14.7	14.0	13.0	12.2	11.4	93.9
17 00	12.0	13.6	14.1	14.2	13.7	13.2	12.7	93.5
18 00	10.2	11.8	12.7	13.4	13.6	13.6	13.5	88.8
19 00	8.6	9.9	10.7	11.8	12.6	13.4	13.8	80.8
20 00	7.9	8.4	9.0	10.1	11.1	12.4	13.4	72.3
21 00	8.2	7.9	7.7	8.4	9.4	10.9	12.4	64.9
22 00	9.6	8.5	7.3	7.3	8.1	9.4	10.9	61.1
23 00	11.8	10.0	7.9	7.1	7.1	8.0	9.4	61.3
Sums.	242.5	255.5	251.7	243.2	237.8	238.0	244.0	1 712.7

Hourly tidat ordinates.

Station, Sitka, Alaska.
Observer, Fremont Morse.
Tabulator, A. F. Z.
Kind of time used, mean local civil.

Lat. 57° 4′ N., long. 135° 20′ W.
Date, 1893.
Tide gauge No. 34; scale $\frac{1}{16}$.
Readings are reduced to staff.

Day of month.	July 22	23	24	25	26	27	28	Horizontal sums.
Day of series.	22	23	24	25	26	27	28	
h. m.	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	
0 00	8·0	8·8	10·1	11·4	12·9	14·9	14·8	80·9
1 00	7·2	7·6	8·4	9·7	11·3	13·5	14·3	72·0
2 00	6·8	6·5	6·9	7·8	9·2	11·6	12·5	61·3
3 00	7·0	6·0	5·8	6·2	7·2	9·2	10·3	51·7
4 00	7·7	6·2	5·4	5·1	5·5	7·1	7·8	44·8
5 00	8·8	7·0	5·7	4·8	4·8	5·4	5·7	42·2
6 00	9·8	8·1	6·5	5·3	4·7	4·7	4·3	43·4
7 00	10·6	9·3	7·9	6·6	5·8	5·1	4·0	49·3
8 00	11·0	10·2	9·2	8·2	7·4	6·4	4·9	57·3
9 00	10·9	10·8	10·4	9·7	9·0	8·2	6·6	65·6
10 00	10·5	10·9	11·1	11·0	10·8	10·1	8·8	73·2
11 00	9·9	10·7	11·3	11·7	12·2	12·0	10·9	78·7
Noon.	9·3	10·2	10·9	11·7	12·6	13·0	12·5	80·2
13 00	9·0	9·7	10·4	11·4	12·5	13·2	13·2	79·4
14 00	9·1	9·3	9·8	10·6	11·8	12·6	13·0	76·2
15 00	9·6	9·3	9·3	9·8	10·9	11·6	12·1	72·6
16 00	10·5	9·7	9·3	9·3	10·1	10·3	10·7	69·9
17 00	11·5	10·5	9·7	9·3	9·6	9·4	9·4	69·4
18 00	12·6	11·6	10·6	10·0	9·9	9·1	8·6	72·4
19 00	13·3	12·6	11·7	11·1	10·8	9·5	8·6	77·6
20 00	13.3	13·2	12·8	12·4	12·2	10·7	9·4	84·0
21 00	12·9	13·3	13·4	13·4	13·7	12·3	10·9	89·9
22 00	11·9	12·8	13·4	14·0	14·9	13·8	12·7	93·5
23 00	10·4	11·6	12·7	13·8	15·3	14·9	14·2	92·9
Sums.	241·6	235·9	232·7	234·3	245·1	248·6	240·2	1 678·4

Hourly tidal ordinates.

Station, Sitka, Alaska.
Observer, Fremont Morse.
Tabulator, A. F. Z.
Kind of time used, mean local civil.

Lat. 57° 4′ N., long. 135° 20′ W.
Date, 1893.
Tide gauge No. 34; scale $\frac{1}{16}$.
Readings are reduced to staff.

Day of month.	July 29	Day of month.	July 29	Day of month.	July 29	Day of month.	July 29
Day of series.	29	Day of series.	29	Day of series.	29	Day of series.	29
h. m.	*Feet.*	*h. m.*	*Feet.*	*h. m.*	*Feet.*	*h. m.*	*Feet.*
0 00	15·0	7 00	3·8	14 00	13·4	20 00	8·6
1 00	14·9	8 00	4·0	15 00	12·8	21 00	9·8
2 00	13·8	9 00	5·4	16 00	11·5	22 00	11·7
3 00	11·8	10 00	7·5	17 00	10·0	23 00	13·4
4 00	9·1	11 00	9·8	18 00	8·8		
5 00	6·6	Noon.	11·9	19 00	8·2	Sums.	239·6
6 00	4·7	13 00	13·1				

Hourly sums.

Station, Sitka, Alaska. Lat. 57° 4′ N., long. 135° 20′ W.

Component, M.
Computer, D. S. B.
Kind of time used, mean local civil.

	Year.	Month.	Day.	Hour.
Observations begin	1893	July	1	0
Observations end	1893	July	29	23

Page.	0h	1h	2h	3h	4h	5h	6h	7h	8h	9h	10h	11h
1	89·3	91·4	88·0	78·9	73·6	50·8	39·9	34·5	42·0	43·6	55·8	69·1
2	62·9	75·6	83·9	98·9	82·7	75·2	66·4	67·8	57·2	62·2	71·8	84·9
3	80·1	105·9	99·0	98·1	90·8	79·2	75·8	48·1	44·5	46·0	60·6	62·6
4	102·6	82·5	84·1	82·1	86·5	63·4	51·2	48·7	39·8	41·2	40·0	48·2
5		15·0	14·9	13·8	11·8	9·1	6·6	4·7	3·8	4·0	5·4	7·5
Sums.	334·9	370·4	369·9	371·8	345·4	277·7	239·9	203·8	187·3	197·0	233·6	272·3
Divisors.	29	29	28	29	30	28	29	29	29	29	29	28
Means.	11·55	12·77	13·21	12·82	11·51	9·92	8·27	7·03	6·46	6·79	8·06	9·72
Residuals.	+1·66	+2·88	+3·32	+2·93	+1·62	+0·03	−1·62	−2·86	−3·43	−3·10	−1·83	−0·17

Page.	12h	13h	14h	15h	16h	17h	18h	19h	20h	21h	22h	23h
1	81·0	103·0	103·5	86·0	76·9	66·4	50·7	54·8	47·4	52·6	60·4	81·0
2	112·9	92·3	92·4	86·0	73·7	66·3	46·7	32·9	24·9	25·2	38·9	47·9
3	72·0	93·9	97·3	104·8	95·0	66·4	52·0	41·4	37·4	44·9	51·3	65·6
4	59·0	81·8	79·9	83·6	82·9	79·0	82·4	75·7	64·3	66·3	72·3	80·9
5	9·8	11·9	13·1	13·4	12·8	11·5	10·0	8·8	8·2	8·6	21·5	13·4
Sums.	334·7	382·9	386·2	373·8	341·3	289·6	241·8	213·6	182·2	197·6	244·4	288·8
Divisors.	29	30	29	29	29	29	29	30	28	29	30	29
Means.	11·54	12·76	13·32	12·89	11·77	9·99	8·34	7·12	6·51	6·81	8·15	9·96
Residuals.	+1·65	+2·87	+3·43	+3·00	+1·88	+0·10	−1·55	−2·77	−3·38	−3·08	−1·74	+0·07

Sum of means = 237·27
Mean = $9{\cdot}88\frac{15}{24}$

COMPONENT M.

Harmonic analysis of tides.

Station, Sitka, Alaska. Lat., 57° 4′ N.; long., 135° 20′ W.

Beginning of obs'ns, July 1, 1893, 0^h. Middle, July 15, 1893, 12^h.

h.	1.	h.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.
				1−2	2d ½ 3 reversed				3−4	5×8 12 c_1	6×8 12 c_3	7×8 12 c_5				3+4	12×15 12 s_1	13×15 12 s_3	14×15 12 s_5
0	+1·66	12	+1·65	+0·01	......	1	1	1	+0·01	+0·010	+0·010	+0·010	0	0	0	+0·01	......	......	......
1	+2·88	13	+2·87	+0·01	−0·24	·966	·707	·259	+0·25	+0·242	+0·177	+0·065	·259	·707	·966	−0·23	−0·060	−0·163	−0·222
2	+3·32	14	+3·43	−0·11	−0·09	·866	0	−·866	−0·02	−0·017	......	+0·017	·5	1	·5	−0·20	−0·100	−0·200	−0·100
3	+2·93	15	+3·00	−0·07	−0·02	·707	−·707	−·707	−0·05	−0·035	+0·035	+0·035	·707	·707	−·707	−0·09	−0·064	−0·064	+0·064
4	+1·62	16	+1·88	−0·26	−0·05	·5	−1	·5	−0·21	−0·105	+0·210	−0·105	·866	0	−·866	−0·31	−0·268	......	+0·268
5	+0·03	17	+0·10	−0·07	−0·09	·259	−·707	·966	+0·02	+0·005	−0·014	+0·019	·966	−·707	·259	−0·16	−0·155	+0·113	−0·041
6	−1·62	18	−1·55	−0·07	......	0	0	0	−0·07	......	......	......	1	−1	1	−0·07	−0·070	+0·070	−0·070
7	−2·86	19	−2·77	−0·09						+0·100	+0·418	+0·041					−0·717	−0·244	−0·101
8	−3·43	20	−3·38	−0·05						+0·0083	+0·0348	+0·0034					−0·0598	−0·0203	−0·0084
9	−3·10	21	−3·08	−0·02															
10	−1·83	22	−1·74	−0·09															
11	−0·17	23	+0·07	−0·24															

19.	20.	21.	22.	23.	24.
	19×11 12 c_7		21×18 12 s_7	1+2	2d half of 23
+	+0·010	0	....	+3·31	−3·17
−	−0·065	+	−0·222	+5·75	−5·63
+	+0·017	−	+0·100	+6·75	−6·81
−	−0·035	+	+0·064	+5·93	−6·18
+	−0·105	−	−0·268	+3·50	−3·57
−	−0·019	+	−0·041	+0·13	−0·10
0	....	−	+0·070	−3·17	
	−0·197		−0·297	−5·63	
	−0·0164		−0·0248	−6·81	
				−6·18	
				−3·57	
				−0·10	

25.	26.	27.	28.	29.	30.	31.	32.	33.	34.	35.
				23−24	25×29 12 c_2	26×29 12 s_2	27×29 12 $c6$	28×29 12 $s6$	23+24 24 δh	2d half of 34
1	0	+	0	+ 6·48	+ 6·480	....	+ 6·480	....	+0·14	−0·25
·866	·5	0	+	+11·38	+ 9·855	+ 5·690	....	+11·380	+0·12	−0·07
·5	·866	−	0	+13·56	+ 6·780	+11·743	−13·560	....	−0·06	+0·03
0	1	0	−	+12·11	....	+12·110	....	−12·110	−0·25	
−·5	·866	+	0	+ 7·07	− 3·535	+ 6·123	+ 7·070	....	−0·07	
−·866	·5	0	+	+ 0·23	− 0·199	+ 0·115	....	+ 0·230	+0·03	
					+19·381	+35·781	− 0·010	− 0·500	....	
					+ 1·6151	+ 2·9818	− 0·0008	− 0·0417	....	

36.	37.	38.	39.	40.	41.	42.	43.	44.	45.
		34−35	36×38 12 c_4	37×38 12 s_4			34+35	41×43 12 c_8	42×43 12 s_8
1	0	+0·39	+0·390	....	1	0	−0·11	−0·110	....
·5	·866	+0·19	+0·095	+0·165	−·5	·866	+0·05	−0·025	+0·043
−·5	·866	−0·09	+0·045	−0·078	−·5	−·866	−0·03	+0·015	+0·026
			+0·530	+0·087				−0·120	+0·0698
			+0·0442	+0·0072				−0·0100	+0·005

	M1	M2		M4		M6		
Log. s..........	8·77670n	0·47448	....	7·85733	....	8·62014n	....	....
Log. c..........	7·91908	0·20820	....	8·64345	....	6·90309n	....	....
Log. tan ζ.....	0·85762	0·26628	....	9·21388	....	1·71705	....	....
ζ.....	277°·9	61°·6	....	9°·3	....	268°·9	....	....
V_0+u.........	291°·3	302°·3	....	244°·6	....	186°·9	....	....
κ'.........	209°·2	3°·9	....	253°·9	....	95°·8	....	....
κ.........	209°·2	3°·9	....	253°·9	....	95°·8	....	....
Log. [cos] sin ζ.	9·99586	9·94414	....	9·20815	....	9·99992	....	....
	8·78084	0·53034	....	8·64918	....	8·62022	....	....
Log. aug. fac...	0·00124	0·00497	....	0·02003	....	0·04560	....	....
Log. R.........	8·78208	0·53531	....	8·66921	....	8·66582	....	....
R.........	0·06055	3·43012	....	0·04669	....	0·04633	....	....
Log. F.........	9·92037	0·01483	....	0·02966	....	0·04449	....	....
Log. H.........	8·70245	0·55014	....	8·69887	....	8·71031	....	....
H.........	0·05040	3·54928	....	0.04999	....	0·05132	....	....

$\tan \zeta = \frac{s}{c}$.

$\kappa = \zeta + V_0 + u$.

$R = \frac{s}{\sin \zeta} \times$ aug. fact. $= \frac{c}{\cos \zeta} \times$ aug. fact.

$H = R \times F$.

$\kappa = \kappa' + n(S-L)$
[when local V_0+u is used; observations in ~~local~~ standard time].

$\kappa = \kappa' + 15\,p(S-L)$
[when standard meridian (S) V_0+u is used; observations in standard time].

$\kappa = \kappa' + nS - 15\,pL$
[when Greenwich V_0+u is used; observations in standard or local time].

Local V_0+u at local midnight = Greenwich V_0+u + correction, Table 5, which is $L\,(n-15p)$.
Local V_0+u at midnight standard time = Greenwich V_0+u + correction, Table 5, $+\,n\,(S-L)$; or $L\,(n-15p)+n\,(S-L), = n\,S-15\,pL$
For diurnals, $p=1$; for semidiurnals, $p=2$, etc.; for long-period tides, $p=0$.

62. *Formulæ for inferring amplitudes and epochs.*

Amplitudes.	Epochs.
$J_1 = \begin{cases} 0·079\ O_1 \\ 0·056\ K_1 \end{cases}$	$J_1°' = K_1° + 0·496\ (K_1° - O_1°)$
$2Q = 0·026\ O_1$	$2Q° = K_1° - 1·992\ (K_1° - O_1°)$
$\rho_1 = 0·038\ O_1$	$\rho_1° = K_1° - 1·429\ (K_1° - O_1°)$
$OO = 0·043\ O_1$	$OO° = 2\ K_1° - O_1°$
$P_1 = 0·331\ K_1$	$P_1° = K_1°$
$Q_1 = 0·194\ O_1$	$Q_1° = K_1° - 1·496\ (K_1° - O_1°)$
$K_2 = 0·272\ S_2$	$K_2° = S_2°$
$L_2 = \begin{cases} 0·145\ N_2 \\ 0·028\ M_2 \end{cases}$	$L_2° = \begin{cases} 2M_2° - N_2° \\ S_2° - 0·464\ (S_2° - M_2°) \end{cases}$
$2N = 0·133\ N_2$	$2N° = 2\ N_2° - M_2°$
$R_2 = 0·008\ S_2$	$R_2° = S_2°$
$T_2 = 0·059\ S_2$	$T_2° = S_2°$
$\lambda_2 = 0·007\ M_2$	$\lambda_2° = S_2° - 0·536\ (S_2° - M_2°)$
$\mu_2 = 0·024\ M_2$	$\mu_2° = 2\ M_2° - S_2°$
$\nu_2 = 0·194\ N_2$	$\nu_2° = M_2° - 0·866\ (M_2° - N_2°)$

63. *Clearance from the effects of other components.*

Computation of equilibrium arguments V_0+u.

Sitka, Alaska, July 1–29, 1893.

ELEMENTS.

Beginning of series, Tables 3 and 4. Middle of series, Tables 6, 7, 8, and 9.

h	s	p	p_1	N	I	P	ν	ξ	ν'	$2\nu''$	R	Q
99°·27	303°·09	69°·79	281°·11	24°·152	28°·225	67°·405	4°·540	4°·090	3°·238	6°·856	11°·97	50°·23

Component.	V_0+u. From Table 1 (Greenwich) $t=0$, midnight.	V_0+u. Numerical values.		Correction from Table 5.	Local V_0+u.	Log F. Tables 10, 11, 12, 13.
		° ° ° ° ° ° °	°	°	°	
J_1	$h+s-p+90°-\nu$	99·27+303·09− 69·79+ 90·00−4·54	58·03	+ 4·27	62·30	
2 Q	$h-4s+2p-90°+2\xi-\nu$	99·27−132·36+139·58− 90·00+8·18− 4·54	20·13	−19·31	0·82	
ρ_1	$3h-3s-p-90°+2\xi-\nu$	297·81−189·27− 69·79− 90·00+8·18− 4·54	312·39	−13·76	298·63	
K_1	$h+90°-\nu'$	99·27+ 90·00− 3·24	186·03	+ 0·37	186·40	9·95652
O_1	$h-2s-90°+2\xi-\nu$	99·27−246·18− 90·00+ 8·18−4·54	126·73	− 9·52	117·21	9·93166
OO	$h+2s+90°-2\xi-\nu$	99·27+246·18+ 90·00− 8·18−4·54	62·73	+10·26	72·99	
P_1	$-h-90°$	− 99·27− 90·00	170·33	− 0·37	169·96	0·00000
Q_1	$h-3s+p-90°+2\xi-\nu$	99·27−189·27+ 69·79− 90·00+8·18− 4·54	253·43	−14·42	239·01	9·93166
K_2	$2h-2\nu''$	198·54− 6·86	191·68	+ 0·74	192·42	9·88937
L_2	$2h-s-p+180°+2\xi-2\nu-R$	198·54−303·09− 69·79+180·00+8·18− 9·08−11·97	352·79	− 4·25	348·73	9·92319
M_1	$h-s+90°+\xi-\nu+Q$	99·27−303·09+ 90·00+ 4·09−4·54+50·23	295·87	− 4·57	291·30	9·92037
M_2	$2h-2s+2\xi-2\nu$	198·54−246·18+ 8·18− 9·08	311·46	− 9·15	302·31	0·01483
M_4	$4h-4s+4\xi-4\nu$	37·08−132·36+ 16·36− 18·16	262·92	−18·28	244·64	0·02966
M_6	$6h-6s+6\xi-6\nu$	235·62− 18·54+ 24·54− 27·24	214·38	−27·50	186·88	0·04449
N_2	$2h-3s+p+2\xi-2\nu$	198·54−189·27+ 69·79+ 8·18−9·08	78·16	−14·04	64·12	0·01483
2 N	$2h-4s+2p+2\xi-2\nu$	198·54−132·36+139·58+ 8·18−9·08	204·86	−18·94	185·92	
S_2	0	0	0·00	0·00	0·00	0·00000
T_2	$-h+p_1$	− 99·27+281·11	181·84	− 0·37	181·47	
μ_2	$4h-4s+2\xi-2\nu$	37·08−132·36+ 8·18− 9·08	263·82	−18·28	245·54	0·01483
ν_2	$4h-3s-p+2\xi-2\nu$	37·08−189·27− 69·79+ 8·18−9·08	137·12	−13·39	123·73	0·01483

When the series analyzed is practically a calendar year, the value of V_0+u can be taken, without computation, from Table 3, and adapted to the time meridiau by Table 5; the F and f can be taken, without modification, from Table 10.

Component.	From auxiliary tables.			From analysis and inference.		R_c	R	ζ_c	ζ
	V_0+u	F	f	H	κ	From analysis	$f\times H$	From analysis	$\kappa-(V_0+u)$
				ft.				°	°
J_1	62·30	0·86569	1·15515	0·079 O_1 =0·0755	$K_1{}^\circ+0·496\,(K_1{}^\circ-O_1{}^\circ)$ =137°·7		0·0872		75·4
2 Q	00·82	0·85439	1·17043	0·026 O_1 =0·0249	$K_1{}^\circ-1·992\,(K_1{}^\circ-O_1{}^\circ)$ = 83°·2		0·0291		82·4
ρ_1	298·63	0·85439	1·17043	0·038 O_1 =0·0363	$K_1{}^\circ-1·430\,(K_1{}^\circ-O_1{}^\circ)$ = 95°·5		0·0425		156·9
K_1	186·40	0·90474	1·10528	1·8173×0·8230 =1·4956	136°·3−9°·5 =126°·8	2·0087	1·6531	309·9	300·4
O_1	117·21	0·85439	1·17043	0·9561	104°·9	1·1190	1·1190	347·7	347 7
OO	72·99	0·58221	1·71759	0·043 O_1 =0·0411	$2\,K_1{}^\circ-O_1{}^\circ$ =148°·7		0·0706		75·7
P_1	169·96	1·00000	1·00000	0·331 K_1 =0·4948	$P_1{}^\circ=K_1{}^\circ$ =126°·8	1·8772	0·4948	278·7	316·8
Q_1	239·01	0·85439	1·17043	0·194 O_1 =0·1855	$K_1{}^\circ-1·495\,(K_1{}^\circ-O_1{}^\circ)$ = 94°·1	0·2113	0·2171	238·7	215·1
K_2	192·42	0·77512	1·29014	0·272 S_2 =0·2933	$K_2{}^\circ=S_2{}^\circ$ = 39°·0	1·2792	0·3784	85·4	206·6
L_2	348·73	0·83790	1·19346	0·145 N_2 =0·0897	$2\,M_2{}^\circ-N_2{}^\circ$ = 33°·3	0·1762	0·1071	56·5	44·6
M_2	302·31	1·03475	0·96641	3·5493	3°·9	3·4301	3·4301	61·6	61·6
N_2	64·12	1·03475	0·96641	0·6183	334°·5	0·5976	0·5976	270·4	270·4
2 N	185·92	1·03475	0 96641	0·133 N_2 =0·0822	$2\,N_2{}^\circ-M_2{}^\circ$ =305°·1		0·0794		119·2
S_2	0·00	1·00000	1·00000	0·8046×1·3408 =1·0788	53°·6−14°·6 = 39°·0	0·8046	1·0788	53·6	39·0
T_2	181·47	1·00000	1·00000	0·059 S_2 =0·0636	$T_2{}^\circ=S_2{}^\circ$ = 39°·0		0·0636		217·5
μ_2	245·54	1·03475	0·96641	0·024 M_2 =0·0852	$2\,M_2{}^\circ-S_2{}^\circ$ =328°·8	0·1523	0·0823	133·6	83·3
ν_2	123·73	1·03475	0·96641	0·194 N_2 =0·1200	$M_2{}^\circ-0·868\,(M_2{}^\circ-N_2{}^\circ)$ =338°·4	1·0559	0·1160	277·4	214·7

From Table 31, mean of four values, June 30–July 30.

Acceleration of K_1 due to P_1	$=-10°·45\times F\,(K_1)=-9°·455$
Resultant amplitude K and P	$=[(1·2378-1·)\times F\,(K_1)]+1·=1·2151$; recip. $=0·8230$
Acceleration of S_2 due to K_2	$=-11°·825\times f(K_2)=-15·256$
" " S_2 due to T_2	$=+0°·675$
" " S_2 due to K_2 and T_2	$=-15°·256+0°·675=-14°·581$
Resultant amplitude S_2 and K_2	$=[(0·8475-1·)\times f(K_2)]+1=0·8033$
" " S_2 and T_2	$=0·9425$
" " S_2 and K_2 and T_2	$=0·8033+0·9425-1=0·7458$; recip. $=1·3408$

64. *Application of elimination tables.*

$$
\begin{aligned}
-\delta\overline{K}_1 = {} & +0·050\ \ J_1'\ \cos[\ 9°+\zeta_c\,(K_1)-\zeta\,(\ J_1\ ]+0·0565\ O_1'\cos[338°+\zeta_c\,(K_1)-\zeta\,(O_1)]\\
& +0·056\ OO'\ \cos[22°+\zeta_c\,(K_1)-\zeta\,(OO)]+0·959\ \ P_1\cos[331°+\zeta_c\,(K_1)-\zeta\,(P_1)]\\
& +0·052\ \ Q_1'\ \cos[328°+\zeta_c(K_1)-\zeta\,(\ Q_1)]+0·049(2Q')\cos[319°+\zeta_c\,(K_1)-\zeta(2Q)]\\
& +0·011\ \ \rho_1'\ \cos[354°+\zeta_c(K_1)-\zeta\,(\ \rho_1)].
\end{aligned}
$$

$$
\begin{aligned}
\delta\overline{\overline{K}}_1 = {} & +0·050\ J_1'\ \sin[9°\ +\zeta_c\,(K_1)-\zeta\,(\ J_1)]+0·0565\ O_1'\sin[338°+\zeta_c\,(K_1)-\zeta\,(\ O_1)]\\
& +0·056OO'\sin[22°\ +\zeta_c\,(K_1)-\zeta\,(OO)]+0·959\ \ P_1\sin[331°+\zeta_c\,(K_1)-\zeta\,(\ P_1)]\\
& +0·052\ Q_1'\ \sin[328°+\zeta_c\,(K_1)-\zeta\,(\ Q_1)]+0·049(2Q')\sin[319°+\zeta_c\,(K_1)-\zeta\,(2Q)]\\
& +0·011\ \ \rho_1'\ \sin[354°+\zeta_c\,(K_1)-\zeta\,(\ \rho_1)].
\end{aligned}
$$

Here J_1' signifies the R of J_1, or $R\,(J_1)$, and not J_1 (or the H of J_1) nor $R_c\,(J_1)$, which means the R direct from analysis and so before the effects of the other scheduled components upon it have been eliminated. Similarly for O_1', P_1, Q_1', etc.

$$
\begin{aligned}
-\delta\overline{K}_1 = {} & +0·050\times0·0872\cos(\ \ 9°+310°-\ 75°)+0·0565\times1·1190\cos(338°+310°-348°)\\
& +0·056\times0·0706\cos(\ 22°+310°-\ 76°)+0·959\ \ \times0·4948\cos(331°+310°-317°)\\
& +0·052\times0·2171\cos(328°+310°-215°)+0·049\ \ \times0·0291\cos(319°+310°-\ 82°)\\
& +0·011\times0·0425\cos(354°+310°-157°).
\end{aligned}
$$

$$
\begin{aligned}
= {} & +0·0043\cos244°+0·0632\cos300°+0·0040\cos256°+0·4745\cos324°+0·0114\cos63°+\\
& 0·0014\cos187°+0·0005\cos147°.
\end{aligned}
$$

$$
\begin{aligned}
= {} & -0·0043\times0·438+0·0632\times0·500-0·0040\times0·242+0·4745\times0·809+0·0114\times0·454-0·0014\times\\
& 0·993-0·0005\times0·839=-0·0019+0·0316-0·0009+0·3839+0·0051-0·0014-0·0004=\\
& +0·4160.
\end{aligned}
$$

$\delta\overline{\overline{\mathrm{K}}}_1 = +0{\cdot}0043 \sin 244^\circ + 0{\cdot}0632 \sin 300^\circ + 0{\cdot}0040 \sin 256^\circ + 0{\cdot}4745 \sin 324^\circ + 0{\cdot}0114 \sin 63^\circ + 0{\cdot}0014 \sin 187^\circ + 0{\cdot}0005 \sin 147^\circ.$

$= -0{\cdot}0043 \times 0{\cdot}899 - 0{\cdot}0632 \times 0{\cdot}866 - 0{\cdot}0040 \times 0{\cdot}970 - 0{\cdot}4745 \times 0{\cdot}588 + 0{\cdot}0114 \times 0{\cdot}891 - 0{\cdot}0014 \times 0{\cdot}122 + 0{\cdot}0005 \times 0{\cdot}545 = -0{\cdot}0040 - 0{\cdot}0547 - 0{\cdot}0039 - 0{\cdot}2790 + 0{\cdot}0100 - 0{\cdot}0002 + 0{\cdot}0003 = -0{\cdot}3315.$

$$\mathrm{Tan}\ \delta\zeta = \frac{-0{\cdot}3315}{2{\cdot}0087 - 0{\cdot}4160} = \frac{-0{\cdot}3315}{1{\cdot}5927} = -0{\cdot}2081;\ \delta\zeta = -11^\circ{\cdot}8;\ \cos\delta\zeta = 0{\cdot}9791.$$

$$R\ (\mathrm{K}_1) = \frac{1{\cdot}5927}{0{\cdot}9791} = 1{\cdot}6267;\ \mathrm{K}_1 = R \times F = 1{\cdot}6267 \times 0{\cdot}9047 = 1{\cdot}4717 \text{ feet.}$$

$$\zeta = 309^\circ{\cdot}9 - 11^\circ{\cdot}8 = 298^\circ{\cdot}1;\ \kappa = \zeta + V_0 + u = 298^\circ{\cdot}1 + 186^\circ{\cdot}4 = 124^\circ{\cdot}5.$$

Form for clearing the component A of the effects of other components B.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
A	B	Coefficients			Angles				$-\delta\overline{A}$ or $-\delta c$, $\delta\overline{\overline{A}}$ or δs			
		Tab. 41	$R(B)$	(3) × (4)	Tab. 41	$\zeta(A)$	$\zeta_c(B)$	(6)+(7)−(8)	cos (9)	sin (9)	(5)×(10)	(5)×(11)
K_1												
	J_1	·050	·0872	·0044	9	310	75	244	− ·438	− ·899	− ·0019	− ·0040
	$2\mathrm{Q}$	·049	·0291	·0014	319	310	82	187	− ·993	− ·122	− ·0014	− ·0002
	ρ_1	·011	·0425	·0005	354	310	157	147	− ·839	+ ·545	− ·0004	+ ·0003
	O_1	·056	1·1190	·0632	338	310	348	300	+ ·500	− ·866	+ ·0316	− ·0547
	OO	·056	·0706	·0040	22	310	76	256	− ·242	− ·970	− ·0009	− ·0039
	P_1	·959	·4948	·4745	331	310	317	324	+ ·809	− ·588	+ ·3839	− ·2790
	Q_1	·052	·2171	·0113	328	310	215	63	+ ·454	+ ·891	+ ·0051	+ ·0100
											+0·4160	−0·3315

$\tan\delta\zeta = \dfrac{(13)}{R_c(A) - (12)}.$

$\zeta = \zeta_c + \delta\zeta.$

$\kappa' = +V_0 + u.$

$R(A) = \dfrac{R_c(A) - (12)}{\cos\delta\zeta}.$

R is always positive.

$H(A) = R(A) \times F(A).$

$R_c\ (\mathrm{K}_1) = 2{\cdot}0087$ feet.

$\zeta_c\ (\mathrm{K}_1) = 309^\circ{\cdot}9.$

$\tan\delta\zeta = -\dfrac{0{\cdot}3315}{2{\cdot}0087 - {\cdot}4160} = -\dfrac{0{\cdot}3315}{1{\cdot}5927} = -0{\cdot}2081.$

$\delta\zeta = -11^\circ{\cdot}8,\ \cos\delta\zeta = 0{\cdot}9791.$

$R\ (\mathrm{K}_1) = \dfrac{1{\cdot}5927}{0{\cdot}9791} = 1{\cdot}6267$ feet.

$\mathrm{K}_1 = 1{\cdot}6267 \times 0{\cdot}9047 = 1{\cdot}4717$ feet.

$\zeta = 309^\circ{\cdot}9 - 11^\circ{\cdot}8 = 298^\circ{\cdot}1.$

$\mathrm{K}_1{}^\circ = 298^\circ{\cdot}1 + 186^\circ{\cdot}4 = 124^\circ{\cdot}5.$

65. *Results from harmonic analyses of hourly ordinates.*

SITKA, ALASKA.

From 29 days—July 1–29, 1893. From 1 year, 1893.

Component.	All ordinates used.				No doublings up or omissions on successive component hours.							
	Direct from analysis.		Corrected for other components.		Direct from analysis.		Corrected for other components.		Direct from analysis.		Corrected for other components.	
	H	κ	H	κ	H	κ	H	κ	H	κ	H	κ
K_1	1·8173	136·3	1·472	125	1·8236	136·5	1·475	125	1·507	125	1·504	125
K_2	0·9915	227·8	0·281	340	0·9800	277·7	0·273	342	0·315	21	0·320	22
L_2	0·1476	45·2	0·121	26	0·1468	45·4	0·116	29	0·155	37	0·109	28
M_1	0·0504	209·2			0·0469	204·4			0·029	150		
M_2	3·5493	3·9	3·589	3·9	3·5589	3·9	3·597	3·9	3·583	2·5	3·591	2·8
M_4	0·0500	253·9			0·0132	117·7			0·013	140		
M_6	0·0513	95·8			0·0194	144·0			0·002	94		
N_2	0·6183	334·5	0·747	336	0·6220	336·7	0·746	338	0·687	340	0·758	335
O_1	0·9561	104·9	0·937	109	0·9522	105·2	0·939	110	0·905	109	0·905	110
P_1	1·8772	88·7	0·345	109	1·8751	88·6	0·337	113	0·465	124	0·450	124
Q_1	0·1806	117·7	0·150	97	0·1606	122·0	0·128	98	0·136	107	0·157	98
S_2	0·8046	53·6	1·272	38	0·8046	53·6	1·177	39	1·137	34	1·145	34
μ_2	0·1573	19·1	0·097	349	0·1586	22·4	0·077	349	0·073	334	0·085	321
ν_2	1·0924	41·1	0·179	31	1·0945	41·5	0·188	27	0·040	79	0·142	343

The results of the above computation show that for so short a series as 29 days, K_2, P_1 and ν_2 should be inferred from the final values of S_2, K_1, and N_2. In fact, it would be unreasonable to suppose that the elimination formulæ should give good results in these cases.

Comparison between results obtained by using all hourly heights and those obtained by so using the hourly heights that the successive component hours are represented once and only once—in other words, paying attention to the arrows in Table 42. There can be no reasonable doubt but that greater accuracy is generally attained by following the second of these two modes of procedure. Usually the difference of the results will be small, especially in the case of the predominating component. However, at most places one or more of the other components are a considerable fraction of M_2 and so M_4 and M_6 may be sensibly affected by the putting of two heights now and then upon the same component hour.

The following is a good example of this, since at Port Townsend K_1 is even larger than M_2. The differences between the two sets of results for the month at Sitka are not great, but they show that some accuracy is gained by paying attention to the arrows in Table 42.

Port Townsend, Wash.

From 29 days' hourly ordinates July 1–29, 1874, using all tabulated heights:	From 29 days' hourly ordinates July 1-29, 1874, omitting certain hourly heights, in accordance with Table 42:	From 3 years' hourly ordinates 1874–6:
$M_4 = 0{\cdot}198$	$M_4 = 0{\cdot}128$	$M_4 = 0{\cdot}131$
$M_4{}^\circ = 275$	$M_4{}^\circ = 297$	$M_4{}^\circ = 296$
$M_6 = 0{\cdot}105$	$M_6 = 0{\cdot}048$	$M_6 = 0{\cdot}03$[illegible]
$M_6{}^\circ = 193$	$M_6{}^\circ = 206$	$M_6{}^\circ = 242$

66. *On the abbreviation of summations.*

Any two components A and B having very nearly equal speeds separate but a small amount n a week's time, which is the time covered by each page of tabulated hourly heights. (See § 61.)

Suppose a page of such heights to be summed with the A stencil. These sums may be regarded as a series of 24 component (A) hourly heights falling on and among the 24 hours of the middle, or fourth, day of the week. Had the B stencil been used instead, the B sums might likewise have been regarded as a series of 24 component (B) hourly heights falling on and among the 24 hours of the fourth day. In fact such times would have been the times of occurrence of the B hours of that day, reckoned from the beginning of the series. A comparison of the (solar) times of the several A sums with the (solar) times of the occurrence of the B hours will show what B hour each A sum lies nearest to. Therefore if the 24 A sums be copied into a form one argument of which is the week or page and the other the hours from 0 to 23, it is very easy to construct a *secondary stencil** which shall sort the A sums according to the B hours.

Since it is assumed that one page is given to each week of hourly heights, the number of pages to be summed in a year's series is 52 or 53. The partial or seven-day sums for a year can be conveniently written upon about 7 pages; and this is the number of pages which have to be summed with the secondary stencil.

If complete stencils be used for the first row of components, secondary stencils may be applied to the resulting partial sums, and abbreviated summations obtained for the components written underneath:

J	L	M	N	O	OO	Q	2 Q	S	MN	MK
2 SM	λ, MS		ν	2 N, μ		ρ		K, P, R, T	2 MK	

This abbreviation of the work has its principal advantages in the case of a long series of observations; for, the irregularities introduced thereby gradually disappear, and, of course, the amount of labor saved in summation is about proportional to the time covered by the series.

In analyzing the B sums obtained by aid of the secondary stencils, the A augmenting factor should be applied because the first summation was according to A time while the observations are in solar time. Another augmenting factor is required because the secondary stencils are constructed with reference to B time. The reason for this can be readily seen if to the S sums we apply a secondary K stencil for obtaining the component K. Clearly the S sums do not fit the K hours; they may diverge as much as half of a K hour when all the S sums are used, and used

* First suggested by F. M. Little, of this Survey.

once, or half of an S hour when each K hour has one, and only one, partial S sum assigned to it. In either case the K augmenting factor can be used, because the speed of K is very nearly but not exactly 15° per hour.

Another way of lessening the labor of summation is to use either bi-hourly ordinates or the hourly heights combined in pairs. Such combination can be made, once for all, upon the hourly-ordinate sheets.

On the harmonic analysis of tides of long period.

67. The principal tides of long period are: The lunar fortnightly, Mf, period ½ tropical month; the luni-solar fortnightly, MSf, period ½ synodic month; the lunar monthly, Mm, period 1 anomalistic month; the annual, Sa, and the semiannual, Ssa.

In the treatment of these components it is convenient to make use of the daily sums (or means) of the hourly heights, distributing them in accordance with the annual and the several monthly periods. That is to say, the various months and the year are each supposed to be divided into 24 equal parts analogous to the subdivision of the day, so that the entire time covered by the observations can be divided up with reference to such periods and their twenty-fourth parts and the position of each daily sum with respect to them be determined. These sums, however taken, will belong to a certain hour of the day. We shall suppose, as usual, that the hourly heights belonging to 0^h, 1^h, 2^h . . . 23^h, are simply added together, thus making the sum pertain to $11^h\ 30^m$ a. m. of each day. Table 43 shows where each daily sum falls for the four kinds of distribution here contemplated.

Suppose the length of series used to be four years, or 1,461 days. It contains 106·95 periods of Mf; 98·95 periods of MSf; 53·02 periods of Mm; 4 periods of Sa, and 8 periods of Ssa. All of these numbers are integers, or very nearly integers. It would seem, therefore, that the daily heights, when once purified of the short-period tides, might be summed and analyzed according to the methods employed with the hourly heights for tides of short period. Moreover, for a series four years in length it seems probable that no elimination on account of the disturbing effects of these components upon one another will be necessary.

It is here proposed to use the uncorrected daily sums or means. The component S_2 is completely eliminated from each daily sum: K_1 and K_2 are very nearly eliminated, because the K day is very nearly equal to the S day. N_2 and O_1 have considerable effects upon the daily sums; but since their synodic periods with S_2 and S_1 are not commensurable with any of the long-period components sought, the latter cannot, in the long run, be greatly disturbed by the long-period waves which are due to N_2 and O_1. M_2 has a direct effect upon MSf, which it is necessary to determine.

Let L denote the wave, whose speed is equal to the speed of MSf, due to the imperfect elimination of M_2 from the daily sums or means. Then by (440) its amplitude is the numerical value of

$$\frac{M_2}{24\ m_2} \mid 2 \sin 12\ m_2 \mid = 0{\cdot}08237 M_2 \times 2 \sin (347° \ 48\tfrac{1}{2}') = 0{\cdot}03479 M_2. \tag{448}$$

or using (442),

$$L = 0{\cdot}03516 M_2.$$

By (441),

$$\zeta (L) = \zeta (M_2) \mp 180°. \tag{449}$$

Since L has the same speed as has MSf, the latter may be cleared of the former by means of equations (432) and (433), or

$$- \delta\ \overline{\mathrm{MSf}} = L' \cos [\zeta_c (\mathrm{MSf}) - \zeta (L)], \tag{450}$$

$$\delta\ \overline{\overline{\mathrm{MSf}}} = L' \sin [\zeta_c (\mathrm{MSf}) - \zeta (L)], \tag{451}$$

and equations (427), (428), and (429) after replacing A by MSf.

Since the hourly heights used each day are for 0^h, 1^h, 2^h . . . 23^h, the daily sum pertains to $11^h\ 30^m$; but the ζ of MSf, obtained in accordance with Table 43, refers to 0^h.

The factor f for the computed L is the same as that for M_2 because the amplitudes of the two waves bear a fixed ratio to each other; and so $L' = 0{\cdot}03620\ M_2'$.

The augmenting factor for MSf, as obtained from the analysis and uncorrected for L, is

$$\frac{\text{arc}\ (s_2 - m_2)\ \tau}{\text{chord}\ (s_2 - m_2)\ \tau} \tag{452}$$

where $s_2 - m_2 = 1^\circ{\cdot}016$, and $\tau = 24$ hours; this becomes

$$\frac{\text{arc}\ 24^\circ\ 23'}{\text{chord}\ 24^\circ\ 23'} = 1{\cdot}00758, \tag{453}$$

the logarithm of which is

$$0{\cdot}00328.$$

For a method of determining the annual, the semiannual, and other tides of very long period, see § 28, Part III.

68. The following treatment of long-period tides is taken bodily from Darwin's 1883 report. It supposes the time to be reckoned from noon instead of midnight, and the series to extend over a period of one year. Darwin has prepared and published special forms* for the reduction of these tides. Reference may also be made to Baird's Manual, pages 41, 52–54.

For the purpose of determining these tides we have to eliminate the oscillations of water-level arising from the tides of short period. As the quickest of these tides has a period of many days, the height of mean water at one instant for each day gives sufficient data. Thus there will in a year's observations be 365 heights to be submitted to harmonic analysis. In leap-years the last day's observation must be dropped, because the treatment is adapted for analysing 365 values.

To find the daily mean for any day it has hitherto been usual to take the arithmetic mean of 24 consecutive hourly values, beginning with the height at noon. This height will then apply to the middle instant of the period from 0^h to 23^h: that is to say, to $11^h\ 30^m$ at night. We shall propose some new modes of treating the observations, and in the first of them it will probably be more convenient that the mean for the day should apply to midnight instead of to $11^h\ 30^m$. For finding a mean applicable to midnight we take the 25 consecutive heights for 0^h to 24^h, and add the half of the first value to the 23 intermediate and to the half of the last and divide by 24. It would probably be sufficiently accurate if we took $\frac{1}{25}$ of the sum of the 25 consecutive values, if it is found that the division of every 24th hourly value into two halves materially increases the labour of computing the daily means. The three plans for finding the daily mean are then

$$\left.\begin{array}{ll} \frac{1}{24}\,(h_0 + h_1 + \quad . \quad . \quad . \quad + h_{23}) & \text{(i)} \\ \frac{1}{24}\,(\frac{1}{2}\,h_0 + h_1 + \quad . \quad . \quad . \quad + h_{23} + \frac{1}{2}\,h_{24}) & \text{(ii)} \\ \frac{1}{25}\,(h_0 + h_1 + \quad . \quad . \quad . \quad + h_{23} + h_{24}) & \text{(iii)} \end{array}\right\} \tag{454}$$

And they will be denoted as methods (i), (ii), (iii) respectively. It does not, however, seem very desirable to use the third method. Major Baird considers that the use of method (i) is most convenient for the computers.

The formation of a daily mean does not obliterate the tidal oscillations of short period, because none of the tides, excepting those of the principal solar series, have commensurable periods in mean solar time.

A correction, or 'clearance of the daily mean,' has therefore to be applied for all the important tides of short period, excepting for the solar tides.

Let $R \cos (nt - \zeta)$ be the expression for one of the tides of short period as evaluated by the harmonic analysis for the same year, and let α be the value of $nt - \zeta$ at any noon. Then the 25 consecutive hourly heights of water, beginning with that noon, are—

$$R \cos \alpha,\ R \cos (n + \alpha),\ R \cos (2\,n + \alpha) \quad . \quad . \quad . \quad R \cos (23\,n + \alpha),\ R \cos (24\,n + \alpha). \tag{455}$$

In the method (i) of taking the daily mean it is obvious that the 'clearance' is

$$-\frac{1}{24}\,R\,\frac{\sin 12\,n}{\sin \frac{1}{2}\,n} \cos (\alpha + 11\tfrac{1}{2}\,n)$$

In the method (ii) it is easily proved to be

$$-\frac{1}{24}\,R\,\frac{\sin 12\,n}{\tan \frac{1}{2}\,n} \cos (\alpha + 12\,n) \tag{456}$$

and in method (iii) it is

$$-\frac{1}{25}\,R\,\frac{\sin \frac{25}{2}\,n}{\sin \frac{1}{2}\,n} \cos (\alpha + 12\,n)$$

The clearance, as written here, is additive.

* For sale by the Cambridge Scientific Instrument Company, St. Tibbs Row, Cambridge.

It was found practically in the computation for these tides that only three tides of short period exercise an appreciable effect, so that clearances for them have to be applied. These tides are the M_2, N, O tides. It was usual to compute these three clearances for every day in the year, and to correct the daily values accordingly. But in following this plan a great deal of unnecessary labour has been incurred, and when a simpler plan is followed it may perhaps be worth while to include more of the short-period tides in the clearances.

Professor J. C. Adams suggests the use of the tide-predicting machine for the evaluation of the sum of the clearances, and if this plan is not found to inconveniently delay operations in India, it may perhaps be tried.

In explaining the process we will suppose that method (i) has been followed; if either of the other plans be adopted it will be easy to change the formulæ accordingly.

It is clear that $R\cos(\alpha + 11\frac{1}{2}n)$ is the height of the tide n at $11^h\ 30^m$; and the same is true for each such tide. Hence if we use the tide-predicter to run off a year of fictitious tides with the semi-range of each tide equal to $\frac{1}{24}\sin 12n/\sin\frac{1}{2}n$ of its true semi-range, and with all the solar series and the annual and semi-annual tides put at zero, the height given at each $11^h\ 30^m$ in the year is the sum for each day of all the clearances to be *subtracted.* The scale to which the ranges are set may of course be chosen so as to give the clearances to a high degree of accuracy.

In the other process of clearance, which will be explained below, a single correction for each short-period tide is applied to each of the final equations, instead of to each daily mean.

We next take the 365 daily means, and find their mean value. This gives the mean height of water for the year. If the daily means be uncleared, the result can not be sensibly vitiated.

We next subtract the mean height from each of the 365 values, and find 365 quantities δh giving the daily height of water above the mean height.

These quantities are to be the subject of the harmonic analysis; and the tides chosen for evaluation are those which have been denoted above as Mm, Mf, MSf, Sa, Ssa.

Let

$$\left.\begin{aligned}\delta h = &\ A\cos(\sigma-\varpi)t + B\sin(\sigma-\varpi)t\\ &+ C\cos 2\sigma t + D\sin 2\sigma t\\ &+ C'\cos 2(\sigma-\eta)t + D'\sin 2(\sigma-\eta)t\\ &+ E\cos\eta t + F\sin\eta t\\ &+ G\cos 2\eta t + H\sin 2\eta t\end{aligned}\right\}\tag{457}$$

where t is time measured from the first $11^h\ 30^m$.

Now suppose l_1, l_2 are the increments in 24 m. s. hours of any two of the five arguments $(\sigma-\varpi)t$, $2\sigma t$, $2(\sigma-\eta)t$, ηt, $2\eta t$, and that A_1, B_1; A_2, B_2, are the corresponding coefficients of the cosine and sine in the expression for δh.

Then if δh_i be the value of δh at the $(i+1)^{th}\ 11^h\ 30^m$ in the year, we may write

$$\delta h_i = A_1\cos l_1 i + B_1\sin l_1 i + A_2\cos l_2 i + B_2\sin l_2 i + \ldots\tag{458}$$

And therefore

$$\begin{aligned}\delta h_i\cos l_1 i = &\ \tfrac{1}{2}A_2\{\cos(l_1+l_2)i + \cos(l_1-l_2)i\}\\ &+ \tfrac{1}{2}B_2\{\sin(l_1+l_2)i - \sin(l_1-l_2)i\} + \ldots\end{aligned}\tag{459}$$

$$\begin{aligned}\delta h_i\sin l_1 i = &\ \tfrac{1}{2}A_2\{\sin(l_1+l_2)i + \sin(l_1-l_2)i\}\\ &+ \tfrac{1}{2}B_2\{-\cos(l_1+l_2)i + \cos(l_1-l_2)i\} + \ldots\end{aligned}\tag{460}$$

Now let

$$\phi(x) = \frac{1}{2}\frac{\sin\frac{365}{2}x}{\sin\frac{1}{2}x},\tag{461}$$

so that

$$\phi(l_1\pm l_2) = \frac{1}{2}\frac{\sin\frac{365}{2}(l_1\pm l_2)}{\sin\frac{1}{2}(l_1\pm l_2)}.\tag{462}$$

We may observe that

$$\phi(x) = \phi(-x),\ \text{and}\ \phi(0) = 182\tfrac{1}{2}.$$

If therefore Σ denotes summation for the 365 values from $i=0$ to $i=364$, we have

$$\left.\begin{aligned}\Sigma\delta h\cos l_1 i = &\ [\phi(l_1+l_2)\cos 182(l_1+l_2) + \phi(l_1-l_2)\cos 182(l_1-l_2)]A_2\\ &+ [\phi(l_1+l_2)\sin 182(l_1+l_2) - \phi(l_1-l_2)\sin 182(l_1-l_2)]B_2 + \ldots,\\ \Sigma\delta h\sin l_1 i = &\ [\phi(l_1+l_2)\sin 182(l_1+l_2) + \phi(l_1-l_2)\sin 182(l_1-l_2)]A_2\\ &+ [-\phi(l_1+l_2)\cos 182(l_1+l_2) + \phi(l_1-l_2)\cos 182(l_1-l_2)]B_2 + \ldots\end{aligned}\right\}\tag{463}$$

In these equations there is always one pair of terms in which l_2 is identical with l_1, and since $\phi(l_1-l_1) = 182\frac{1}{2}$, and $\cos 182(l_1-l_1) = 1$, it follows that there is one term in each equation in which there is a coefficient nearly equal to 182·5. In the cosine series it will be a coefficient of an A; in the sine series, of a B.

The following are the equations (copied from the Report for 1872) with the coefficients inserted, as computed from these formulæ, or their equivalents:—

[P.]

Final Equations for Tides of Long Period.

	Coefft. of A.	Coefft. of B.	Coefft. of C.	Coefft. of D.	Coefft. of C′.	Coefft. of D′.	Coefft. of E.	Coefft. of F.	Coefft. of G.	Coefft. of H.
$\Sigma\,\delta\,h \times \cos\,(\sigma-\varpi)\,t =$	+183·05	+ 2·14	+ 0·73	+ 4·29	+ 0·77	+ 5·04	+ 4·88	− 0·34	+ 4·96	− 0·69
$\times \sin\,(\sigma-\varpi)\,t =$	+ 2·14	+181·95	− 4·15	+ 1·02	− 4·90	+ 1·07	+ 3·80	+ 0·34	+ 3·88	+ 0·69
$\times \cos\,2\,\sigma\,t =$	+ 0·73	− 4·15	+183·18	+ 0·88	+ 0·61	+ 0·92	− 1·50	− 0·10	− 1·51	− 0·19
$\times \sin\,2\,\sigma\,t =$	+ 4·29	+ 1·02	+ 0·88	+181·82	+ 0·92	− 0·75	+ 3·05	− 0·08	+ 3·06	− 0·17
$\times \cos\,2\,(\sigma-\eta)\,t =$	+ 0·77	− 4·90	+ 0·61	+ 0·92	+183·19	+ 0·97	− 1·68	− 0·11	− 1·70	− 0·23
$\times \sin\,2\,(\sigma-\eta)\,t =$	+ 5·04	+ 1·07	+ 0·92	− 0·75	+ 0·97	+181·81	+ 3·25	− 0·10	+ 3·27	− 0·23
$\times \cos\,\eta t =$	+ 4·88	+ 3·80	− 1·50	+ 3·05	− 1·68	+ 3·25	+182·43	+ 0·00	− 0·14	+ 0·00
$\times \sin\,\eta t =$	− 0·34	+ 0·34	− 0·10	− 0·08	− 0·11	− 0·10	+ 0·00	+182·57	+ 0·00	+ 0·00
$\times \cos\,2\,\eta t =$	+ 4·96	+ 3·88	− 1·51	+ 3·06	− 1·70	+ 3·27	− 0·14	+ 0·00	+182·43	+ 0·00
$\times \sin\,2\,\eta t =$	− 0·69	+ 0·69	− 0·19	+ 0·17	− 0·23	− 0·23	+ 0·00	+ 0·00	+ 0·00	+182·57

If the daily means have been cleared by the use of the tide-predicter as above described, these ten equations are to be solved by successive approximation, and we are then furnished with the two component semiamplitndes, say A_ι, B_ι of the five long-period tides. But the initial instant of time is the first $11^h\,30^m$ in the year instead of the first noon. Hence if as before we put $R^2 = A_\iota^2 + B_\iota^2$, and $\tan\zeta_\iota = B_\iota/A_\iota$, we must, in order to reduce the results to the normal form in which noon of the first day is the initial instant of time, add to ζ_ι the increment of the corresponding argument for $11^h\,30^m$, according to method (i), or for 12 hours according to methods (ii) or (iii).

69. If, however, the daily means have not been cleared, then before solution of the final equations corrections for clearance will have to be applied, which we shall now proceed to evaluate.

For this process we still suppose method (i) to be adopted.

Let n be the speed of a short-period tide in degrees per m. s. hour, and let $\psi\,(n) = \frac{1}{24}\,\frac{\sin 12\,n}{\sin\frac{1}{2}\,n}$. Then we have already seen that the clearance to $\delta\,h_i$, the mean height of water at $11^h\,30^m$ of the $(i+1)^{th}$ day, will be

$$-\psi\,(n)\,R\cos\,[n\,\{24\,i + 11\tfrac{1}{2}\} - \zeta]. \tag{464}$$

If we write $m = 24\,n$ (so that m is the daily increase of argument of the tide of short period), and $\beta = n \times 11\frac{1}{2} - \zeta$, this becomes

$$-\psi\,(n)\,R\cos\,(mi + \beta). \tag{465}$$

Hence the clearance for $\delta\,h_i \cos\,li$ is

$$-\tfrac{1}{2}\,\psi\,(n)\,R\,\{\cos[(m+l)i+\beta] + \cos\,[(m-l)i+\beta]\}, \tag{466}$$

and for $\delta\,h_i \sin\,li$ is

$$-\tfrac{1}{2}\,\psi\,(n)R\,\{\sin\,[(m+l)\,i+\beta] - \sin\,[(m-l)i+\beta]\}. \tag{467}$$

Summing the series of 365 terms we find that the additive clearance for $\Sigma\,\delta\,h\cos\,li$ is

$$-R\,\psi\,(n)\,\{\phi\,(m+l)\cos\,[182\,(m+l)+\beta] + \phi\,(m-l)\cos\,[182\,(m-l)+\beta]\}, \tag{468}$$

where as before

$$\phi\,(x) = \tfrac{1}{2}\,\frac{\sin\frac{365}{2}\,x}{\sin\frac{1}{2}\,x}. \tag{469}$$

If $\varDelta\,n$ denotes the increase of the argument nt in $182^d\,11^h\,30^m$, this may now be written

$$-R\,\psi(n)\,\{\phi(m+l)\cos\,[\varDelta n + 182\,l - \zeta] + \phi\,(m-l)\cos\,[\varDelta n - 182\,l - \zeta]\}. \tag{470}$$

If therefore $R\cos\zeta = A$, $R\sin\zeta = B$, so that A and B are the component semi-ranges of the tide n as immediately deduced from the harmonic analysis for the tides of short period, we have for the clearance to $\Sigma\delta h\cos\,li$

$$\begin{aligned}&-[\psi\,(n)\,\phi\,(m+l)\cos\,(\varDelta n+182\,l) + \psi\,(n)\,\phi\,(m-l)\cos\,(\varDelta n-182\,l)]\,A\\ &-[\psi\,(n)\,\phi\,(m+l)\sin\,(\varDelta n+182\,l) + \psi\,(n)\,\phi\,(m-l)\sin\,(\varDelta n-182\,l)]\,B.\end{aligned} \tag{471}$$

In precisely the same manner we find the clearance for $\Sigma\delta h\sin\,li$ to be

$$\begin{aligned}&-[\psi\,(n)\,\phi\,(m+l)\sin\,(\varDelta n+182\,l) - \psi\,(n)\,\phi\,(m-l)\sin\,(\varDelta n-182\,l)]\,A\\ &+[\psi\,(n)\,\phi\,(m+l)\cos\,(\varDelta n+182\,l) - \psi\,(n)\,\phi\,(m-l)\cos\,(\varDelta n-182\,l)]\,B.\end{aligned} \tag{472}$$

These coefficients may be written in a form more convenient for computation. For

$$\phi\,(m \pm l) = \frac{\sin \frac{365}{2}\,(m \pm l)}{2 \sin \frac{1}{2}\,(m \pm l)} = \tfrac{1}{2} \cos 182\,(m \pm l) + \tfrac{1}{2} \sin 182\,(m \pm l) \cot \tfrac{1}{2}\,(m \pm l). \tag{473}$$

Then let

$$\left.\begin{aligned} K\,(n, l) &= \phi\,(m+l) + \phi\,(m-l) \\ Z\,(n, l) &= \phi\,(m+l) - \phi\,(m-l) \end{aligned}\right\} \tag{474}$$

Also let

$$\left.\begin{aligned} \psi\,(n) \cos \varDelta n &= \tfrac{1}{24} \frac{\sin 12\,n}{\sin \frac{1}{2}\,n} \cos \varDelta n = C\,(n) \\ \psi\,(n) \sin \varDelta n &= S\,(n) \end{aligned}\right\} \tag{475}$$

The functions K (n, l), Z (n, l), C (n), S (n) may be easily computed from (473), (474), (475).

Then if we denote the additive clearance for $\Sigma \delta h \cos li$ by

$$[A, n, l, \cos]\,A + [B, n, l, \cos]\,B, \tag{476}$$

and that for $\Sigma \delta h \sin li$ by

$$[A, n, l, \sin]\,A + [B, n, l, \sin]\,B. \tag{477}$$

We have

$$\left.\begin{aligned} [A, n, l, \cos] &= -C\,(n)\,K\,(n, l) \cos 182\,l + S\,(n)\,Z\,(n, l) \sin 182\,l \\ [B, n, l, \cos] &= -S\,(n)\,K\,(n, l) \cos 182\,l - C\,(n)\,Z\,(n, l) \sin 182\,l \\ [A, n, l, \sin] &= -S\,(n)\,Z\,(n, l) \cos 182\,l - C\,(n)\,K\,(n, l) \sin 182\,l \\ [B, n, l, \sin] &= C\,(n)\,Z\,(n, l) \cos 182\,l - S\,(n)\,K\,(n, l) \sin 182\,l \end{aligned}\right\} \tag{478}$$

We must remark that if $\frac{1}{2}\,(m + l) = 360^\circ$, $\phi\,(m + l)$ is equal to 182·5.

This case arises when l is the tide MSf of speed $2(\sigma - \eta)$, and m the tide M_2 of speed $2\,(\gamma - \sigma)$, for $m + l$ is then $24 \times 2\,(\gamma - \eta) = 720^\circ$.

The clearance of the long-period tide l from the effects of the short-period tide n requires the computation of these four coefficients. For the clearance of the five long-period tides from the effects of the three tides M_2, N, O, it will be necessary to compute 60 coefficients.

If it shall be found convenient to make the initial instant or epoch for the tides of long period different from that chosen in the reductions of those of short period, it will, of course, be necessary to compute the values which A and B would have had if the two epochs had been identical. A and B are, of course, the component semi-ranges of the tide of short period at the epoch chosen for the tides of long period; to determine them it is necessary to multiply R by the cosine and sine of $V + u - \kappa$ at the epoch.

[Q.]

Schedule of Coefficients for Clearance of Daily Means in the Final Equations.

$l =$	$\sigma - \varpi$	2σ	$2(\sigma - \eta)$	η	2η
(M_2) $n = 2(\gamma - \sigma)$.					
[A, n, l, cos]	−0·05557	+0·00302	+5·7393	−0·10410	−0·01465
[B, n, l, cos]	−0·17036	−0·03773	−2·9228	−0·07525	−0·07546
[A, n, l, sin]	−0·17075	+0·04170	−2·8400	−0·00176	−0·00353
[B, n, l, sin]	+0·04410	+0·01052	−5·7271	+0·00476	+0·00958
(N) $n = 2\gamma - 3\sigma + \varpi$.					
[A, n, l, cos]	−0·05884	+0·03680	+0·02938	−0·01760	−0·01760
[B, n, l, cos]	−0·07758	−0·22337	−0·19384	+0·00254	+0·00254
[A, n, l, sin]	−0·02059	−0·15245	−0·12210	+0·00020	+0·00041
[B, n, l, sin]	+0·11381	−0·08544	−0·08081	+0·00007	+0·00015
(O) $n = \gamma - 2\sigma$.					
[A, n, l, cos]	−0·06485	+0·01673	+0·01582	−0·19240	−0·19340
[B, n, l, cos]	−0·34765	−0·07788	−0·08158	−0·18260	−0·18311
[A, n, l, sin]	−0·34523	+0·08418	+0·08748	−0·00460	−0·00926
[B, n, l, sin]	+0·04052	+0·03379	+0·03295	+0·00897	+0·01802

It may happen from time to time that the tide-gauge breaks down for a few days, from the stoppage of the clock, the choking of the tube, or some other such accident. In this case there will be a hiatus in the values of δh. Now, the whole process employed depends on the existence of 365 continuous values of δh. Unless, therefore, the year's observations are to be sacrificed, this hiatus must be filled. If not more than three or four days' observations are wanting, it will be best to plot out the values of δh graphically on each side of the hiatus, and filling in the gap with a curve drawn by hand, use the values of δh given by the conjectural curve. If the gap is somewhat longer, several plans may be suggested, and judgment must be used as to which of them is to be adopted.

If there is another station of observation in the neighborhood, the values of δh for that station may be inserted.

The values of δh for another part of the year, in which the moon's and sun's declinations are as nearly as may be the same as they were during the gap, may be used.

It may be, however, that the hiatus is of considerable length, so that the preceding methods are inapplicable: as when in 1882 the tidal record for Vizagapatam is wanting for 67 days. The following method of treatment will then be applicable:—

We find approximate values of the tidal constituents of long period, and fill in the hiatus, so as to complete the 365 values, with the computed height of the tide during the hiatus.

To find these approximate values we form $\Sigma\delta h \cos lt$ and $\Sigma\delta h \sin lt$ for the days of observation; next, in the ten final equations of Schedule P we neglect all the terms with small coefficients, and in the terms whose coefficients are approximately 182·5, we substitute a coefficient equal to 182·5 diminished by half the number of days of hiatus. For example, for Vizagapatam in 1882 we have $182{\cdot}5-\frac{1}{2}\times 67=149$, and, e. g., $\Sigma\delta h \cos(\sigma-\varpi)t=149\text{A}$ approximately. After the approximate values of A, B, C, D, &c., have been found, it is easy to find the approximate height of tide for the days of the hiatus. This plan will also apply where the hiatus is of short duration.

It may be pursued whether or not we are working with cleared daily means; for if the daily means are uncleared, as will henceforth be the case, we import with the numbers by which the hiatus is filled exactly those fictitious tides of long period which are cleared away by the use of the "clearance coefficients," in preparing the ten final equations for solution.

Other methods of treating a stoppage of the record may be devised. If the stoppage be near the beginning of the year, or near the end, we may neglect the observations before or after the gap, and compute afresh the 100 coefficients of Schedule P, and the clearance coefficients of Schedule Q for the number of days remaining. If the gap is in the middle we might compute the values of the coefficients of Schedules P and Q as though the days of hiatus were days of observation, bearing in mind that the formulæ are to be altered by the consideration that time is to be measured from the initial $11^{\text{h}}\ 30^{\text{m}}$ of the year, instead of from the initial $11^{\text{h}}\ 30^{\text{m}}$ of the days of hiatus.

The so computed coefficients are then to be subtracted from the values given in Schedules P and Q, and the amended final equations and amended clearance coefficients to be used.

It must remain a matter of judgment as to which of these various methods is to be adopted in each case.

70. *Method of Equivalent Multipliers for the Harmonic Analysis for the Tides of Long Period.*

Up to the present time the harmonic analysis for these tides has been conducted on a plan which seems to involve a great deal of unnecessary labour. If l be the speed of any one of the five tides for which the analysis has been carried out, in degrees per m. s. day, the values of $\cos lt$ and $\sin lt$ have been computed for $t=0, 1, 2 \ldots 364$, so that there are 730 values for each of the five tides. These 730 values have then been multiplied by the 365 δh's corresponding to each value of t, and the summations gave $\Sigma\delta h \cos lt$ and $\Sigma\delta h \sin lt$, the numerical results being the left-hand sides of one pair of the ten final equations explained in § 68. Now, it appears that this labour may be largely abridged, without any substantial loss of accuracy.

The plan proposed by Professor Adams is that of equivalent multipliers. The values of $\cos lt$ may be divided into eleven groups, according as they fall nearest to 1·0, ·9, ·8, ·7 . . . . . ·2, ·1, 0. Then, as all the values of δh are to be multiplied by some value of $\cos lt$, and that value of $\cos lt$ must fall into one of these groups, we collect together all the values of δh which belong to one of these groups, sum them, and multiply the sum by the corresponding multiplier, 1·0, ·9, ·8, &c., as the case may be. Since there are as many values of $\cos lt$ which are negative as positive, we must change the sign of half of the δh's. This changing of sign may be effected mechanically as follows:—In the spaces for entry of the δh's, those δh's whose sign is to be unchanged are to be entered on the left side of the space if positive, and to the right if negative; when the sign is to be altered this order of entry is to be reversed. Thus in the column corresponding to each multiplier we shall have two sub-columns, on the left all the δh's which, when the signs are appropriately altered, are +, and on the right those which are —. The sub-columns are to be separately summed, and their difference gives the total of the column, which is to be multiplied by the multiplier appropriate to the column. The treatment for the formation of $\Sigma\delta h \sin lt$ is precisely similar.

The annexed form [Schedule R] is designed for entry for determination of $\Sigma\delta h \cos(\sigma-\eta)t$.

The entries of δh are to be made continuously in the marked squares from left to right, and back again from right to left. The numbers in the squares, which in the computation forms are to be printed small and put in the corner, indicate the days of observation. The rows are arranged in sets of four corresponding to each complete period of $2(\sigma-\eta)$. In the middle pair for each period the + values of δh are to be written on the right, and in the rest on the left. The word 'change' opposite half the rows is to show the computer that he is to change the mode of entry. Each column, excepting that for zero, is to be summed at the foot of the page, and multiplied by the multiplier corresponding to its column. A pair of forms is required for each tide of long period; they are very easily prepared from the existing forms, in which the values of the multipliers are already computed.

[R]

Form for Reduction of the Tide MSf.

		+ − 1·0	+ − ·9	+ − ·8	+ − ·7	+ − ·6	+ − ·5	+ − ·4	+ − ·3	+ − ·2	+ − ·1	No entries.	
1	→	0	1		2				3				
	←	7		6			5				4		change.
	→	8		9				10				11	change.
	←	14			13			12					
2	→	15	16			17				18			
	←		21			20				19			change.
	→	22	23		24			25					change.
	←	29		28			27					26	
3	→	30		31			32				33		
	←		36		35				34				change.
	→	37	38			39			40				change.
	←	44	43			42				41			
4	→	45		46				47					
	←	51		50				49				48	change.
	→	52	53			54				55			change.
	←		58			57			56				
5	→	59	60		61				62				
	←	66		65			64				63		change.
	→	67		68			69				70		change.
	←	74	73		72			71					
Total + Total −			&c.				&c.				&c.	No entries.	
Total Multiply Results		×1·0	×·9	×·8	×·7	×·6	×·5	×·4	×·3	×·2	×·1	×·0	

Sum laterally . . . . Sum of + = ... Sum of − = ...

$\Sigma\delta h \cos 2\,(\sigma-\eta)\,t =$ . . .

71. *To reproduce the quantities harmonically analyzed.*

Having analyzed a set of partial or hour sums and determined the c's and s's, or R's and ζ's, it is sometimes desirable to recombine the partial tides of the analysis sheet into a resultant curve in order to see how this curve compares with the curve obtained by plotting the hour sums; or it may be done for other purposes. The nature of the case will suggest how many of the harmonics should be retained.

The same process is applicable to inequality curves, certain constants having been found from the analysis of the heights or intervals involving the particular inequality in question. (See § 54, Part I.) The accompanying form will show how this combination may be made.

Form for reproducing observed quantities from the results of an analysis.

$$y = y_0 + c_1 \cos x + s_1 \sin x + c_2 \cos 2x + s_2 \sin 2x + c_3 \cos 3x + s_3 \sin 3x + \;.\;.\;.\;.$$

A	$x=0$		$x=1^h=15°$		$x=2^h=30°$		$x=3^h=45°$		$x=4^h=60°$		$x=5^h=75°$		$x=6^h=90°$		$x=7^h=105°$	
	(0)	A × (0)	(1)	A × (1)	(2)	A × (2)	(3)	A × (3)	(4)	A × (4)	(5)	A × (5)	(6)	A × (6)	(7)	A × (7)
$c_1 =$	1		·966		·866		·707		·5		·259		0		− ·259	
$s_1 =$	0		·259		·5		·707		·866		·966		1		·966	
$c_2 =$	1		·866		·5		0		− ·5		− ·866		−1		− ·866	
$s_2 =$	0		·5		·866		1		·866		·5		0		− ·5	
$c_3 =$	1		·707		0		− ·707		−1		− ·707		0		·707	
$s_3 =$	0		·707		1		·707		0		− ·707		−1		− ·707	
$c_4 =$	1		·5		− ·5		−1		− ·5		·5		1		·5	
$s_4 =$	0		·866		·866		0		− ·866		− ·866		0		·866	
$c_6 =$	1		0		−1		0		1		0		−1		0	
$s_6 =$	0		1		0		−1		0		1		0		−1	
$c_8 =$	1		− ·5		− ·5		1		− ·5		− ·5		1		− ·5	
$s_8 =$	0		·866		− ·866		0		·866		− ·866		0		·866	
Sum																
$y = y_0 + \text{sum}$																

A	$x=8^h=120°$		$x=9^h=135°$		$x=10^h=150°$		$x=11^h=165°$		$x=12^h=180°$		$x=13^h=195°$		$x=14^h=210°$		$x=15^h=225°$	
	(8)	A × (8)	(9)	A × (9)	(10)	A × (10)	(11)	A × (11)	(12)	A × (12)	(13)	A × (13)	(14)	A × (14)	(15)	A × (15)
$c_1 =$	− ·5		− ·707		− ·866		− ·966		−1		− ·966		− ·866		− ·707	
$s_1 =$	·866		·707		·5		·259		0		− ·259		− ·5		− ·707	
$c_2 =$	− ·5		0		·5		·866		1		·866		·5		0	
$s_2 =$	− ·866		−1		− ·866		− ·5		0		·5		·866		1	
$c_3 =$	1		·707		0		− ·707		−1		− ·707		0		·707	
$s_3 =$	0		·707		1		·707		0		− ·707		−1		− ·707	
$c_4 =$	− ·5		−1		− ·5		·5		1		·5		− ·5		−1	
$s_4 =$	·866		0		− .866		− ·866		0		·866		·866		0	
$c_6 =$	1		0		−1		0		1		0		−1		0	
$s_6 =$	0		1		0		−1		0		1		0		−1	
$c_8 =$	− ·5		1		−5		− ·5		1		− ·5		− ·5		1	
$s_8 =$	− ·866		0		·866		− ·866		0		·866		− ·866		0	
Sum																
$y = y_0 + \text{sum}$																

A	$x=16^h=240°$		$x=17^h=255°$		$x=18^h=270°$		$x=19^h=285°$		$x=20^h=300°$		$x=21^h=315°$		$x=22^h=330°$		$x=23^h=345°$	
	(16)	A × (16)	(17)	A × (17)	(18)	A × (18)	(19)	A × (19)	(20)	A × (20)	(21)	A × (21)	(22)	A × (22)	(23)	A × (23)
$c_1 =$	− ·5		− ·259		0		·259		·5		·707		·866		·966	
$s_1 =$	− ·866		− ·966		−1		− ·966		− ·866		− ·707		− ·5		− ·259	
$c_2 =$	− ·5		− ·866		−1		− ·866		− ·5		0		·5		·866	
$s_2 =$	·866		·5		0		− ·5		− ·866		−1		− ·866		− ·5	
$c_3 =$	1		·707		0		− ·707		−1		− ·707		0		·707	
$s_3 =$	0		·707		1		·707		0		− ·707		−1		− ·707	
$c_4 =$	− ·5		·5		1		·5		− ·5		−1		− ·5		·5	
$s_4 =$	− ·866		− ·866		0		·866		·866		0		− ·866		− ·866	
$c_6 =$	1		0		−1		0		1		0		−1		0	
$s_6 =$	0		1		0		−1		0		1		0		−1	
$c_8 =$	− ·5		− ·5		1		− ·5		− ·5		1		− ·5		− ·5	
$s_8 =$	·866		− ·866		0		·866		− ·866		0		·866		− ·866	
Sum																
$y = y_0 + \text{sum}$																

$$y = y_0 + R_1 \cos(x - \zeta_1) + R_2 \cos(2x - \zeta_2) + R_3 \cos(3x - \zeta_3) + \;.\;.\;.$$

$$R = \sqrt{c^2 + s^2}, \quad \tan\zeta = s/c, \quad c = R\cos\zeta, \quad s = R\sin\zeta.$$

72. *To combine the various tidal components for any given future time.*

This may be illustrated by an example: To find the height of the tide at San Francisco (Fort Point), Cal., at 1 o'clock p. m. (standard time) March 1, 1910, the principal tidal components being

$$\begin{aligned} M_2 &= 1{\cdot}69 \text{ ft.}, & M_2° &= 332, & S_2 &= 0{\cdot}39 \text{ ft.}, & S_2° &= 336°, \\ N_2 &= 0{\cdot}37 \text{ ft.}, & N_2° &= 305°, & K_1 &= 1{\cdot}22 \text{ ft.}, & K_1° &= 107°, \\ O_1 &= 0{\cdot}78 \text{ ft.}, & O_1° &= 0{\cdot}87°, & P_1 &= 0{\cdot}37 \text{ ft.}, & P_1° &= 105°. \end{aligned}$$

We are to find the value of

$$\begin{aligned} y = {} & M_2' \cos(m_2t + \arg_o M_2 - M_2°) + S_2 \cos(s_2t + \arg_o S_2 - S_2°) \\ & + N_2' \cos(n_2t + \arg_o N_2 - N_2°) + K_1' \cos(k_1t + \arg_o K_1 - K_1°) \\ & + O_1' \cos(o_1t + \arg_o O_1 - O_1°) + P_1 \cos(p_1t + \arg_o P_1 - P_1°) \end{aligned} \tag{479}$$

where the accent denotes that the factor f, Table 10, has been applied to the amplitude. Now Table 3 gives $V_0 + u_o$ or arg, for Greenwich midnight, while the above equation supposes it to belong to meridian of San Francisco and the time to be local. According to section 62, Part III, we may use the Greenwich values directly, provided we modify the epochs so as to take into account the longitude of the place and of the time meridian. The epochs modified once for all are

$$344°, 340°, 322°, 108°, 99°, 106°$$

respectively.

By Tables 3 and 4, we have for March 1, 1910,

$$\begin{aligned} \arg_o M_2 &= 243° + 1° = 244°, & \arg_o K_1 &= 3° + 58° = 61°, \\ \arg_o S_2 &= 0 + 0 = 0, & \arg_o O_1 &= 243 + 303 = 186, \\ \arg_o N_2 &= 107 + 311 = 58, & \arg_o P_1 &= 350 + 302 = 292. \end{aligned}$$

By Table 1, we have for 13 hours (midnight to 1 p. m.)

$$\begin{aligned} & m_2t = 17°, s_2t = 30°, n_2t = 10°, k_1t = 196°, o_1t = 181°, p_1t = 194°; \\ \therefore\ & y = M_2' \cos 277° + S_2 \cos 50° + N_2' \cos 106° \\ & \quad + K_1' \cos 149° + O_1' \cos 268° + P_1 \cos 210°. \end{aligned}$$

By Table 10, the values of f are

$$0{\cdot}98, 1{\cdot}00, 0{\cdot}98, 1{\cdot}07, 1{\cdot}12, 1{\cdot}00$$

respectively.

$$\begin{aligned} \therefore\ & M_2' = 1{\cdot}67, S_2 = 0{\cdot}38, N_2' = 0{\cdot}35, K_1' = 1{\cdot}31, O_1' = 0{\cdot}86, P_1 = 0{\cdot}37 \text{ feet.} \\ & y = +0{\cdot}20 + 0{\cdot}24 - 0{\cdot}10 - 1{\cdot}10 - 0{\cdot}03 + 0.35 = -0{\cdot}44 \text{ feet} \end{aligned}$$

as the height of the sea, reckoned from mean sea level, at the given time.

At 2, 3, 4, 5, and 6 o'clock, p.m., the heights are in like manner found to be $-0{\cdot}64$, $-1{\cdot}33$, $-2{\cdot}32$, $-2{\cdot}55$, $-1{\cdot}14$ feet.

73. *Harmonic analysis of a series two weeks in extent.*

In the Manual of Scientific Enquiry* Darwin shows how to analyze a short series of hourly readings extending over a fortnight or a month. Summations are made for M, S, and O. The M and O sums are analyzed in the usual way, giving the amplitudes and epochs of M_2 and O_1. The S sums give an affected S_2 and K_1; the summation extends over slightly different periods in the two cases.

The lengths of series used are as follows:

	d.	h.		d.	h.
For M	14	12	or	29	00,
" S (diurnal)	14	00	"	28	00,
" S (semidiurnal)	15	00	"	30	00,
" O	14	00	"	26	21.

* Or B. A. A. S. Report, 1896.

If, for the moment, we put

$$\tan \psi = \frac{f(\mathrm{K}_2)\, \mathrm{K}_2 \sin (\arg^* \mathrm{K}_2 - \arg \mathrm{S}_2)}{\mathrm{S}_2 \text{ (with } \mathrm{T}_2) + f(\mathrm{K}_2)\, \mathrm{K}_2 \cos (\arg \mathrm{K}_2 - \arg \mathrm{S}_2)}, \tag{480}$$

in which the arguments are to be taken at the middle of the series, we obtain the amount that the S_2 corresponding to a given solar parallax is accelerated by K_2 (see § 2, Part III). From Table 1

$$\frac{\mathrm{S}_2}{\mathrm{K}_2} = 3{\cdot}67, \ \arg \mathrm{K}_2 - \arg \mathrm{S}_2 = 2\,(h - \nu''). \tag{481}$$

The formula for ψ may now be written

$$\tan \psi = \frac{f(\mathrm{K}_2) \sin 2\,(h - \nu'')}{3{\cdot}67\, p_{,} + f(\mathrm{K}_2) \cos 2\,(h - \nu'')} \tag{482}$$

where

$$p_{,} = \left(\frac{\text{sun's parallax}}{\text{sun's mean parallax}}\right)^3 \doteq \frac{\mathrm{S}_2 \text{ (with } \mathrm{T}_2)}{\mathrm{S}_2} \tag{483}$$

$$= \text{tabular value last column, Table 31.}$$

$$\therefore \mathrm{S}_2{}^\circ = \zeta(\mathrm{S}_2) + \psi. \tag{484}$$

The fifth column of Table 31 gives the correction of $\mathrm{S}_2{}^\circ$ due to the direct effect of T_2; Darwin has disregarded this correction. The amplitude of S_2 is the observed amplitude $R\,(\mathrm{S}_2)$ multiplied by

$$\frac{3{\cdot}67 \cos \psi}{3{\cdot}67\, p_{,} + f(\mathrm{K}_2) \cos 2\,(h - \nu'')} \tag{485}$$

The epoch and amplitude of K_2 are obtained from the equations

$$\mathrm{K}_2{}^\circ = \mathrm{S}_2{}^\circ, \tag{486}$$

$$\mathrm{K}_2 = \frac{1}{3{\cdot}67}\, \mathrm{S}_2. \tag{487}$$

In like manner we may put

$$\tan \phi = \frac{\mathrm{P}_1 \sin (\arg \mathrm{P}_1 - \arg \mathrm{K}_1)}{f(\mathrm{K}_1)\, \mathrm{K}_1 + \mathrm{P}_1 \cos (\arg \mathrm{P}_1 - \arg \mathrm{K}_1)} \tag{488}$$

where ϕ is the amount by which K_1 is accelerated because of P_1. From Table 1

$$\frac{\mathrm{K}_1}{\mathrm{P}_1} = 3, \ \arg \mathrm{P}_1 - \arg \mathrm{K}_1 = -\,2\,h + \nu' \pm 180^\circ: \tag{489}$$

$$\therefore \tan \phi = \frac{\sin (2\,h - \nu')}{3\, f(\mathrm{K}_1) - \cos (2\,h - \nu')}. \tag{490}$$

$$\mathrm{K}_1{}^\circ = \zeta(\mathrm{K}_1) + \arg_o \mathrm{K}_1 + \phi, \tag{491}$$

$$\mathrm{K}_1 = \frac{3 \cos \phi}{3\, f(\mathrm{K}_1) - \cos (2\,h - \nu')} \times R\,(\mathrm{K}_1). \tag{492}$$

For P_1 we have

$$\mathrm{P}_1{}^\circ = \mathrm{K}_1{}^\circ, \tag{493}$$

$$\mathrm{P}_1 = \tfrac{1}{3}\, \mathrm{K}_1. \tag{494}$$

74. *Harmonic analysis of high and low waters.*†

The components K_1 and O_1 can be quite accurately obtained by the following process:

Copy the heights of the high and low waters into the form for "hourly ordinates," always putting these values upon the nearest solar hour. Apply the K and O stencils in the usual manner, but also keeping track of the number of high waters and the number of low waters that enter into each partial hourly sum. Then bring the partial hourly sums together, and note the difference between the number of low and high waters. Correct the hourly sums by this difference multiplied by one-half of the mean range of tide for the period of the observations. Analyze the

* I. e., the equilibrium argument, Table 1.

† See Chapter III, Part III.

twenty-four hourly means or residuals in the usual manner. K_1 thus obtained should be corrected for P_1 by Table 31. O_1 does not require correction. The example given below shows the degree of accuracy attained at Sitka from a 29-day series of high and low waters. The corresponding results for Sandy Hook, N. J., are also given.

Station, Sitka, Alaska.	Lat. 57° 4′ N.; long. 135° 20′ W.
	yr. mo. da. hr.
Component, K.	Observations begin 1893, July 1 0.
Computer, D. S. B.	Observations end 1893, July 29 23.
Kind of time used, mean local civil.	

	Page.	0h	1h	2h	3h	4h	5h	6h	7h	8h	9h	10h	11h
Stencil sums	1	0·0	21·2	28·4	13·2	12·5	12·0	11·4	0·0	3·4	7·9	9·3	5·9
No. highs and lows			1 H, 1 L	2 H	1 H	1 H	1 H	1 H		1 L	2 L	2 L	1 L
	2	16·1	16·3	5·8	0·0	5·0	3·9	2·9	4·0	13·0	0·0	11·5	11·9
		1 H	1 H	1 L		1 L	1 L	1 L	2 L	1 H, 1 L		1 H	1 H
	3	7·0	6·8	23·1	16·3	15·7	14·0	12·7	11·6	11·2	2·4	3·6	9·3
		1 L	1 L	1 H, 1 L	1 H	1 H	1 H	1 H	1 H	1 H	1 L	1 L	2 L
	4	27·6	15·3	15·1	6·8	6·0	0·0	5·3	9·4	4·7	15.0	0·0	11·0
		2 H	1 H	1 H	1 L	1 L		1 L	2 L	1 L	1 H, 1 L		1 H
	5	0·0	0·0	15·1	0·0	0·0	0·0	0·0	0·0	0·0	3·8	0·0	0·0
				1 H							1 L		
	Sums	50·7	59·6	87·5	36·3	39·2	29·9	32·3	25·0	32·3	29·1	24·4	38·1

	Page.	12h	13h	14h	15h	16h	17h	18h	19h	20h	21h	22h	23h
Stencil sums	1	6·7	0·0	0·0	25·8	12·8	26·1	13·6	14·0	8·4	16·1	7·6	7·2
No. highs and lows		1 L			2 H	1 H	2 H	1 H	1 H	1 L	2 L	1 L	1 L
	2	0·0	33·0	13·5	22·2	8·6	0·0	16·4	7·8	7·6	14·6	15·0	31·2
			2 H, 1 L	1 H	1 H, 1 L	1 L		2 L	1 L	1 L	1 H	1 H	2 H
	3	13·0	8·3	0·0	0·0	29·7	14·7	28·0	13·7	13·8	7·8	7·9	7·3
		2 L	1 L			2 H	1 H	2 H	1 H	1 H	1 L	1 L	1 L
	4	0·0	11·3	33·5	26·4	9·2	9·3	0·0	18·8	17·6	13·4	13·4	0·0
			1 H	2 H, 1 L	2 H	1 L	1 L		2 L	2 L	1 H	1 H	
	5	0·0	0·0	0·0	0·0	13·5	0·0	0·0	0·0	0·0	8.2	0·0	0·0
						1 H					1 L		
	Sums	19·7	52·6	47·0	74·4	73·8	50·1	58·0	54·3	47·4	60·1	43·9	45·7

	Page.	0h	1h	2h	3h	4h	5h	6h	7h	8h	9h	10h	11h
No. of highs		3	3	5	2	2	2	2	1	2	1	1	2
No. of lows		1	2	2	1	2	1	2	4	3	5	3	3
Sums		50·7	59·6	87·5	36·3	39·2	29·9	32·3	25·0	32·3	29·1	24·4	38·1
½ Mn × diff.		− 7·3	− 3·6	−11·0	− 3·6	0·0	− 3·6	0·0	+11·0	+ 3·6	+14·6	+ 7·3	+3·6
		43·4	56·0	76·5	32·7	39·2	26·3	32·3	36·0	35·9	43·7	31·7	41·7
Divisors		4	5	7	3	4	3	4	5	5	6	4	5
Means		10·85	11·20	10·93	10·90	9·80	8·77	8·08	7·20	7·18	7·29	7·92	8·34
Residuals		+ 1·00	+ 1·35	+ 1·08	+ 1·05	− 0·05	− 1·08	− 1·77	− 2·65	− 2·67	− 2·56	− 1·93	−1.51

	Page.	12h	13h	14h	15h	16h	17h	18h	19h	20h	21h	22h	23h
No. of highs		0	3	3	5	4	3	3	2	1	2	2	2
No. of lows		3	2	1	1	2	1	2	3	4	4	2	2
Sums		19·7	52·6	47·0	74·4	73·8	50·1	58·0	54·3	47·4	60·1	43·9	45·7
½ Mn × diff.		+11·0	− 3·6	− 7·3	−14·6	− 7·3	− 7·3	− 3·6	+ 3·6	+11·0	+ 7·3	0·0	0·0
		30·7	49·0	39·7	59·8	66·5	42·8	54·4	50·7	58·4	67·4	43·9	45·7
Divisors		3	5	4	6	6	4	5	5	5	6	4	4
Means		10·23	9·80	9·92	9·97	11·08	10·70	10·88	10·14	11·68	11·23	10·98	11·42
Residuals		+ 0·38	− 0·05	+ 0·07	+ 0·12	+ 1·23	+ 0·85	+ 1·03	+ 0·29	+ 1·83	+ 1·38	+ 1·13	+1·57

The divisor for each hour = No. of highs + No. of lows.

The sum of the hourly means is 2364·9 and the mean $9·85\frac{9}{24}$.

The correction ½ Mn × diff. is one-half the mean range from the "first reduction" multiplied by the difference between the number of highs and lows entering into each hourly sum; subtracted when the highs are in excess, and added when the lows are in excess. Mn for this month is 7·31 feet.

SITKA, ALASKA.

From 29 days' high and low waters, analyzed as above, July 1–29, 1893

K_1	1·602
$K_1°$	127·3
O_1	0·917
$O_1°$	95·7

From harmonic analysis of hourly ordinates 1 year, 1893–94

K_1	1·508
$K_1°$	125
O_1	0·906
$O_1°$	123

SANDY HOOK, N. J.

From 29 days' high and low waters, analyzed as above, July 1–29, 1893		From harmonic analysis of hourly ordinates 2 years, 1887, 1888	
K_1	0·336	K_1	0·334
$K_1°$	97·4	$K_1°$	101
O_1	0·150	O_1	0·173
$O_1°$	99·6	$O_1°$	98

If, however, before beginning the summation we subtract $\frac{1}{2}$ Mn from all high-water heights and add $\frac{1}{2}$ Mn to all low-water heights, the necessity for keeping count of the number of highs and lows will be avoided and the same result obtained as before.

If the high and low water heights be tabulated upon hourly ordinate forms, and summations made with the K and O stencils even for a very long period, there is no guaranty that the effect of M_2 will be totally eliminated, although it will be nearly so. To understand this, suppose that at the given station the tropic diurnal inequality be great in low-water heights but small in the high-water heights; also, suppose that observations be confined to high waters. Now if these heights be tabulated in their proper places upon the hourly ordinate sheets and the K stencil applied to them, it is clear that an amplitude much too small will be obtained for K_1. In other words, M_2 will have a pronounced effect upon the result. If, now, lows as well as highs be included in the observations, the effect of M_2 upon a diurnal tide will be very much diminished; but there is no reason to suppose that it will ever disappear completely, however long the series.

While it is possible to distribute high and low water heights or times according to known arguments and obtain consistent results for a given station, serious difficulties arise when an attempt is made to interpret the results in terms of the harmonic constants or to accurately obtain these constants by any prescribed distributions.

For instance, the phase inequality in the range of tide is not a simple harmonic increase and decrease of the range even if μ_2 be ignored. Its unsymmetrical character can readily be seen by referring to Table 16. The parallax and declinational inequalities are even more complicated than that of phase, as is pointed out in §§ 46, 47 of Part III.

Having called attention to some of the difficulties encountered in the harmonic analysis of high and low water observations, we pass to a more thorough and systematic procedure. For convenience, the whole work of making the analysis may be divided into six steps or operations:

(1) Making a "first reduction."

(2) Finding the mean amplitudes and the time occurrence of the mean ranges using four consecutive tides.

(3) Obtaining hourly ordinates from these amplitudes by aid of a system of sine curves drawn upon a transparent sheet.

(4) Summing these ordinates with semidiurnal stencils and analyzing the partial sums.

(5) Diminishing the heights of the tides by the ordinates belonging to the times of the tides, and tabulating the heights thus altered upon hourly ordinate forms.

(6) Summing these values with diurnal stencils, and analyzing the partial sums.

(1′) The process of making a "first reduction" is fully described in § 51, Part I, and § 27, Part III. The epoch of M_2 is found from the lunitidal intervals by aid of this last-named paragraph; the amplitude, from the observed mean range of tide by aid of Tables 23 and 14. Where there is much difference between the duration of rise and of fall, the amplitude of M_2 as determined from the range must be affected with M_4. To correct for this, divide the M_2 as obtained above by the quantity

$$\cos v + \sin 2\, v \times \tfrac{1}{8}\,(\text{duration fall} \sim 6^{\text{h}}{\cdot}21) \qquad (495)$$

where

$$v = (\text{duration fall} \sim 6^{\text{h}}{\cdot}21) \times 14^{\circ}{\cdot}492.$$

(2′) In the accompanying tabulation the last column shows the values of the successive mean amplitudes; the fifth, the times of occurrence of the ranges; and the sixth, their duration which is 6·21 hours on an average. The values inclosed in parentheses involve observations prior to July 1; generally they would have been simply inferred from succeeding values.

SITKA, ALASKA.

Date.	Time of tide.	Sum of consecutive four.	Mean.	6 hours, etc., applied.	Difference.	Height of tide.	Range.	Sum of alternate ranges.	Amplitude.
1893.	*h.*	*h.*	*h.*	*h.*	*h.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*
				(22·7)					(3·8)
July 1.	1·1				6·1	14·6			
				(4·8)			11·2		(3·9)
	8·2				6·2	3·4			
		44·0	11·0	11·0			9·4	15·6	3·9
	14·8				6·2	12·8			
		44·9	11·2	17·2			4·4	15·4	3·8
	19·9				6·2	8·4			
		45·4	11·4	23·4			6·0	14·9	3·7
2.	2·0				6·1	14·4			
		46·0	11·5	5·5			10·5	15·1	3·8
	8·7				6·3	3·9			
		46·9	11·7	11·8			9·1	15·2	3·8
	15·4				6·0	13·0			
		47·4	11·8	17·8			4·7	14·8	3·7
	20·8				6·2	8·3			
		48·1	12·0	0·0			5·7	14·7	3·7
3.	2·5				6·2	14·0			
		48·7	12·2	6·2			10·0	14·5	3·6
	9·4				6·1	4·0			
		49·1	12·3	12·3			8·8	15·0	3·8
	16·0				6·1	12·8			
		49·8	12·4	18·4			5·0	14·2	3·6
	21·2				6·2	7·8			
		50·3	12·6	0·6			5·4	13·9	3·5
4.	3·2				6·1	13·2			
		50·9	12·7	6·7			8·9	14·0	3·5
	9·9				6·3	4·3			
		51·9	13·0	13·0			8·6	14·2	3·6
	16·6				6·2	12·9			
		52·8	13·2	19·2			5·3	13·5	3·4
	22·2				6·1	7·6			
		53·3	13·3	1·3			4·9	12·8	3·2
5.	4·1				6·3	12·5			
		54·1	13·5	7·6			7·5	13·1	3·3
	10·4				6·2	5·0			
		55·3	13·8	13·8			8·2	13·5	3·4
	17·4				6·3	13·2			
		56·4	14·1	20·1			6·0	13·0	3·2
	23·4				6·2	7·2			
		57·2	14·3	2·3			4·8	12·1	3·0
6.	5·2				6·1	12·0			
		57·8	14·4	8·4			6·1	12·5	3·1
	11·2				6·3	5·9			
		34·9	8·7	14·7			7·7	13·1	3·3
	18·0				6·3	13·6			
		36·1	9·0	21·0			7·0	12·5	3·1
7.	0·5				6·6	6·6			

(3′). To prepare for interpolating hourly ordinates from the successive ranges, first sel cross-section paper whose smallest divisions are about $\frac{1}{10}$ inch square. Let an hour of time an inch or more, and let a foot of tide, when convenient, correspond to an inch on the sheet. A the upper margin of the sheet number the hours from 0 to 24, or 0, and thence on to 12, maki hours represented. On a sheet of tracing cloth lay out a rectangle, say 10 inches high and enough to represent 6 lunar, or 6·21 solar, hours. The center of this rectangle is the nod system of, say, 10 sine curves whose amplitudes vary from 0 to 5 inches, and are numbe convenient, from 0 to 5. All curves extend over a half period or 6·21 hours. To inte hourly heights, place the node of the sine curves at the time of occurrence of a semidiurnal given in the fifth column of the above tabulation. Select the curve having a number equi to the mean amplitude at the time given in the last column. Read the heights at the points this curve crosses the hour lines. These are the required hourly heights reckoned from mean level. So proceed with each range. If the tide have a very large phase inequality, causi quarter tidal day to depart much from a quarter lunar day, or 6·21 hours, this fact may be

into consideration by reading one or two of the hourly heights on either side of the node as before, and the hourly heights near the times of maxima and minima, when the edges of the rectangle have been made to fall exactly half way between the times given in the fifth column; or more than one permanent set of curves may be used. Probably this refinement is unnecessary.

SITKA, ALASKA, 1893.

Interpolated ordinates of semidiurnal wave.

Day of month.	July 1	2	3	4	5	6	7
Day of series.	1	2	3	4	5	6	7
h. m.	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*
0 00	8·3	7·1	6·0	5·0	4·1	3·3	2·8
1 00	9·5	8·7	7·7	6·7	5·5	4·2	3·2
2 00	9·9	9·5	9·1	8·3	7·1	5·6	3·9
3 00	9·1	9·7	9·6	9·3	8·4	7·1	5·2
4 00	7·6	8·6	9·2	9·4	9·1	8·3	6·5
5 00	5·7	7·0	8·1	8·6	9·2	9·0	7·9
6 00	3·8	5·1	6·4	7·2	8·3	8·9	8·7
7 00	2·5	3·4	4·6	5·5	7·0	8·0	8·7
8 00	2·1	2·3	3·2	3·9	5·3	6·6	8·0
9 00	2·7	2·2	2·5	2·8	3·9	5·0	6·7
10 00	4·1	3·0	2·5	2·5	3·0	3·8	5·3
11 00	6·0	4·5	3·7	3·0	2·6	3·0	4·0
Noon.	7·9	6·4	5·5	4·2	3·3	2·8	3·3
13 00	9·3	8·2	7·4	6·0	4·6	3·5	3·0
14 00	9·9	9·4	9·0	7·8	6·3	4·9	3·6
15 00	9·4	9·7	9·8	9·0	7·9	6·5	4·8
16 00	8·1	8·9	9·4	9·5	9·0	7·9	6·3
17 00	6·3	7·4	8·3	9·1	9·2	8·9	7·7
18 00	4·5	5·6	6·7	7·9	8·8	9·2	8·7
19 00	3·0	3·9	4·9	6·3	7·7	8·7	9·2
20 00	2·2	2·7	3·4	4·7	6·2	7·5	8·7
21 00	2·6	2·4	2·5	3·3	4·6	6·0	7·8
22 00	3·6	3·0	2·6	2·7	3·5	4·5	6·3
23 00	5·3	4·2	3·5	3·1	2·8	3·3	4·8

(4′) Before summing with the semidiurnal stencils, the hourly heights should have a constant added to them in order to avoid negative quantities; in the accompanying tabulation 6 feet has been added. The summation and analysis are then to be carried out as in the case of true hourly ordinates.

(5′) The values of the semidiurnal ordinates referred to mean water level are now known for each hour; they are therefore known for the times of the true high and low waters. Subtracting the appropriate semidiurnal ordinates, we have four heights per lunar day which lie upon the diurnal curve, very nearly.

(6′) The diurnal heights are then summed with the stencils, and analyzed for K_1 and O_1 in the usual way excepting that the augmenting factors 1·00287 and 1·00249 are to be used twice instead of once.

When the series is very short, and so the divisors generally quite unequal, it may be advisable to write each height of the tide twice, i. e., to regard it as the hourly ordinate immediately preceding and immediately following the time of tide.

Below is appended the results obtained from analyzing a series of tidal observations one month in extent. They show how the results obtained from high and low waters agree with the results obtained from regular hourly readings. Using the notation of §§ 59, 64, the uncorrected amplitudes and angles are really the R_c's and ζ_c's, since no corrections have been applied to them on account of the disturbing components.

SITKA, ALASKA, JULY, 1893.

Component.	From ordinates obtained from observed high and low waters. Uncorrected amplitude, R.	Uncorrected angle, ζ.	From observed hourly ordinates. Uncorrected amplitude, R.	Uncorrected angle, ζ.	Period analyzed.
	Feet.	°	Feet.	°	d. h.
S_2	0·897	49	0·873	53	29 13
μ_2	0·179	127	0·191	108	29 13
N_2	0·818	263	0·785	264	27 13
L_2	0·055	284	0·041	338	27 13
K_1	1·907	308	2·018	309	27 8
O_1	1·017	355	1·097	352	27 8

75. *Interpolation of hourly heights from tabulated high and low waters.*

The set of curves drawn upon a transparent sheet and described in the preceding paragraph, may be used for this purpose when the tide is wholly semidiurnal in its character.

The hourly heights of a tide of this kind may be computed by the formula,

$$\left.\begin{array}{l}\text{Depression below high water}\\ \text{or elevation above low water}\end{array}\right\} = \frac{r}{2}\text{ versed sine}\left(180^\circ\frac{t}{d}\right) = \frac{2\,r\frac{t}{d}}{0{\cdot}7+0{\cdot}6\frac{t}{d}},\text{ approximately,} \tag{496}$$

where d is the duration of rise or fall, expressed in minutes, r the corresponding range or amount of the same, and t the number of minutes from high or low water; $t \leqq d$.

If the diurnal inequality is considerable, any process of interpolation is rather laborious and not always accurate, even where the shallow water components are small, because the period of the diurnal wave is not generally exactly twice that of the semidiurnal.

The range of the diurnal wave is, approximately,

$$2\,\Delta_1 = \sqrt{(\text{HW ineq.})^2 + (\text{LW ineq.})^2}. \tag{497}$$

The high-water inequality is found by subtracting a given high-water height from the mean of the two adjacent high-water heights. Similarly for the inequality in the low waters. The range of the semidiurnal wave is, very nearly,

$$2\,\Delta_2 = \text{mean range for the day} - \frac{(\text{HW ineq.})^2 + (\text{LW ineq.})^2}{16 \times \text{mean range for the day}}. \tag{498}$$

The position of the maximum of the semidiurnal wave can be found from values like those given in the fifth and sixth columns of the second tabulation in the preceding paragraph.

These data, with Table 19, enable one to find the position of the diurnal wave with respect to the semidiurnal, also the value of Δ_1.

Rules for determining the quadrant of the HW phase of the diurnal wave, i. e., the angle or phase of the diurnal at the time of HW of the semidiurnal, are as follows:

$$\begin{array}{l}\text{Sequence HHW to LLW}\left\{\begin{array}{l}\text{For HHW, HW phase falls in 1st quadrant.}\\ \text{For LHW, HW phase falls in 3d quadrant.}\end{array}\right.\\ \text{Sequence LLW to HHW}\left\{\begin{array}{l}\text{For HHW, HW phase falls in 4th quadrant.}\\ \text{For LHW, HW phase falls in 2d quadrant.}\end{array}\right.\end{array} \tag{499}$$

The HW phase, when converted into time at the rate of 15° per hour, gives the time by which the HW of the diurnal wave precedes that of the semidiurnal. This angle should be taken between -180° and $+180^\circ$.

The sum of the hourly ordinates of the diurnal and semidiurnal waves give the hourly ordinates of the tide; the height of tide at any time t is

$$\Delta_2 \cos 29^\circ\,(t - t_2) + \Delta_1 \cos 15^\circ\,(t - t_1) \tag{500}$$

where t_2, t_1 denote the times of high water of the two waves.

76. *Remarks upon published results and tables.*

The effects of the motion of the moon's node upon the amplitudes and epochs of the components, as brought out by the harmonic analysis, were not allowed for in the earlier published results. Consequently, determinations from successive years were not *inter se* comparable. In this connection see the British Association Reports, 1878, p. 481, note, and 1883, p. 91.

Ferrel corrected the components M_2, K_2, K_1, and O_1 for such effects, but omitted the corrections for the other components similarly affected. His tables for this purpose are given upon page 304 of the Survey Report for 1878. It is to be noted that the numerical values of his $\varDelta\epsilon$'s for K_1, M_2, and K_2 have the wrong signs prefixed; he seems, however, to have always corrected this in making analyses.

Initial equilibrium arguments.—In the Survey Report for 1878, Ferrel uses c, and in the Report for 1883 he uses k, to denote V_0, or the uniformly varying portion of $V_0 + u$. The k's (or c's) of K_1, K_2, and λ_2, and perhaps others, are sometimes wrong by 90° or 180°. To ascertain this, compare the k's (or c's), Report for 1878, pages 270, 303, and for 1883, page 267, with the $V_0 + u$ of Table 3. A leap year is not so convenient as a common year for this purpose, because the k's then refer to the 2d instead of the 1st of January. The smaller discrepancies shown by such comparison are due to the u of Table 1. That is

$$V_0 + u - k = u = -\,\delta\varepsilon^* \text{ or } + \varDelta\epsilon,^\dagger \text{ very nearly.} \tag{501}$$

In this manual the origin of the day is taken as midnight. Consequently, unless otherwise stated, V_0 refers to midnight instead of noon as contemplated by the British Tidal Committee.

In Tables 1 and 3 the initial equilibrium argument of R_2 is in error by 180°.

E. Roberts has noted that the lower half of Table 8 has been incorrectly formed from the upper half. The tabular values will still hold good if for ($P =$) 95°, 100°, 105°, etc., there be substituted 175°, 170°, 165°, etc.

In using Table 31, enter columns $\frac{2,\ 6}{3,\ 7}$ as many days before the given date as there are degrees in $\frac{\frac{1}{2}\nu'}{\nu'}$; Tables 6, 7.

Enter Table 32 as many days before the given date as there are degrees in $\frac{1}{2}\nu'$; Tables 6, 7.
Enter Table 33 as many days before the given date as there are degrees in ν'; Tables 6, 7.

77. *Harmonic analysis of tidal currents.*

The height of the tide or the vertical displacement of the surface of the sea at a given point is usually assumed to have for its expression

$$\begin{aligned} y \text{ or } h = {} & M_2' \cos(m_2t + \arg_0 M_2 - M_2^\circ) + S_2 \cos(s_2t + \arg_0 S_2 - S_2^\circ) \\ & + N_2' \cos(n_2t + \arg_0 N_2 - N_2^\circ) + \ . \ . \ . \\ & + K_1' \cos(k_1t + \arg_0 K_1 - K_1^\circ) + O_1' \cos(o_1t + \arg_0 O_1 - O_1^\circ) \\ & + P_1' \cos(p_1t + \arg_0 P_1 - P_1^\circ) + \ . \ . \ . \end{aligned} \tag{502}$$

In a canal of indefinite length it would be reasonable (according to § 22, Part I) to assume the horizontal displacement (ξ) to be of the above form, save that sines take the place of cosines, and that the amplitudes are the above amplitudes ~~all multiplied by the same constant~~. The velocity of the current is $d\xi/dt$, an expression involving cosines instead of sines, and having the horizontal displacement amplitudes multiplied by the respective speeds. ~~This shows that~~ a diurnal component of the velocity is only about one-half as great as a semidiurnal when the two corresponding tidal components, or partial tides, are equal. On the other hand, the quarter diurnals and the sixth diurnals have a tendency to become more pronounced in the velocity. The theoretical ratio between two velocity amplitudes is not the same as the theoretical ratios between the two corresponding tidal coefficients, even though both partial tides are diurnal or both semidiurnal; but this ratio must be multiplied by the speed ratio.

each divided by a constant proportional to its speed

In certain straits (See §34, Part I)

*$\delta\varepsilon$ as used by Ferrel is the alteration in the epoch of a component which will adapt it to a particular year. It is tabulated for M_2, K_2, K_1, and O_1 upon page 268 of the Report for 1883.

† $\varDelta\epsilon$ is the correction in the observed epoch of a component, due to the motion of the moon's node. It is tabulated upon page 304 of the Report for 1878, as already stated.

It is here supposed that we have hourly observations upon the velocity and direction of the current extending over one or more months. These velocities can be resolved with reference to two fixed directions—say, north and east. The velocities may be denoted by the letters used to denote the partial tides, each with a dot written above each: e. g., $\dot{M}_2$, $\dot{S}_2$, $\dot{N}_2$. The resolved portions may be distinguished by the subscripts n, e, s, or w, according to the direction—north, east, south, or west—to which they refer. The resolution of the velocities into two portions is readily effected by aid of a circle drawn upon cross-section paper and divided into degrees, together with a scale whose zero is at the center of the circle and whose straight edge always falls upon a radius.

The summations and analyses are made in the same manner as for the partial tides, because the velocity in a given direction (north, say) is assumed to be written

$$
\begin{aligned}
v_n = {} & \dot{M}_{2_n}{}' \cos (m_2 t + \arg_0 M_2 - \dot{M}_{2_n}{}^\circ) + \dot{S}_{2_n} \cos (s_2 t + \arg_0 S_2 - \dot{S}_{2_n}{}^\circ) \\
& + \dot{N}_{2_n}{}' \cos (n_2 t + \arg_0 N_2 - \dot{N}_{2_n}{}^\circ) + \ . \ . \ . \\
& + \dot{K}_{1_n}{}' \cos (k_1 t + \arg_0 K_1 - \dot{K}_{1_n}{}^\circ) + \dot{O}_{1_n}{}' \cos (o_1 t + \arg_0 O_1 - \dot{O}_{1_n}{}^\circ) \\
& + \dot{P}_{1_n} \cos (p_1 t + \arg_0 P_1 - \dot{P}_{1_n}{}^\circ) \quad + \ . \ . \ . \ . \qquad (503)
\end{aligned}
$$

For the easterly portion of the velocity e simply replaces n.

Where the water follows a fixed channel

$$\dot{M}_2 = \sqrt{\dot{M}_{2_n}{}^2 + \dot{M}_{2_e}{}^2}. \qquad (504)$$

If a direction indicated by the subscript approximately coincides with the direction of flood, then for long tidal rivers we should expect $\dot{M}_2{}^\circ$ to be approximately equal to $M_2{}^\circ$; but for a small bay or harbor it should be nearer $M_2{}^\circ - 90^\circ$. If the direction indicated by the subscript approximately coincides with the direction of the ebb, $\dot{M}_2{}^\circ$ should be approximately equal to $M_2{}^\circ \pm 180^\circ$ for a long tidal river and $M_2{}^\circ + 90^\circ$ for a small bay or harbor. The same statements hold true when M_2 is replaced by S_2, N_2, K_1, etc.

In tidal rivers or in straits the currents have a nearly fixed line of motion, and so it is hardly necessary to decompose the velocities into two parts, as indicated above.

78. *To combine the various current components for any given future time.*

Combine, as in § 72, all partial north-and-south currents; likewise combine all partial east-and-west currents. With the two velocities thus obtained a rectangle of velocities can be constructed whose diagonal represents the direction and velocity of the total current.

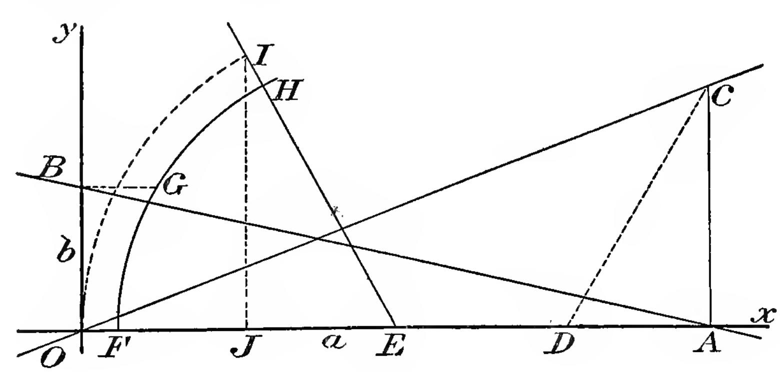


FIG. 12.

Prediction of currents.—Suppose all partial north-and-south currents to have been properly combined, and a curve drawn representing the result from hour to hour. A tide predictor which traces a continuous curve can be used for this purpose and so the labor of computation be avoided. Suppose the partial east-and-west currents to have been similarly combined and a

curve drawn. Let the two curves be placed side by side. At any given instant the direction and velocity of the total current become known upon constructing a rectangle of velocities by aid of the two curves.

The time of minimum velocity can be found either by trial or directly by the following process:

Upon glancing at the two curves the approximate time of minimum velocity can be found; it will generally lie near the point where the curve of greater amplitude crosses its axis. A straight line can be drawn closely coinciding with this curve for some distance either way from the assumed time of minimum velocity. Another straight line can be drawn nearly coinciding with the other curve for the time in question. The question then reduces to the simple geometrical problem (which has an application in § 36, Part III):

In a plane are given two straight lines referred to rectangular coördinates; it is required to find geometrically an abscissa such that the sum of the squares of the two corresponding ordinates shall be a minimum.

Suppose the equation

$$y = mx \tag{505}$$

to represent the line having the greater inclination to the x-axis and the equation

$$\frac{x}{a} + \frac{y}{b} = 1, \tag{506}$$

the other line.

When

$$x = \frac{b^2 a}{a^2 m^2 + b^2}, \tag{507}$$

the sum of the squares of the two y's becomes a minimum. The problem may now be stated: Given a, b, and m, to find x geometrically.

Let OC and AB denote the given lines; draw AC parallel to the y-axis and lay off $AD = BO = b$. Bisect AO, thus determining E; then with E as center and CD as radius, describe an arc FGH. Draw BG parallel to the x-axis; take $GH = FG$ and draw the line EHI. Take $EI = EO = \frac{1}{2}\, a$ and project I upon the x-axis in J: then is OJ the required abscissa.

79. *Rules governing the choice between Roman and Italic letters in tidal work.*

Considerable confusion having already arisen among writers upon tides in regard to the notation employed, it has been thought best to here state certain rules which have been generally followed in this manual.

Roman letters are used to denote—

1st. Quantities which are in themselves particular or definite tidal quantities at a given station. E. g., $\mathrm{M_2}$, $\mathrm{M^\circ_2}$, $\mathrm{S_2}$, $\mathrm{S_2{}^\circ}$, Mn, HWI.

2d. Definite quantities intimately connected or associated with those of the kind just referred to. E. g., $\mathrm{m_2}$, $\mathrm{s_2}$, meaning speeds; M, S, denoting particular series of lunar and solar tides.

Italic letters are used to denote—

1st. Quantities not tidal. E. g., S = the longitude in time of the time meridian; a = the earth's radius; r = the moon's distance; I, i, = inclinations of lunar orbit; V = potential.

2d. Indefinite tidal quantities; i. e., such as must be connected with definite tidal quantities before they have a meaning. E. g., H or R used for amplitude, $V_0 + u$, F, f.

3d. Temporary or general symbols whether tidal or not. E. g., X, Y, Z, x, y, z, A, B, C, a, b, c, A°, C_2, c_2.

AUXILIARY TABLES

FOR THE

REDUCTION AND PREDICTION

OF

TIDES.

[Tables 1 to 35 are appended to Part III, Appendix No. 7, Report for 1894.]

TABLE 36.—*Shallow-water components.*

[Terms from y'^2.]

SEMIDIURNAL COMPONENTS.

Designation of component.		Primitive amplitude.	Speed.		Argument.	Primitive epoch.
(K_1K_1)	K_2	$\frac{1}{2}K_1^2$	$k_1+k_1=k_2$	30.0821372	2 arg K_1	$2K_1°$
(K_1O_1)	M_2	K_1O_1	$k_1+o_1=m_2$	28.9841042	arg K_1 + arg O_1	$K_1°+O_1°$
(K_1P_1)	S_2	K_1P_1	$k_1+p_1=s_2$	30.0000000	arg K_1 + arg P_1	$K_1°+P_1°$
(O_1O_1)	O_2	$\frac{1}{2}O_1^2$	$o_1+o_1=o_2$	27.8860712	2 arg O_1	$2O_1°$
(O_1P_1)		O_1P_1	o_1+p_1	28.9019670	arg O_1 + arg P_1	$O_1°+P_1°$
(P_1P_1)	P_2	$\frac{1}{2}P_1^2$	$p_1+p_1=p_2$	29.9178628	2 arg P_1	$2P_1°$
COMPONENTS OF LONG PERIOD.						
$(K_1\sim K_1)$		$\frac{1}{2}K_1^2$	$k_1-k_1=0$	0	0	0
$(K_1\sim O_1)$	Mf	K_1O_1	$k_1-o_1=mf$	1.0980330	arg K_1 − arg O_1	$K_1°-O_1°$
$(K_1\sim P_1)$	Ssa	K_1P_1	$k_1-p_1=ssa$	0.0821372	arg K_1 − arg P_1	$K_1°-P_1°$
$(O_1\sim O_1)$		$\frac{1}{2}O_1^2$	$o_1-o_1=0$	0	0	0
$(O_1\sim P_1)$	MSf	O_1P_1	$p_1-o_1=msf$	1.0158958	arg P_1 − arg O_1	$P_1°-O_1°$
$(P_1\sim P_1)$		$\frac{1}{2}P_1^2$	$p_1-p_1=0$	0	0	0
TERDIURNAL COMPONENTS.						
(M_2K_1)	MK	M_2K_1	$m_2+k_1=mk$	44.0251728	arg M_2 + arg K_1	$M_2°+K_1°$
(M_2O_1)	2MK	M_2O_1	m_2+o_1	42.9271398	arg M_2 + arg O_1	$M_2°+O_1°$
(M_2P_1)		M_2P_1	m_2+p_1	43.9430356	arg M_2 + arg P_1	$M_2°+P_1°$
(S_2K_1)		S_2K_1	s_2+k_1	45.0410686	arg S_2 + arg K_1	$S_2°+K_1°$
(S_2O_1)		S_2O_1	s_2+o_1	43.9430356	arg S_2 + arg O_1	$S_2°+O_1°$
(S_2P_1)		S_2P_1	s_2+p_1	44.9589314	arg S_2 + arg P_1	$S_2°+P_1°$
(N_2K_1)		N_2K_1	n_2+k_1	43.4807982	arg N_2 + arg K_1	$N_2°+K_1°$
(N_2O_1)		N_2O_1	n_2+o_1	42.3827652	arg N_2 + arg O_1	$N_2°+O_1°$
(N_2P_1)		N_2P_1	n_2+p_1	43.3986610	arg N_2 + arg P_1	$N_2°+P_1°$
(K_2K_1)	K_3	K_2K_1	$3k_1$	45.1232058	3 arg K_1	$3K_1°$
(K_2O_1)	MK	K_2O_1	$k_2+o_1=mk$	44.0251728	arg K_2 + arg O_1	$K_2°+O_1°$
(K_2P_1)		K_2P_1	k_2+p_1	45.0410686	arg K_2 + arg P_1	$K_2°+P_1°$
(L_2K_1)		L_2K_1	l_2+k_1	44.5695474	arg L_2 + arg K_1	$L_2°+K_1°$
(L_2O_1)		L_2O_1	l_2+o_1	43.4715144	arg L_2 + arg O_1	$L_2°+O_1°$
(L_2P_1)		L_2P_1	l_2+p_1	44.4874102	arg L_2 + arg P_1	$L_2°+P_1°$
DIURNAL COMPONENTS.						
$(M_2\sim K_1)$	O_1	M_2K_1	$m_2-k_1=o_1$	13.9430356	arg M_2 − arg K_1	$M_2°-K_1°$
$(M_2\sim O_1)$	K_1	M_2O_1	$m_2-o_1=k_1$	15.0410686	arg M_2 − arg O_1	$M_2°-O_1°$
$(M_2\sim P_1)$		M_2P_1	m_2-p_1	14.0251728	arg M_2 − arg P_1	$M_2°-P_1°$
$(S_2\sim K_1)$	P_1	S_2K_1	$s_2-k_1=p_1$	14.9589314	arg S_2 − arg K_1	$S_2°-K_1°$
$(S_2\sim O_1)$		S_2O_1	s_2-o_1	16.0569644	arg S_2 − arg O_1	$S_2°-O_1°$
$(S_2\sim P_1)$	K_1	S_2P_1	$s_2-p_1=k_1$	15.0410686	arg S_2 − arg P_1	$S_2°-P_1°$
$(N_2\sim K_1)$	Q_1	N_2K_1	$n_2-k_1=q_1$	13.3986610	arg N_2 − arg K_1	$N_2°-K_1°$
$(N_2\sim O_1)$	$[M_1]$	N_2O_1	$n_2-o_1=[m_1]$	14.4966940	arg N_2 − arg O_1	$N_2°-O_1°$
$(N_2\sim P_1)$		N_2P_1	n_2-p_1	13.4807982	arg N_2 − arg P_1	$N_2°-P_1°$
$(K_2\sim K_1)$	K_1	K_2K_1	$k_2-k_1=k_1$	15.0410686	arg K_2 − arg K_1	$K_2°-K_1°$
$(K_2\sim O_1)$		K_2O_1	k_2-o_1	16.1391016	arg K_2 − arg O_1	$K_2°-O_1°$
$(K_2\sim P_1)$		K_2P_1	k_2-p_1	15.1232058	arg K_2 − arg P_1	$K_2°-P_1°$
$(L_2\sim K_1)$		L_2K_1	l_2-k_1	14.4874102	arg L_2 − arg K_1	$L_2°-K_1°$
$(L_2\sim O_1)$	J_1	L_2O_1	$l_2-o_1=j_1$	15.5854432	arg L_2 − arg O_1	$L_2°-O_1°$
$(L_2\sim P_1)$		L_2P_1	l_2-p_1	14.5695474	arg L_2 − arg P_1	$L_2°-P_1°$

For a description of this table, see § 48, Part II.

For sake of clearness we have supposed (AB) to denote a component whose speed is $a+b$, and $(A\sim B)$ a component whose speed is $a\sim b$.

TABLE 36.—*Shallow-water components*—Continued.

[Terms from y'^2.]

QUARTER-DIURNAL COMPONENTS.

Designation of component.		Primitive amplitude.	Speed.		Argument.	Primitive epoch.
(M_2M_2)	M_4	$\frac{1}{2} M_2^2$	$m_2 + m_2 = m_4$	57·9682084	2 arg M_2	2 M_2°
(M_2S_2)	MS	M_2S_2	$m_2 + s_2$	58·9841042	arg M_2 + arg S_2	$M_2^\circ + S_2^\circ$
(M_2N_2)	MN	M_2N_2	$m_2 + n_2 = mn$	57·4238338	arg M_2 + arg N_2	$M_2^\circ + N_2^\circ$
(M_2K_2)		M_2K_2	$m_2 + k_2$	59·0662414	arg M_2 + arg K_2	$M_2^\circ + K_2^\circ$
(M_2L_2)		M_2L_2	$m_2 + l_2$	58·5125830	arg M_2 + arg L_2	$M_2^\circ + L_2^\circ$
(S_2S_2)	S_4	$\frac{1}{2} S_2^2$	$s_2 + s_2 = s_4$	60·0000000	2 arg S_2	2 S_2°
(S_2N_2)		S_2N_2	$s_2 + n_2$	58·4397296	arg S_2 + arg N_2	$S_2^\circ + N_2^\circ$
(S_2K_2)	R_4	S_2K_2	$s_2 + k_2 = r_4$	60·0821372	arg S_2 + arg K_2	$S_2^\circ + K_2^\circ$
(S_2L_2)		S_2L_2	$s_2 + l_2$	59·5284788	arg S_2 + arg L_2	$S_2^\circ + L_2^\circ$
(N_2N_2)	N_4	$\frac{1}{2} N_2^2$	$n_2 + n_2 = n_4$	56·8794592	2 arg N_2	2 N_2°
(N_2K_2)		N_2K_2	$n_2 + k_2$	58·5218668	arg N_2 + arg K_2	$N_2^\circ + K_2^\circ$
(N_2L_2)		N_2L_2	$n_2 + l_2$	57·9682084	arg N_2 + arg L_2	$N_2^\circ + L_2^\circ$
(K_2K_2)	K_4	$\frac{1}{2} K_2^2$	$k_2 + k_2 = k_4$	60·1642744	2 arg K_2	2 K_2°
(K_2L_2)		K_2L_2	$k_2 + l_2$	59·6106160	arg K_2 + arg L_2	$K_2^\circ + L_2^\circ$
(L_2L_2)	L_4	$\frac{1}{2} L_2^2$	$l_2 + l_2 = l_4$	59·0569576	2 arg L_2	$2_2L_2^\circ$

COMPONENTS OF LONG PERIOD.

$(M_2 \sim M_2)$		$\frac{1}{2} M_2^2$	$m_2 - m_2 = 0$	0	0	0
$(M_2 \sim S_2)$	MSf	M_2S_2	$s_2 - m_2 = msf$	1·0158958	arg S_2 − arg M_2	$S_2^\circ - M_2^\circ$
$(M_2 \sim N_2)$	Mm	M_2N_2	$m_2 - n_2 = mm$	0·5443746	arg M_2 − arg N_2	$M_2^\circ - N_2^\circ$
$(M_2 \sim K_2)$	Mf	M_2K_2	$m_2 - k_2 = mf$	1·0980330	arg M_2 − arg K_2	$M_2^\circ - K_2^\circ$
$(M_2 \sim L_2)$	Mm	M_2L_2	$l_2 - m_2 = mm$	0·5443746	arg L_2 − arg M_2	$L_2^\circ - M_2^\circ$
$(S_2 \sim S_2)$		$\frac{1}{2} S_2^2$	$s_2 - s_2 = 0$	0	0	0
$(S_2 \sim N_2)$		S_2N_2	$s_2 - n_2$	1·5602704	arg S_2 − arg N_2	$S_2^\circ - N_2^\circ$
$(S_2 \sim K_2)$	Ssa	S_2K_2	$k_2 - s_2 = ssa$	0·0821372	arg K_2 − arg S_2	$K_2^\circ - S_2^\circ$
$(S_2 \sim L_2)$		S_2L_2	$s_2 - l_2$	0·4715212	arg S_2 − arg L_2	$S_2^\circ - L_2^\circ$
$(N_2 \sim N_2)$		$\frac{1}{2} N_2^2$	$n_2 - n_2 = 0$	0	0	0
$(N_2 \sim K_2)$		N_2K_2	$k_2 - n_2$	1·6424076	arg K_2 − arg N_2	$K_2^\circ - N_2^\circ$
$(N_2 \sim L_2)$		K_2L_2	$l_2 - n_2$	1·0887492	arg L_2 − arg N_2	$L_2^\circ - N_2^\circ$
$(K_2 \sim K_2)$		$\frac{1}{2} K_2^2$	$k_2 - k_2 = 0$	0	0	0
$(K_2 \sim L_2)$		K_2L_2	$k_2 - l_2$	0·5536584	arg K_2 − arg L_2	$K_2^\circ - L_2^\circ$
$(L_2 \sim L_2)$		$\frac{1}{2} L_2^2$	$l_2 - l_2 = 0$	0	0	0

TABLE 36.—*Shallow-water components*—Continued.

[Terms from y'^3 or $y' \times y'^2$.]

ONE-SIXTH-DIURNAL COMPONENTS.

Designation of component.		Primitive amplitude.	Speed.		Argument.	Primitive epoch.
$(M_2M_2M_2)$	M_6	$\frac{1}{2}M_2^3$	$3\ m_2 = m_6$	86·9523126	3 arg M_2	$3\ M_2°$
$(M_2M_2S_2)$		$M_2^2S_2$	$2\ m_2 + s_2$	87·9682084	2 arg M_2 + arg S_2	$2\ M_2° + S_2°$
$(M_2M_2N_2)$		$M_2^2N_2$	$2\ m_2 + n_2$	86·4079380	2 arg M_2 + arg N_2	$2\ M_2° + N_2°$
$(M_2M_2K_2)$		$M_2^2K_2$	$2\ m_2 + k_2$	88·0503456	2 arg M_2 + arg K_2	$2\ M_2° + K_2°$
$(M_2M_2L_2)$		$M_2^2L_2$	$2\ m_2 + l_2$	87·4966872	2 arg M_2 + arg L_2	$2\ M_2° + L_2°$
$(M_2S_2S_2)$		$\frac{1}{2}\ M_2S_2^2$	$2\ s_2 + m_2$	88·9841042	2 arg S_2 + arg M_2	$M_2' + 2\ S_2°$
$(M_2S_2N_2)$		$M_2S_2N_2$	$m_2 + s_2 + n_2$	87·4238338	arg M_2 + arg S_2 + arg N_2	$M_2° + S_2° + N_2°$
$(M_2S_2K_2)$		$M_2S_2K_2$	$m_2 + s_2 + k_2$	89·0662414	arg M_2 + arg S_2 + arg K_2	$M_2° + S_2° + K_2°$
$(M_2S_2L_2)$		$M_2S_2L_2$	$m_2 + s_2 + l_2$	88·5125830	arg M_2 + arg S_2 + arg L_2	$M_2° + S_2° + L_2°$
$(S_2M_2M_2)$		$\frac{1}{2}S_2M_2^2$	$2\ m_2 + s_2$	87·9682084	2 arg M_2 + arg S_2	$2\ M_2° + S_2°$
$(S_2M_2S_2)$		$M_2S_2^2$	$m_2 + 2\ s_2$	88·9841042	arg M_2 + 2 arg S_2	$M_2° + 2\ S_2°$
$(S_2M_2N_2)$		$M_2S_2N_2$	$m_2 + s_2 + n_2$	87·4238338	arg M_2 + arg S_2 + arg N_2	$M_2° + S_2° + N_2°$
$(S_2M_2K_2)$		$M_2S_2K_2$	$m_2 + s_2 + k_2$	89·0662414	arg M_2 + arg S_2 + arg K_2	$M_2° + S_2° + K_2°$
$(S_2M_2L_2)$		$M_2S^2L_2$	$m_2 + s_2 + l_2$	88·5125830	arg M_2 + arg S_2 + arg L_2	$M_2° + S_2° + L_2°$
$(S_2S_2S_2)$	S_6	$\frac{1}{2}S_2^3$	$3\ s_2 = s_6$	90·0000000	3 arg S_2	$3\ S_2°$
$(S_2S_2N_2)$		$S_2^2N_2$	$2\ s_2 + n_2$	88·4397296	2 arg S_2 + arg N_2	$2\ S_2° + N_2°$
$(S_2S_2K_2)$		$S_2^2K_2$	$2\ s_2 + k_2$	90·0821372	2 arg S_2 + arg K_2	$2\ S_2° + K_2°$
$(S_2S_2L_2)$		$S_2^2L_2$	$2\ s_2 + l_2$	89·5284788	2 arg S_2 + arg L_2	$2\ S_2° + L_2°$

SEMIDIURNAL COMPONENTS.

$(M_2 \sim M_2M_2)$	M_2	$\frac{1}{2}M_2^3$	m_2	28·9841042	arg M_2	$M_2°$
$(M_2 \sim M_2S_2)$	S_2	$M_2^2S_2$	s_2	30·0000000	arg S_2	$S_2°$
$(M_2 \sim M_2N_2)$	N_2	$M_2^2N_2$	n_2	28·4397296	arg N_2	$N_2°$
$(M_2 \sim M_2K_2)$	K_2	$M_2^2K_2$	k_2	30·0821372	arg K_2	$K_2°$
$(M_2 \sim M_2L_2)$	L_2	$M_2^2L_2$	l_2	29·5284788	arg L_2	$L_2°$
$(M_2 \sim S_2S_2)$	2 SM	$\frac{1}{2}M_2S_2^2$	$2\ s_2 - m_2$	31·0158958	2 arg S_2 − arg M_2	$2\ S_2° - M_2°$
$(M_2 \sim S_2N_2)$	λ_2	$M_2S_2N_2$	$s_2 + n_2 - m_2 = \lambda_2$	29·4556254	arg S_2 + arg N_2 − arg M_2	$S_2° + N_2° - M_2°$
$(M_2 \sim S_2K_2)$		$M_2S_2K_2$	$s_2 + k_2 - m_2$	31·0980330	arg S_2 + arg K_2 − arg M_2	$S_2° + K_2° - M_2°$
$(M_2 \sim S_2L_2)$		$M_2S_2L_2$	$s_2 + l_2 - m_2$	30·5443746	arg S_2 + arg L_2 − arg M_2	$S_2° + L_2° - M_2°$
$(S_2 \sim M_2M_2)$	2 MS	$\frac{1}{2}S_2M_2^2$	$2\ m_2 - s_2 = \mu_2$	27·9682084	2 arg M_2 − arg S_2	$2\ M_2° - S_2°$
$(S_2 \sim M_2S_2)$		M_2S_2	m_2	28·9841042	arg M_2	$M_2°$
$(S_2 \sim M_2N_2)$		$M_2S_2N_2$	$m_2 + n_2 - s_2$	27·4238338	arg M_2 + arg N_2 − arg S_2	$M_2° + N_2° - S_2°$
$(S_2 \sim M_2K_2)$		$M_2S_2K_2$	$m_2 + k_2 - s_2$	29·0662414	arg M_2 + arg K_2 − arg S_2	$M_2° + K_2° - S_2°$
$(S_2 \sim M_2L_2)$	ν_2	$M_2S_2L_2$	$m_2 + l_2 - s_2 = \nu_2$	28·5125830	arg M_2 + arg K_2 − arg S_2	$M_2° + L_2° - S_2°$
$(S_2 \sim S_2S_2)$	S_2	$\frac{1}{2}S_2^3$	s_2	30·0000000	arg S_2	$S_2°$
$(S_2 \sim S_2N_2)$	N_2	$S_2^2N_2$	n_2	28·4397296	arg N_2	$N_2°$
$(S_2 \sim S_2K_2)$	K_2	$S_2^2K_2$	k_2	30·0821372	arg K_2	$K_2°$
$(S_2 \sim S_2L_2)$	L_2	$S_2^2L_2$	l_2	29·5284788	arg L_2	$L_2°$

[Terms from y'^4 or $y' \times y'^3$.]

ONE-EIGHTH-DIURNAL COMPONENTS.

(M_2M_6)	M_8	$\frac{1}{2}\ M_2^4$	$4\ m_2 = m_8$	115·9364168	4 arg M_2	$4\ M_2°$
(M_2S_6)		$\frac{1}{2}\ M_2S_2^3$	$3\ s_2 + m_2$	118·9841042	3 arg S_2 + arg M_2	$3\ S_2° + M_2°$
(S_2M_6)		$\frac{1}{2}\ S_2M_2^3$	$3\ m_2 + s_2$	116·9523126	3 arg M_2 + arg S_2	$3\ M_2° + S_2°$
(S_2S_6)	S_8	$\frac{1}{2}\ S_2^4$	$4\ s_2 = s_8$	120·0000000	4 arg S_2	$4\ S_2°$

QUARTER-DIURNAL COMPONENTS.

$(M_2 \sim M_6)$	M_4	$\frac{1}{2}\ M_2^4$	$2\ m_2 = m_4$	57·9682084	2 arg M_2	$2\ M_2°$
$(M_2 \sim S_6)$		$\frac{1}{2}\ M_2S_2^3$	$3\ s_2 - m_2$	61·0158958	3 arg S_2 − arg M_2	$3\ S_2° - M_2°$
$(S_2 \sim M_6)$		$\frac{1}{2}\ S_2M_2^3$	$3\ m_2 - s_2$	56·9523126	3 arg M_2 − arg S_2	$3\ M_2° - S_2°$
$(S_2 \sim S_6)$	S_4	$\frac{1}{2}\ S_2^4$	$2\ s_2 = s_4$	60·0000000	2 arg S_2	$2\ S_2°$

TABLE 37.—*The theoretical amplitudes of some of the more important components for every 5 degrees of latitude.*

λ	$\cos^2\lambda$	$\sin 2\lambda$	$\frac{1}{2}-\frac{3}{2}\sin^2\lambda$	M_2	N_2	S_2	K_1	O_1	P_1	Mf
°				*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*
+90	0·0000	0·0000	−1·0000	0·000	0·000	0·000	0·000	0·000	0·000	−0·138
+85	0·0076	0·1736	−0·9886	0·006	0·001	0·003	+0·081	+0·058	+0·027	−0·136
+80	0·0301	0·3420	−0·9548	0·024	0·005	0·011	+0·160	+0·114	+0·053	−0·132
+75	0·0670	0·5000	−0·8995	0·054	0·010	0·025	+0·233	+0·166	+0·077	−0·124
+70	0·1170	0·6428	−0·8245	0·094	0·018	0·044	+0·300	+0·213	+0·099	−0·114
+65	0·1786	0·7660	−0·7321	0·143	0·028	0·066	+0·358	+0·254	+0·118	−0·101
+60	0·2500	0·8660	−0·6250	0·200	0·039	0·093	+0·404	+0·287	+0·134	−0·086
+55	0·3290	0·9397	−0·5065	0·263	0·051	0·122	+0·439	+0·312	+0·145	−0·070
+50	0·4132	0·9848	−0·3802	0·330	0·064	0·154	+0·460	+0·327	+0·152	−0·052
+45	0·5000	1·0000	−0·2500	0·400	0·077	0·186	+0·467	+0·332	+0·154	−0·034
+40	0·5868	0·9848	−0·1198	0·469	0·091	0·218	+0·460	+0·327	+0·152	−0·017
+35	0·6711	0·9397	+0·0066	0·537	0·104	0·250	+0·439	+0·312	+0·145	+0·001
+30	0·7500	0·8660	+0·1250	0·600	0·116	0·279	+0·404	+0·287	+0·134	+0·017
+25	0·8214	0·7660	+0·2321	0·657	0·127	0·306	+0·358	+0·254	+0·118	+0·032
+20	0·8830	0·6428	+0·3245	0·706	0·137	0·329	+0·300	+0·213	+0·099	+0·045
+15	0·9330	0·5000	+0·3995	0·746	0·145	0·347	+0·233	+0·166	+0·077	+0·055
+10	0·9698	0·3420	+0·4547	0·776	0·150	0·361	+0·160	+0·114	+0·053	+0·063
+ 5	0·9924	0·1736	+0·4886	0·794	0·154	0·369	+0·081	+0·058	+0·027	+0·067
0	1·0000	0·0000	+0·5000	0·800	0·155	0·372	+0·000	+0·000	+0·000	+0·069
− 5	0·9924	−0·1736	+0·4886	0·794	0·154	0·369	−0·081	−0·058	−0·027	+0·067
−10	0·9698	−0·3420	+0·4547	0·776	0·150	0·361	−0·160	−0·114	−0·053	+0·063
−15	0·9330	−0·5000	+0·3995	0·746	0·145	0·347	−0·233	−0·166	−0·077	+0·055
−20	0·8830	−0·6428	+0·3245	0·706	0·137	0·329	−0·300	−0·213	−0·099	+0·045
−25	0·8214	−0·7660	+0·2321	0·657	0·127	0·306	−0·358	−0·254	−0·118	+0·032
−30	0·7500	−0·8660	+0·1250	0·600	0·116	0·279	−0·404	−0·287	−0·134	+0·017
−35	0·6711	−0·9397	+0·0066	0·537	0·104	0·250	−0·439	−0·312	−0·145	+0·001
−40	0·5868	−0·9848	−0·1198	0·469	0·091	0·218	−0·460	−0·327	−0·152	−0·017
−45	0·5000	−1·0000	−0·2500	0·400	0·077	0·186	−0·467	−0·332	−0·154	−0·034
−50	0.4132	−0·9848	−0·3802	0·330	0·064	0·154	−0·460	−0·327	−0·152	−0·052
−55	0·3290	−0·9397	−0·5065	0·263	0·051	0·122	−0·439	−0·312	−0·145	−0·070
−60	0·2500	−0·8660	−0·6250	0.200	0·039	0·093	−0·404	−0·287	−0·134	−0·086
−65	0·1786	−0·7660	−0·7321	0·143	0·028	0·066	−0·358	−0·254	−0·118	−0·101
−70	0·1170	−0·6428	−0·8245	0·094	0·018	0·044	−0·300	−0·213	−0·099	−0·114
−75	0·0670	−0·5000	−0·8995	0·054	0·010	0·025	−0·233	−0·166	−0·077	−0·124
−80	0·0301	−0·3420	−0·9548	0·024	0·005	0·011	−0·160	−0·114	−0·053	−0·132
−85	0·0076	−0·1736	−0·9886	0·006	0·001	0·003	−0·081	−0·058	−0·027	−0·136
−90	0·0000	0·0000	−1·0000	0·000	0·000	0·000	0·000	0·000	0·000	−0·138

Tabular value $=\left[\frac{3}{2}\frac{M}{E}\left(\frac{a}{c}\right)^3 a = 1·760\right]\times$ latitude factor $\times$ coefficient. The latitude factor is given in column 2, 3, or 4; the coefficient in Table 1. For this table it is assumed that $\frac{M}{E}=\frac{1}{81·07}$, $\frac{a}{c}=\frac{1}{60·34}$, $a = 20\,902\,000$ feet, according to Harkness, Solar Parallax, pages 138, 140, using a mean radius of the earth instead of the equatorial radius. The negative amplitude signifies that the phase of the tide is altered by 180°. The north latitude is +, the south −.

TABLE 38.—*Augmenting factors.*

Components.		Subscript. 1	2	3	4	5	6	7	8
S		1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
		Group covers one solar hour.							
J	2 SM	1·00307 0·001331	1·01231 0·005313						
K	P, R, T	1·00287 0·001246	1·01158 0·004998	1·02632 0·011281	1·04746 0·020138				
L	λ, MS	1·00273 0·001196	1·01116 0·004819	1·02534 0·010868	1·04568 0·019400				
M		1·00266 0·001153	1·01075 0·004644	1·02440 0·010470	1·04396 0·018683	1·06989 0·029339	1·10283 0·042507	1·14363 0·058286	1·19343 0·076797
N	ν	1·00256 0·001111	1·01033 0·004464						
O	2 N, μ	1·00249 0·001081	1·00994 0·004295	1·02256 0·009691	1·04300 0·018285				
OO		1·00333 0·001442							
Q	ρ	1·00227 0·000983							
2 Q		1·00209 0·000906							
MN	2 MK	1·00261 0·001132	1·01055 0·004557	1·02394 0·010274	1·04311 0·018331				
MK		1·00274 0·001189	1·01102 0·004760	1·02503 0·010739					
		Group covers one component hour.							
All		1·00286 0·001240	1·01152 0·004974	1·02617 0·011219	1·04720 0·020030	1·07513 0·031461	1·11072 0·045605	1·15496 0·062571	1·20920 0·082498

The tabular value for any component other than S is

$$\frac{\text{arc } c\,\tau}{\text{chord } c\,\tau}$$

where τ = the length of the group. It is a solar hour when each component hour receives one, and only one, hourly height; it may be regarded as a component hour when all hourly heights are used in the summation. (See § 57, Part II.)

Tides of long period.

When all daily means are used, the factors given under the heading "Group covers one component hour" are to be applied to the long-period tides, the subscripts referring to the year or month, instead of the day as in the case of tides of short period. When attention is paid to the arrows, Table 43, in making the summations, the augmenting factors due to using solar instead of component time, are given by the above formula by putting τ = one solar day, and c = mf, msf, mm. The results are: 1.00887 (log. = 0.003835), 1.00759 (log. = 0.003282), 1.00217 (log. = 0.000941). In case of any long-period tide there is, besides the augmenting factor proper, what might be called a group factor, due to using the mean of 24 heights each day. The numerical values just given are also the group factors for Mf, MSf, and Mm.

TABLE 39.—*Values of b—a and of 24×(b—a).*

DIURNALS.

A	*B*									
	J_1	K_1	M_1	O_1	OO	P_1	Q_1	2Q	S_1	ρ_1
J_1	0 0	− 0·5443747 −13·064993	− 1·0933912 −26·241389	− 1·6424077 −39·417785	+ 0·5536583 +13·287799	− 0·6265119 −15·036286	− 2·1867824 −52·482778	− 2·7311571 −65·547770	− 0·5854433 −14·050639	2·1139289 - 50·734294
K_1	+ 0·5443747 +13·064993	0 0	− 0·5490165 −13·176396	− 1·0980330 −26·352792	+ 1·0980330 +26·352792	− 0·0821372 − 1·971293	− 1·6424077 −39·417785	− 2·1867824 −52·482778	− 0·0410686 − 0·985646	− 1·5695542 − 37·669301
M_1	+ 1·0933912 +26·241389	+ 0·5490165 +13·176396	0 0	− 0·5490165 −13·176396	+ 1·6470495 +39·529188	+ 0·4668793 +11·205103	− 1·0933912 −26·241389	− 1·6377659 −39·306382	+ 0·5079479 +12·190750	− 1·0205377 −24·492905
O_1	+ 1·6424077 +39·417785	+ 1·0980330 +26·352792	+ 0·5490165 +13·176396	0 0	+ 2·1960660 +52·705584	+ 1·0158958 +24·381499	− 0·5443747 −13·064993	− 1·0887494 −26·129986	+ 1·0569644 +25·367146	− 0·4715212 −11·316509
OO	− 0·5536583 −13·287799	− 1·0980330 −26·352792	− 1·6470495 −39·529188	− 2·1960660 −52·705584	0 0	− 1·1801702 −28·324085	− 2·7404407 −65·770577	− 3·2848154 −78·835570	− 1·1391016 −27·338438	− 2·6675872 --64·022093
P_1	+ 0·6265119 +15·036286	+ 0·0821372 + 1·971293	− 0·4668793 −11·205103	− 1·0158958 −24·381499	+ 1·1801702 +28·324085	0 0	− 1·5602705 −37·446492	− 2·1046452 −50·511485	+ 0·0410686 + 0·985646	− 1·4874170 −35·698008
Q_1	+ 2·1867824 +52·482778	+ 1·6424077 +39·417785	+ 1·0933912 +26·241389	+ 0·5443747 +13·064993	+ 2·7404407 +65·770577	+ 1·5602705 +37·446492	0 0	− 0·5443747 −13·064993	+ 1·6013391 +38·432138	+ 0·0728535 + 1·748484
2Q	+ 2·7311571 +65·547770	+ 2·1867824 +52·482778	+ 1·6377659 +39·306382	+ 1·0887494 +26·129986	+ 3·2848154 +78·835570	+ 2·1046452 +50·511485	+ 0·5443747 +13·064993	0 0	+ 2·1457138 +51·497131	+ 0·6172282 +14·813477
S_1	+ 0·5854433 +14·050639	+ 0·0410686 + 0·985646	− 0·5079479 −12·190750	− 1·0569644 −25·367146	+ 1·1391016 +27·338438	− 0·0410686 − 0·985646	− 1·6013391 −38·432138	− 2·1457138 −51·497131	0 0	− 1·5284856 −36·683654
ρ_1	+ 2·1139289 +50·734294	+ 1·5695542 +37·669301	+ 1·0205377 +24·492905	+ 0·4715212 +11·316509	+ 2·6675872 +64·022093	+ 1·4874170 +35·698008	− 0·0728535 − 1·748484	− 0·6172282 −14·813477	+ 1·5284856 +36·683654	0 0

TABLE 39.—*Values of b—a and of 24×(b—a)*—Continued.

SEMIDIURNALS.

A	*B*											
	K_2	L_2	M_2	N_2	2N	R_2	S_2	T_2	λ_2	μ_2	ν_2	2SM
K_2	0 0	− 0·5536584 −13·287802	− 1·0980330 −26·352792	− 1·6424076 −39·417782	− 2·1867824 −52·482778	− 0·0410686 − 0·985646	− 0·0821372 − 1·971293	− 0·1232058 − 2·956939	− 0·6265118 −15·036283	− 2·1139288 −50·734291	− 1·5695542 −37·669301	+ 0·9337586 +22·410206
L_2	+ 0·5536584 +13·287802	0 0	− 0·5443746 −13·064990	− 1·0887492 −26·129981	− 1·6331240 −39·194976	+ 0·5125898 +12·302155	+ 0·4715212 +11·316509	+ 0·4304526 +10·330862	− 0·0728534 − 1·748482	− 1·5602704 −37·446490	− 1·0158958 −24·381499	+ 1·4874170 +35·698008
M_2	+ 1·0980330 +26·352792	+ 0·5443746 +13·064990	0 0	− 0·5443746 −13·064990	− 1·0887494 −26·129986	+ 1·0569644 +25·367146	+ 1·0158958 +24·381499	+ 0·9748272 +23·395853	+ 0·4715212 +11·316509	− 1·0158958 −24·381499	− 0·4715212 −11·316509	+ 2·0317916 +48·762998
N_2	+ 1·6424076 +39·417782	+ 1·0887492 +26·129981	+ 0·5443746 +13·064990	0 0	− 0·5443748 −13·064995	+ 1·6013390 +38·432136	+ 1·5602704 +37·446490	+ 1·5192018 +36·460843	+ 1·0158958 +24·381499	− 0·4715212 −11·316509	+ 0·0728534 + 1·748482	+ 2·5761662 +61·827989
2N	+ 2·1867824 +52·482778	+ 1·6331240 +39·194976	+ 1·0887494 +26·129986	+ 0·5443748 +13·064995	0 0	+ 2·1457138 +51·497131	+ 2·1046452 +50·511485	+ 2·0635766 +49·525838	+ 1·5602706 +37·446494	+ 0·0728536 + 1·748486	+ 0·6172282 +14·813477	+ 3·1205410 +74·892984
R_2	+ 0·0410686 + 0·985646	− 0·5125898 −12·302155	− 1·0569644 −25·367146	− 1·6013390 −38·432136	− 2·1457138 −51·497131	0 0	− 0·0410686 − 0·985646	− 0·0821372 − 1·971293	− 0·5854432 −14·050637	− 2·0728602 −49·748645	− 1·5284856 −36·683654	+ 0·9748272 +23·395853
S_2	+ 0·0821372 + 1·971293	− 0·4715212 −11·316509	− 1·0158958 −24·381499	− 1·5602704 −37·446490	− 2·1046452 −50·511485	+ 0·0410686 + 0·985646	0 0	− 0·0410686 − 0·985646	− 0·5443746 −13·064990	− 2·0317916 −48·762998	− 1·4874170 −35·698008	+ 1·0158958 +24·381499
T_2	+ 0·1232058 + 2·956939	− 0·4304526 −10·330862	− 0·9748272 −23·395853	− 1·5192018 −36·460843	− 2·0635766 −49·525838	+ 0·0821372 + 1·971293	+ 0·0410686 + 0·985646	0 0	− 0·5033060 −12·079344	− 1·9907230 −47·777352	− 1·4463484 −34·712362	+ 1·0569644 +25.367146
λ_2	+ 0·6265118 +15·036283	+ 0·0728534 + 1·748482	− 0·4715212 −11·316509	− 1·0158958 −24·381499	− 1·5602706 −37·446494	+ 0·5854432 +14·050637	+ 0·5443746 +13·064990	+ 0·5033060 +12·079344	0 0	− 1·4874170 −35·698008	− 0·9430424 −22·633018	+ 1·5602704 +37·446490
μ_2	+ 2·1139288 +50·734291	+ 1·5602704 +37·446490	+ 1·0158958 +24·381499	+ 0·4715212 +11·316509	− 0·0728536 − 1·748486	+ 2·0728602 +49·748645	+ 2·0317916 +48·762998	+ 1·9907230 +47·777352	+ 1·4874170 +35·698008	0 0	+ 0·5443746 +13·064990	+ 3·0476874 +73·144498
ν_2	+ 1·5695542 +37·669301	+ 1·0158958 +24·381499	+ 0·4715212 +11·316509	− 0·0728534 − 1·748482	− 0·6172282 −14·813477	+ 1·5284856 +36·683654	+ 1·4874170 +35·698008	+ 1·4463484 +34·712362	+ 0·9430424 +22·633018	− 0·5443746 −13·064990	0 0	+ 2·5033128 +60·079507
2SM	− 0·9337586 −22·410206	− 1·4874170 −35·698008	− 2·0317916 −48·762998	− 2·5761662 −61·827989	− 3·1205410 −74·892984	− 0·9748272 −23·395853	− 1·0158958 −24·381499	− 1·0569644 −25·367146	− 1·5602704 −37·446490	− 3·0476874 −73·144498	− 2·5033128 −60·079507	0 0

In this table *a*, *b* denote the hourly speeds of the components *A*, *B*.

TABLE 40.—*Synodic periods in days and hours.*

DIURNALS.

A	B: J_1	K_1	M_1	O_1	OO	P_1	Q_1	2 Q	S_1	ρ_1
J_1	∞ ∞									
K_1	27·55455 661·3092	∞ ∞								
M_1	13·71879 329·2509	27·32158 655.7180	∞ ∞							
O_1	9·13293 219·1904	13·66079 327·8590	27·32158 655·7180	∞ ∞						
OO	27·09252 650·2205	13·66079 327·8590	9·10719 218·5727	6·83040 163·9295	∞ ∞					
P_1	23·94208 574·6100	182·62127 4382·9105	32·12822 771·0772	14·76529 354·3671	12·71003 305·0407	∞ ∞				
Q_1	6·85939 164·6254	9·13293 219·1904	13·71879 329·2509	27·55455 661·3092	5·47357 131·3657	9·61372 230·7292	∞ ∞			
2 Q	5·49218 131·8123	6·85939 164·6254	9·15882 219·8116	13·77728 330·6546	4·56647 109·5952	7·12709 171·0502	27·55455 661·3092	∞ ∞		
S_1	25·62161 614·9186	365·24255 8765·8211	29·53059 708·7341	14·19158 340·5980	13·16827 316·0385	365·24255 8765·8211	9·36716 224·8118	6·99068 167·7763	∞ ∞	
ρ_1	7·09579 170·2990	9·55685 229·3645	14·69813 352·7552	31·81193 763·4863	5·62306 134·9534	10·08460 242·0304	205·89265 4941·4235	24·30219 583·2527	9·81364 235·5272	∞ ∞

SEMIDIURNALS.

A	B: K_2	L_2	M_2	N_2	2 N	R_2	S_2	T_2	λ_2	μ_2	ν_2	2 SM
K_2	∞ ∞											
L_2	27·09252 650·2204	∞ ∞										
M_2	13·66079 327·8590	27·55456 661·3094	∞ ∞									
N_2	9·13293 219·1904	13·77728 330·6547	27·55456 661·3094	∞ ∞								
2 N	6·85939 164·6254	9·18485 220·4364	13·77728 330·6546	27·55455 661 3092	∞ ∞							
R_2	365·24255 8765·8211	29·26317 702·3160	14·19158 340·5980	9·36716 224·8118	6·99068 167·7763	∞ ∞						
S_2	182·62127 4382·9105	31·81193 763·4863	14·76529 354·3671	9·61372 230·7292	7·12709 171·0502	365·24255 8765·8211	∞ ∞					
T_2	121·74751 2921·9403	34·84704 836·3290	15·38734 369·2962	9·87361 236·9665	7·26893 174·4544	182·62127 4382·9105	365·24255 8765·8211	∞ ∞				
λ_2	23·94208 574·6100	205·89265 4941·4235	31·81193 763·4863	14·76529 354·3671	9·61372 230·7292	25·62161 614·9186	27·55456 661·3094	29·80294 715·2706	∞ ∞			
μ_2	7·09579 170·2990	9·61372 230·7292	14·76529 354·3671	31·81193 763·4863	205·89236 4941·4165	7·23638 173·6731	7·38265 177·1835	7·53495 180·8388	10·08460 242·0304	∞ ∞		
ν_2	9·55685 229·3645	14·76529 354·3671	31·81193 763·4863	205·89265 4941·4235	24·30216 583·2526	9·81364 235·5272	10·08460 242·0304	10·37095 248·9027	15·90597 381·7432	27·55456 661·3094	∞ ∞	
2 SM	16·06411 385·5386	10·08460 242·0304	7·38265 177·1835	5·82261 139·7425	4·80686 115·3646	15·38734 369·2962	14·76529 354·3671	14·19158 340·5980	9·61372 230·7292	4·92176 118·1224	5·99206 143·8094	∞ ∞

Synodic period $= \frac{15}{b \sim a}$ days or $\frac{360}{b \sim a}$ hours.

TABLE 41.—*For clearing one component of the effects of others.*

[Length of series, 29 days.]

Component sought. (A)	Disturbing components (B, C, etc.).									
	J_1	K_1	M_1	O_1	OO	P_1	Q_1	2 Q	S_1	ρ_1
J_1		·0497 351 π	·053 339	·0525 328 π	·065 13 π	·162 322 π	·049 319	·046 310 π	·113 336 π	·021 344
K_1	·050 9 π		·057 349 π	·0565 338	·056 22	·959 331	·052 328 π	·049 319	·990 346	·011 354 π
M_1	·053 21	·0575 11 π		·0575 349 π	·055 33 π	·106 162	·053 339	·050 330 π	·018 177	·014 185 π
O_1	·052 32 π	·0565 22	·057 11 π		·052 44	·018 174 π	·050 351 π	·049 341	·021 8	·096 196
OO	·065 347 π	·0565 338	·055 327 π	·0523 316		·108 309	·048 306 π	·045 297	·086 324	·029 332 π
P_1	·162 38 π	·9590 29	·106 198	·0182 186 π	·108 51		·005 357 π	·017 348	·990 14	·042 202
Q_1	·049 41	·0525 32 π	·053 21	·0497 9 π	·048 54 π	·005 3 π		·050 351	·031 17 π	·968 25
2 Q	·046 50 π	·0494 41	·050 30 π	·0489 19	·045 63	·017 12	·050 9 π		·034 27	·152 35 π
S_1	·113 24 π	·9902 14	·018 183	·0212 352	·086 36	·990 346	·031 343 π	·034 333		·015 188
ρ_1	·021 16	·0113 6 π	·014 175 π	·0958 164	·029 28 π	·042 158	·968 335	·152 325 π	·015 172	

Component sought. (A)	Disturbing components (B, C, etc.).											
	K_2	L_2	M_2	N_2	2 N	R_2	S_2	T_2	λ_2	μ_2	ν_2	2 SM
K_2		·065 347 π	·0565 338	·052 328 π	·049 319	·990 346	·9590 331	·909 317	·162 322 π	·021 344	·011 354 π	·101 145 π
L_2	·065 13 π		·0497 351 π	·049 341	·048 332 π	·009 178	·0958 164	·192 150	·968 335	·005 357 π	·018 186 π	·042 158
M_2	·056 22	·050 9 π		·050 351 π	·049 341	·021 8	·0182 174 π	·060 159	·096 164	·018 186 π	·096 196	·018 67 π
N_2	·052 32 π	·049 19	·0497 9 π		·050 351 π	·031 17 π	·0055 3 π	·021 169	·018 174 π	·096 196	·968 25	·004 177
2 N	·049 41	·048 28 π	·0489 19	·050 9 π		·034 27	·0168 12	·003 178 π	·005 3 π	·968 25	·152 35 π	·005 6
R_2	·990 14	·009 182	·0212 352	·031 343 π	·034 333		·9902 346	·959 331	·113 336 π	·002 359	·015 188	·060 159 π
S_2	·959 29	·096 196	·0182 186 π	·005 357 π	·017 348	·990 14		·990 346	·050 351 π	·018 293 π	·042 202	·018 174 π
T_2	·909 43	·192 210	·0599 201 π	·021 191	·003 182 π	·959 29	·9902 14		·028 185	·038 207 π	·068 217	·021 8
λ_2	·162 38 π	·968 25	·0958 196	·018 186 π	·005 357 π	·113 24 π	·0497 9 π	·028 175		·042 202	·092 212 π	·005 3 π
μ_2	·021 16	·005 3 π	·0182 174 π	·096 164	·968 335	·002 1	·0181 67 π	·038 153 π	·042 158		·050 9 π	·018 161 π
ν_2	·011 6 π	·018 174 π	·0958 164	·968 335	·152 325 π	·015 172	·0422 158	·068 143	·092 148 π	·050 351 π		·032 151
2 SM	·101 215 π	·042 202	·0181 293 π	·004 183	·005 354	·060 201 π	·0182 186 π	·021 352	·005 357 π	·018 199 π	·032 209	

TABLE 41—*For clearing one component of the effects of others*—Continued.

[Length of series, 369 days.]

Component sought. (A)	Disturbing components (B, C, etc.).									
	J_1	K_1	M_1	O_1	OO	P_1	Q_1	2 Q	S_1	ρ_1
J_1		·0224 290 π	·004 198	·0075 287	·022 112 π	·020 286 π	·004 217 π	·003 326 π	·021 288	·000 180 π
K_1	·022 70 π		·024 269 π	·0004 358 π	·000 2 π	·010 356	·007 287	·004 217 π	·010 358 π	·008 250
M_1	·004 162	·0236 91 π		·0236 269 π	·008 93	·028 87 π	·004 198	·006 308	·025 89	·004 341 π
O_1	·008 73	·0004 2 π	·024 91 π		·000 4	·000 178	·022 290 π	·007 219	·000 180 π	·026 252 π
OO	·022 248 π	·0004 358 π	·008 267	·0004 356		·001 354 π	·005 285 π	·002 215	·001 356	·004 248 π
P_1	·020 74 π	·0102 4	·028 273 π	·0004 182	·001 6 π		·008 291	·004 221 π	·010 2 π	·008 254
Q_1	·004 143 π	·0075 73	·004 162	·0224 70 π	·005 75 π	·008 69		·022 290 π	·008 71 π	·108 143 π
2 Q	·003 34 π	·0036 143 π	·006 52	·0075 141	·002 145	·004 139 π	·022 70 π		·004 141	·011 33 π
S_1	·021 72	·0102 2 π	·025 271	·0000 180 π	·001 4	·010 358 π	·008 289 π	·004 219		·008 252 π
ρ_1	·000 180 π	·0078 110	·004 19 π	·0261 108 π	·004 112 π	·008 106	·108 217 π	·011 327 π	·008 108 π	

Component sought. (A)	Disturbing components (B, C, etc.).											
	K_2	L_2	M_2	N_2	2 N	R_2	S_2	T_2	λ_2	μ_2	ν_2	2 SM
K_2		·022 248 π	·0000 358 π	·008 287	·004 217 π	·010 358 π	·0102 356	·010 354 π	·020 286 π	·000 180 π	·008 250	·001 175
L_2	·022 112 π		·0224 290 π	·007 219	·004 329	·024 110	·0261 108 π	·029 106	·108 217 π	·008 291	·000 182	·008 106
M_2	·000 2 π	·022 70 π		·022 290 π	·007 219	·000 180 π	·0004 178	·001 177 π	·026 108 π	·000 182	·026 252 π	·000 177 π
N_2	·008 73	·007 141	·0224 70 π		·022 290 π	·008 71 π	·0077 69	·008 67 π	·000 178	·026 252 π	·108 143 π	·005 67 π
2N	·004 143 π	·004 31	·0075 141	·022 70 π		·004 141	·0040 139 π	·004 138	·008 69	·108 143 π	·011 33 π	·003 138
R_2	·010 2 π	·024 250	·0000 180 π	·008 289 π	·004 219		·0102 358 π	·010 356	·021 288	·000 181	·008 252 π	·001 177 π
S_2	·010 4	·026 252 π	·0004 182	·008 291	·004 221 π	·010 2 π		·010 358 π	·022 290 π	·000 183 π	·008 254	·000 178
T_2	·010 6 π	·029 254	·0008 183 π	·008 293 π	·004 222	·010 4	·0102 2 π		·024 291	·001 185	·009 256 π	·000 180 π
λ_2	·020 74 π	·108 143 π	·0261 252 π	·000 182	·008 291	·021 72	·0224 70 π	·024 69		·008 254	·008 324 π	·008 69
μ_2	·000 180 π	·008 69	·0004 178	·026 108 π	·108 217 π	·000 179	·0004 177 π	·001 175	·008 106		·022 70 π	·000 175
ν_2	·008 110	·000 178	·0261 108 π	·108 217 π	·011 327 π	·008 108 π	·0084 106	·009 104 π	·008 36 π	·022 290 π		·005 105 π
2 SM	·001 185	·008 254	·0004 183 π	·005 293 π	·003 222	·001 183 π	·0004 182	·000 180 π	·008 291	·000 185	·005 255 π	

TABLE 42.—*Component hours derived from solar hours.*

Day of series.	J	K	L	M	N		2N		O		OO		P
1	13+1 ←	....	....	15-1 ↑	10-1 ↑	....	8-1 ←	22-2 ←	8-1 ←	22-2 ←	7+1 ←	20+2 ←	....
2	15+2 ↑	....	8-1 ↑	21-2 ←	5-2 ↑	....	12-3 ↑	....	12-3 ←	....	9+3 ←	23+4 ↑	....
3	17+3 ↑	....	....	....	1-3 ←	20-4 ←	2-4 ↑	17-5 ←	2-4 ↑	16-5 ↑	12+5 ↑	....	....
4	18+4 ←	....	....	2-3 ↑	15-5 ↑	....	7-6 ←	21-7 ↑	7-6 ←	21-7 ←	1+6 ↑	14+7 ←	....
5	20+5 ↑	....	0-2 ←	8-4 ←	10-6 ↑	....	11-8 ↑	....	11-8 ←	....	3+8 ←	16+9 ←	....
6	21+6 ←	....	....	13-5 ↑	5-7 ↑	....	2-9 ←	16-10←	1-9 ↑	15-10↑	6+10↑	19+11↑	....
7	23+7 ←	....	16-3 ←	19-6 ←	1-8 ←	20-9 ←	6-11↑	20-12↑	6-11←	20-12←	8+12↑	21-11←	....
8	....	15+1 ←	....	....	15-10↑	....	11+11←	....	10+11←	....	10-10←	23-9 ←	15-1 ↑
9	1+8 ↑	....	....	0-7 ↑	10-11↑	....	1+10←	15+9 ↑	0+10↑	14+9 ↑	13-8 ↑	....	....
10	2+9 ←	....	7-4 ↑	6-8 ←	6-12←	....	5+8 ↑	20+7 ←	4+8 ↑	19+7 ←	2-7 ↑	15-6 ↑	....
11	4+10↑	....	....	12-9 ←	1+11←	20+10↑	10+6 ←	....	9+6 ←	23+5 ↑	4-5 ←	17-4 ←	....
12	6+11↑	....	23-5 ←	17-10↑	15+9 ↑	....	0+5 ↑	14+4 ↑	13+4 ↑	....	6-3 ←	20-2 ↑	....
13	7+12←	....	....	23-11←	11+8 ←	....	5+3 ←	19+2 ←	3+3 ↑	18+2 ←	9-1 ↑	22-0 ↑	....
14	9-11↑	....	....	....	6+7 ←	....	9+1 ↑	23+0 ↑	8+1 ←	22+0 ↑	11+1 ←	....	....
15	10-10←	....	14-6 ↑	4-12↑	1+6 ←	20+5 ↑	14-1 ←	....	12-1 ↑	....	0+2 ←	13+3 ←	....
16	12-9 ←	....	....	10+11←	15+4 ↑	....	4-2 ←	18-3 ↑	2-2 ↑	17-3 ←	3+4 ↑	16+5 ↑	....
17	14-8 ↑	....	....	15+10↑	11+3 ←	....	8-4 ↑	23-5 ←	7-4 ←	21-5 ←	5+6 ↑	18+7 ←	....
18	15-7 ←	....	6-7 ↑	21+9 ←	6+2 ←	....	13-6 ←	....	11-6 ↑	....	7+8 ←	20+9 ←	....
19	17-6 ↑	....	....	....	1+1 ↑	20+0 ↑	3-7 ↑	18-8 ←	1-7 ↑	16-8 ←	10+10↑	23+11↑	....
20	18-5 ←	....	22-8 ←	2+8 ↑	16-1 ←	....	8-9 ←	22-10↑	6-9 ←	20-10←	12+12↑	....	....
21	20-4 ←	....	....	8+7 ←	11-2 ←	....	12-11↑	....	10-11↑	....	1-11←	14-10←	....
22	22-3 ↑	....	....	13+6 ↑	6-3 ↑	....	3-12←	17+11←	0-12↑	14+11↑	3-9 ←	17-8 ↑	....
23	23-2 ←	20+2 ←	13-9 ↑	19+5 ←	1-4 ↑	20-5 ↑	7+10↑	21+9 ↑	5+10←	19+9 ←	6-7 ↑	19-6 ↑	20-2 ↑
24	....	....	....	....	16-6 ←	....	12+8 ←	....	9+8 ↑	23+7 ↑	8-5 ←	21-4 ←	....
25	1-1 ↑	....	....	0+4 ↑	11-7 ←	....	2+7 ←	16+6 ↑	13+6 ↑	....	10-3 ←	....	....
26	3-0 ↑	....	5-10←	6+3 ←	6-8 ↑	....	6+5 ↑	21+4 ←	4+5 ←	18+4 ←	0-2 ↑	13-1 ↑	....
27	4+1 ←	....	....	11+2 ↑	1-9 ↑	21-10←	11+3 ←	....	8+3 ↑	22+2 ↑	2-0 ↑	15+1 ←	....
28	6+2 ↑	....	21-11←	17+1 ←	16-11←	....	1+2 ↑	15+1 ↑	12+1 ↑	....	4+2 ←	17+3 ←	....
29	7+3 ←	....	....	22+0 ↑	11-12↑	....	6+0 ←	20-1 ←	3+0 ←	17-1 ←	7+4 ↑	20+5 ↑	....
30	9+4 ←	....	....	....	6+11↑	....	10-2 ↑	....	7-2 ←	21-3 ↑	9+6 ←	22+7 ←	....
31	11+5 ↑	....	12-12↑	4-1 ←	2+10←	21+9 ←	0-3 ↑	15-4 ←	11-4 ↑	....	11+8 ←	....	....
32	12+6 ←	....	....	10-2 ←	16+8 ←	....	5-5 ←	19-6 ↑	2-5 ←	16-6 ←	1+9 ↑	14+10↑	....
33	14+7 ↑	....	....	15-3 ↑	11+7 ↑	....	9-7 ↑	....	6-7 ←	20-8 ↑	3+11↑	16+12←	....
34	16+8 ↑	....	4+11←	21-4 ←	6+6 ↑	....	0-8 ←	14-9 ←	10-9 ↑	....	5-11←	18-10←	....
35	17+9 ←	....	....	....	2+5 ←	21+4 ←	4-10↑	18-11↑	1-10←	15-11←	8-9 ↑	21-8 ↑	....
36	19+10↑	....	19+10↑	2-5 ↑	16+3 ↑	....	9-12←	23+11←	5-12←	19+11↑	10-7 ↑	23-6 ←	....
37	20+11←	....	....	8-6 ←	11+2 ↑	....	13+10↑	....	9+10↑	23+9 ↑	12-5 ←	....	....
38	22+12←	....	....	13-7 ↑	7+1 ←	....	3+9 ↑	18+8 ←	14+8 ←	....	1-4 ←	15-3 ↑	....
39	....	2+3 ↑	11+9 ↑	19-8 ←	2+0 ←	21-1 ↑	8+7 ←	22+6 ↑	4+7 ←	18+6 ↑	4-2 ↑	17-1 ↑	2-3 ←
40	0-11↑	....	....	....	16-2 ↑	....	12+5 ↑	....	8+5 ↑	22+4 ↑	6-0 ←	19+1 ←	....
41	1-10←	....	....	0-9 ↑	11-3 ↑	....	3+4 ←	17+3 ←	13+3 ←	....	8+2 ←	22+3 ↑	....
42	3-9 ↑	....	3+8 ←	6-10←	7-4 ←	....	7+2 ↑	21+1 ↑	3+2 ←	17+1 ↑	11+4 ↑	....	....
43	5-8 ↑	....	....	11-11↑	2-5 ←	21-6 ↑	12+0 ←	....	7+0 ↑	21-1 ↑	0+5 ↑	13+6 ←	....
44	6-7 ←	....	18+7 ↑	17-12←	16-7 ↑	....	2-1 ←	16-2 ↑	12-2 ←	....	2+7 ←	15+8 ←	....
45	8-6 ↑	....	....	22+11↑	12-8 ←	....	6-3 ↑	21-4 ←	2-3 ←	16-4 ←	5+9 ↑	18+10↑	....
46	9-5 ←	....	....	....	7-9 ←	....	11-5 ←	....	6-5 ↑	20-6 ↑	7+11↑	20+12←	....
47	11-4 ←	....	10+6 ←	4+10←	2-10↑	21-11↑	1-6 ↑	15-7 ↑	11-7 ←	....	9-11←	22-10←	....
48	13-3 ↑	....	....	9+9 ↑	17-12←	....	6-8 ←	20-9 ←	1-8 ←	15-9 ←	12-9 ↑	....	....
49	14-2 ←	....	....	15+8 ←	12+11←	....	10-10↑	....	5-10↑	19-11↑	1-8 ↑	14-7 ↑	....
50	16-1 ↑	....	2+5 ←	20+7 ↑	7+10←	....	0-11↑	15-12←	9-12↑	....	3-6 ←	16-5 ←	....
51	18-0 ↑	....	....	....	2+9 ↑	21+8 ↑	5+11←	19+10↑	0+11←	14+10←	5-4 ←	19-3 ↑	....
52	19+1 ←	....	17+4 ↑	2+6 ↑	17+7 ←	....	9+9 ↑	....	4+9 ↑	18+8 ↑	8-2 ↑	21-1 ↑	....
53	21+2 ↑	....	....	8+5 ←	12+6 ←	....	0+8 ←	14+7 ←	8+7 ↑	23+6 ←	10-0 ←	23+1 ←	....
54	22+3 ←	7+4 ↑	....	13+4 ↑	7+5 ↑	....	4+6 ↑	19+5 ←	13+5 ←	....	12+2 ←	....	7-4 ←
55	....	....	9+3 ←	19+3 ←	2+4 ↑	22+3 ←	9+4 ←	23+3 ↑	3+4 ↑	17+3 ↑	2+3 ↑	15+4 ↑	....
56	0+4 ←	....	....	....	17+2 ←	....	13+2 ↑	....	7+2 ↑	22+1 ←	4+5 ↑	17+6 ←	....
57	2+5 ↑	....	....	0+2 ↑	12+1 ↑	....	4+1 ←	18+0 ←	12+0 ←	....	6+7 ←	19+8 ←	....
58	3+6 ←	....	0+2 ↑	6+1 ←	7+0 ↑	....	8-1 ↑	22-2 ↑	2-1 ←	16-2 ↑	9+9 ↑	22+10↑	....
59	5+7 ↑	....	....	11+0 ↑	2-1 ↑	22-2 ←	13-3 ←	....	6-3 ↑	21-4 ←	11+11↑	....	....
60	6+8 ←	....	16+1 ↑	17-1 ←	17-3 ←	....	3-4 ←	17-5 ↑	11-5 ←	....	0+12←	13-11←	....
61	8+9 ←	....	....	22-2 ↑	12-4 ↑	....	7-6 ↑	22-7 ←	1-6 ←	15-7 ↑	2-10←	16-9 ↑	....
62	10+10↑	....	....	....	7-5 ↑	....	12-8 ←	....	5-8 ↑	20-9 ←	5-8 ↑	18-7 ↑	....
63	11+11←	....	8+0 ←	4-3 ←	3-6 ←	22-7 ←	2-9 ↑	16-10↑	10-10←	....	7-6 ←	20-5 ←	....
64	13+12↑	....	....	9-4 ↑	17-8 ↑	....	7-11←	21-12←	0-11←	14-12↑	9-4 ←	23-3 ↑	....
65	15-11↑	....	23-1 ↑	15-5 ←	12-9 ↑	....	11+11↑	....	4+11↑	18+10↑	12-2 ↑	....	....
66	16-10←	....	....	20-6 ↑	8-10←	....	1+10↑	16+9 ←	9+9 ←	23+8 ←	1-1 ↑	14-0 ←	....
67	18-9 ↑	....	....	....	3-11←	22-12←	6+8 ←	20+7 ↑	13+7 ↑	....	3+1 ←	16+2 ←	....
68	19-8 ←	....	15-2 ←	2-7 ←	17+11↑	....	10+6 ↑	....	3+6 ↑	17+5 ↑	6+3 ↑	19+4 ↑	....
69	21-7 ←	12+5 ←	....	7-8 ↑	12+10↑	....	1+5 ←	15+4 ←	8+4 ←	22+3 ←	8+5 ↑	21+6 ←	12-5 ↑
70	23-6 ↑	....	....	13-9 ←	8+9 ←	....	5+3 ↑	19+2 ↑	12+2 ↑	....	10+7 ←	23+8 ←	....
71	....	....	7-3 ←	19-10←	3+8 ←	22+7 ↑	10+1 ←	....	2+1 ↑	16+0 ↑	13+9 ↑	....	....
72	0-5 ←	....	....	....	17+6 ↑	....	0+0 ←	14-1 ↑	7-1 ←	21-2 ←	2+10↑	15+11↑	....
73	2-4 ↑	....	22-4 ↑	0-11↑	13+5 ←	....	4-2 ↑	19-3 ←	11-3 ←	....	4+12←	17-11←	....
74	4-3 ↑	....	....	6-12←	8+4 ←	....	9-4 ←	23-5 ↑	1-4 ↑	15-5 ↑	6-10←	20-9 ↑	....
75	5-2 ←	....	....	11+11↑	3+3 ↑	22+2 ↑	13-6 ↑	....	6-6 ←	20-7 ←	9-8 ↑	22-7 ↑	....
76	7-1 ↑	....	14-5 ←	17+10←	17+1 ↑	....	4-7 ←	18-8 ←	10-8 ←	....	11-6 ←	....	....
77	8-0 ←	....	....	22+9 ↑	13+0 ←	....	8-9 ↑	22-10↑	0-9 ↑	14-10↑	0-5 ←	13-4 ←	....
78	10+1 ←	....	....	....	8-1 ←	....	13-11←	....	5-11←	19-12←	3-3 ↑	16-2 ↑	....
79	12+2 ↑	....	5-6 ↑	4+8 ←	3-2 ↑	22-3 ↑	3-12←	17+11↑	9+11←	23+10↑	5-1 ↑	18-0 ←	....
80	13+3 ←	....	....	9+7 ↑	18-4 ←	....	7+10↑	22+9 ←	13+9 ↑	....	7+1 ←	20+2 ←	....

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	J	K	L	M	N		2N		O		OO		P
81	15+4 ↑		21−7 ↑	15+6 ←	13−5 ←		12+8 ←		3+8 ↑	18+7 ←	10+3 ↑	23+4 ↑	
82	17+5 ↑			20+5 ↑	8−6 ↑		2+7 ↑	16+6 ↑	8+6 ←	22+5 ↑	12+5 ↑		
83	18+6 ←				3−7 ↑	23−8 ←	7+5 ←	21+4 ←	12+4 ↑		1+6 ←	14+7 ←	
84	20+7 ↑	17+6 ←	13−8 ←	2+4 ←	18−9 ←		11+3 ↑		2+3 ↑	17+2 ←	3+8 ←	17+9 ↑	17−6 ↑
85	21+8 ←			7+3 ↑	13−10←		1+2 ↑	16+1 ←	7+1 ←	21+0 ←	6+10↑	19+11↑	
86	23+9 ←			13+2 ←	8−11↑		6+0 ←	20−1 ↑	11−1 ↑		8+12←	21−11←	
87			4−9 ↑	18+1 ↑	3−12↑	23+11←	10−2 ↑		1−2 ↑	16−3 ←	11−10↑		
88	1+10↑				18+10←		1−3 ←	15−4 ←	6−4 ←	20−5 ←	0−9 ↑	13−8 ↑	
89	2+11←		20−10←	0+0 ←	13+9 ↑		5−5 ↑	20−6 ←	10−6 ↑		2−7 ←	15−6 ←	
90	4+12↑			5−1 ↑	8+8 ↑		10−7 ←		0−7 ↑	15−8 ←	4−5 ←	18−4 ↑	
91	6−11↑			11−2 ←	4+7 ←	23+6 ←	0−8 ↑	14−9 ↑	5−9 ←	19−10←	7−3 ↑	20−2 ↑	
92	7−10←		12−11←	17−3 ←	18+5 ↑		5−10←	19−11←	9−11↑	23−12↑	9−1 ←	22−0 ←	
93	9−9 ↑			22−4 ↑	13+4 ↑		9−12↑	23+11↑	13+11↑		11+1 ←		
94	10−8 ←				8+3 ↑		14+10←		4+10←	18+9 ←	1+2 ↑	14+3 ↑	
95	12−7 ←		3−12↑	4−5 ←	4+2 ←	23+1 ←	4+9 ←	18+8 ↑	8+8 ↑	22+7 ↑	3+4 ↑	16+5 ←	
96	14−6 ↑			9−6 ↑	18+0 ↑		8+7 ↑	23+6 ←	12+6 ↑		5+6 ←	18+7 ←	
97	15−5 ←		19+11←	15−7 ←	13−1 ↑		13+5 ←		3+5 ←	17+4 ←	8+8 ↑	21+9 ↑	
98	17−4 ↑			20−8 ↑	9−2 ←		3+4 ↑	17+3 ↑	7+3 ↑	21+2 ↑	10+10↑	23+11←	
99	18−3 ←	23+7 ↑			4−3 ←	23−4 ↑	8+2 ←	22+1 ←	11+1 ↑		12+12←		23−7 ←
100	20−2 ←		10+10↑	2−9 ←	18−5 ↑		12+0 ↑		2+0 ←	16−1 ←	1−11←	15−10↑	
101	22−1 ↑			7−10↑	14−6 ←		2−1 ↑	17−2 ←	6−2 ←	20−3 ↑	4−9 ↑	17−8 ↑	
102	23−0 ←			13−11←	9−7 ←		7−3 ←	21−4 ↑	10−4 ↑		6−7 ←	19−6 ←	
103			2+9 ↑	18−12↑	4−8 ←	23−9 ↑	11−5 ↑		1−5 ←	15−6 ←	8−5 ←	22−4 ↑	
104	1+1 ↑				18−10↑		2−6 ←	16−7 ←	5−7 ←	19−8 ↑	11−3 ↑		
105	3+2 ↑		18+8 ←	0+11←	14−11←		6−8 ↑	20−9 ↑	9−9 ↑		0−2 ↑	13−1 ←	
106	4+3 ←			5+10↑	9−12←		11−10←		0−10←	14−11←	2−0 ←	15+1 ←	
107	6+4 ↑			11+9 ←	4+11↑	23+10↑	1−11←	15−12↑	4−12←	18+11↑	5+2 ↑	18+3 ↑	
108	7+5 ←		9+7 ↑	16+8 ↑	19+9 ←		5+11↑	20+10←	8+10↑	22+9 ↑	7+4 ↑	20+5 ←	
109	9+6 ←			22+7 ←	14+8 ←		10+9 ←		13+8 ←		9+6 ←	22+7 ←	
110	11+7 ↑				9+7 ↑		0+8 ↑	14+7 ↑	3+7 ←	17+6 ↑	12+8 ↑		
111	12+8 ←		1+6 ←	3+6 ↑	4+6 ↑	23+5 ↑	5+6 ←	19+5 ←	7+5 ↑	21+4 ↑	1+9 ↑	14+10↑	
112	14+9 ↑			9+5 ↑	19+4 ←		9+4 ↑	23+3 ↑	12+3 ←		3+11←	16+12←	
113	16+10↑		17+5 ←	15+4 ←	14+3 ←		14+2 ←		2+2 ←	16+1 ←	5−11←	19−10↑	
114	17+11←			20+3 ↑	9+2 ↑		4+1 ←	18+0 ↑	6+0 ↑	20−1 ↑	8−9 ↑	21−8 ↑	
115	19+12↑	4+8 ↑			4+1 ↑		8−1 ↑	23−2 ←	11−2 ←		10−7 ←	23−6 ←	4−8 ←
116	20−11←		8+4 ↑	2+2 ←	0+0 ←	19−1 ←	13−3 ←		1−3 ←	15−4 ←	12−5 ←		
117	22−10←			7+1 ↑	14−2 ↑		3−4 ↑	17−5 ↑	5−5 ↑	19−6 ↑	2−4 ↑	15−3 ↑	
118				13+0 ←	9−3 ↑		8−6 ←	22−7 ←	10−7 ←		4−2 ↑	17−1 ←	
119	0−9 ↑		0+3 ←	18−1 ↑	5−4 ←		12−8 ↑		0−8 ←	14−9 ←	6−0 ←	19+1 ←	
120	1−8 ←				0−5 ←	19−6 ↑	2−9 ↑	17−10←	4−10↑	18−11↑	9+2 ↑	22+3 ↑	
121	3−7 ↑		15+2 ↑	0−2 ←	14−7 ↑		7−11←	21−12↑	8−12↑	23+11←	11+4 ↑		
122	5−6 ↑			5−3 ↑	9−8 ↑		11+11↑		13+10←		0+5 ←	13+6 ←	
123	6−5 ←			11−4 ←	5−9 ←		2+10←	16+9 ←	3+9 ↑	17+8 ↑	2+7 ←	16+8 ↑	
124	8−4 ↑		7+1 ↑	16−5 ↑	0−10←	19−11↑	6+8 ↑	20+7 ↑	7+7 ↑	22+6 ←	5+9 ↑	18+10↑	
125	9−3 ←			22−6 ←	14−12↑		11+6 ←		12+5 ←		7+11←	20+12←	
126	11−2 ←		23+0 ←		10+11←		1+5 ↑	15+4 ↑	2+4 ↑	16+3 ↑	9−11←	23−10↑	
127	13−1 ↑			3−7 ↑	5+10←		6+3 ←	20+2 ←	6+2 ↑	21+1 ←	12−9 ↑		
128	14−0 ←			9−8 ←	0+9 ↑	19+8 ↑	10+1 ↑		11+0 ←		1−8 ↑	14−7 ←	
129	16+1 ↑		14−1 ↑	14−9 ↑	15+7 ←		0+0 ↑	15−1 ←	1−1 ←	15−2 ↑	3−6 ←	16−5 ←	
130	18+2 ↑	9+9 ←		20−10←	10+6 ←		5−2 ←	19−3 ↑	5−3 ↑	20−4 ←	6−4 ↑	19−3 ↑	9−9 ↑
131	19+3 ←				5+5 ←		9−4 ↑		10−5 ←		8−2 ↑	21−1 ←	
132	21+4 ↑		6−2 ←	2−11←	0+4 ↑	19+3 ↑	0−5 ←	14−6 ←	0−6 ←	14−7 ↑	10−0 ←	23+1 ←	
133	22+5 ←			7−12↑	15+2 ←		4−7 ↑	18−8 ↑	4−8 ↑	19−9 ←	13+2 ↑		
134			22−3 ←	13+11←	10+1 ←		9−9 ←	23−10←	9−10←	23−11←	2+3 ↑	15+4 ↑	
135	0+6 ←			18+10↑	5+0 ↑		13−11↑		13−12↑		4+5 ←	17+6 ←	
136	2+7 ↑				0−1 ↑	20−2 ←	3−12↑	18+11←	3+11↑	17+10↑	6+7 ←	20+8 ↑	
137	3+8 ←		13−4 ↑	0+9 ←	15−3 ←		8+10←	22+9 ↑	8+9 ←	22+8 ←	9+9 ↑	22+10↑	
138	5+9 ↑			5+8 ↑	10−4 ↑		12+8 ↑		12+7 ↑		11+11←		
139	6+10←			11+7 ←	5−5 ↑		3+7 ←	17+6 ←	2+6 ↑	16+5 ↑	0+12←	13−11←	
140	8+11←		5−5 ←	16+6 ↑	0−6 ↑	20−7 ←	7+5 ↑	21+4 ↑	7+4 ←	21+3 ←	3−10↑	16−9 ↑	
141	10+12↑			22+5 ←	15−8 ←		12+3 ←		11+2 ↑		5−8 ↑	18−7 ←	
142	11−11←		20−6 ↑		10−9 ↑		2+2 ←	16+1 ↑	1+1 ↑	15+0 ↑	7−6 ←	20−5 ←	
143	13−10↑			3+4 ↑	5−10↑		6+0 ←	21−1 ←	6−1 ←	20−2 ←	10−4 ↑	23−3 ↑	
144	15−9 ↑			9+3 ←	1−11←	20−12←	11−2 ←		10−3 ←		12−2 ←		
145	16−8 ←	14+10←	12−7 ↑	14+2 ↑	15+11↑		1−3 ↑	15−4 ↑	0−4 ↑	14−5 ↑	1−1 ←	14−0 ←	14−10↑
146	18−7 ↑			20+1 ←	10+10↑		6−5 ←	20−6 ←	5−6 ←	19−7 ←	4+1 ↑	17+2 ↑	
147	19−6 ←				6+9 ←		10−7 ↑		9−8 ←	23−9 ↑	6+3 ↑	19+4 ←	
148	21−5 ←		4−8 ←	1+0 ↑	1+8 ←	20+7 ←	0−8 ↑	15−9 ←	13−10↑		8+5 ←	21+6 ←	
149	23−4 ↑			7−1 ←	15+6 ↑		5−10←	19−11↑	3−11↑	18−12←	11+7 ↑		
150			19−9 ↑	12−2 ↑	10+5 ↑		9−12↑		8+11←	22+10↑	0+8 ↑	13+9 ↑	
151	0−3 ←			18−3 ←	6+4 ←		0+11←	14+10←	12+9 ↑		2+10←	15+11←	
152	2−2 ↑				1+3 ←	20+2 ↑	4+9 ↑	18+8 ↑	2+8 ↑	17+7 ←	4+12←	18−11↑	
153	4−1 ↑		11−10←	0−4 ←	15+1 ↑		9+7 ←	23+6 ←	7+6 ←	21+5 ↑	7−10↑	20−9 ↑	
154	5−0 ←			5−5 ↑	11+0 ←		13+5 ↑		11+4 ↑		9−8 ←	22−7 ←	
155	7+1 ↑			11−6 ←	6−1 ←		3+4 ↑	18+3 ←	1+3 ↑	16+2 ←	11−6 ←		
156	8+2 ←		2−11↑	16−7 ↑	1−2 ↑	20−3 ↑	8+2 ←	22+1 ↑	6+1 ←	20+0 ←	1−5 ↑	14−4 ↑	
157	10+3 ←			22−8 ←	15−4 ↑		12+0 ↑		10−1 ↑		3−3 ↑	16−2 ←	
158	12+4 ↑		18−12↑		11−5 ←		3−1 ←	17−2 ←	0−2 ↑	15−3 ←	5−1 ←	18−0 ←	
159	13+5 ←			3−9 ↑	6−6 ←		7−3 ↑	21−4 ↑	5−4 ←	19−5 ←	8+1 ↑	21+2 ↑	
160	15+6 ↑	20+11↑		9−10←	1−7 ↑	20−8 ↑	12−5 ←		9−6 ↑	23−7 ↑	10+3 ↑	23+4 ←	20−11←

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	J	K	L	M	N		2N		O		OO		P
161	17+7 ↑	....	10+11←	14−11↑	16−9 ←	....	2−6 ↑	16−7 ↑	14−8 ←	....	12+5 ←	....	....
162	18+8 ←	....	....	20−12←	11−10←	....	7−8 ←	21−9 ←	4−9 ←	18−10←	1+6 ←	15+7 ↑	....
163	20+9 ↑	....	....	....	6−11↑	....	11−10↑	....	8−11↑	22−12↑	4+8 ↑	17+9 ↑	....
164	21+10←	....	1+10↑	1+11↑	1−12↑	21+11←	1−11↑	16−12←	12+11↑	....	6+10←	19+11←	....
165	23+11←	....	....	7+10←	16+10←	....	6+11←	20+10↑	3+10←	17+9 ←	8+12←	22−11↑	....
166	....	....	17+9 ←	12+9 ↑	11+9 ←	....	10+9 ↑	....	7+8 ↑	21+7 ↑	11−10↑	....	....
167	1+12↑	....	....	18+8 ←	6+8 ↑	....	1+8 ←	15+7 ←	11+6 ↑	....	0−9 ↑	13−8 ←	....
168	2−11←	....	....	23+7 ↑	1+7 ↑	21+6 ←	5+6 ↑	19+5 ↑	2+5 ←	16+4 ←	2−7 ←	15−6 ←	....
169	4−10↑	....	9+8 ←	....	16+5 ←	....	10+4 ←	....	6+3 ↑	20+2 ↑	5−5 ↑	18−4 ↑	....
170	6−9 ↑	....	....	5+6 ←	11+4 ↑	....	0+3 ←	14+2 ↑	10+1 ↑	....	7−3 ↑	20−2 ←	....
171	7−8 ←	....	....	10+5 ↑	6+3 ↑	....	4+1 ↑	19+0 ←	1+0 ←	15−1 ←	9−1 ←	22−0 ←	....
172	9−7 ↑	....	0+7 ↑	16+4 ↑	2+2 ←	21+1 ←	9−1 ←	23−2 ↑	5−2 ←	19−3 ↑	12+1 ↑	....	....
173	10−6 ←	....	....	22+3 ←	16+0 ↑	....	13−3 ↑	....	9−4 ↑	....	1+2 ↑	14+3 ↑	....
174	12−5 ←	....	16+6 ←	....	11−1 ↑	....	4−4 ←	18−5 ←	0−5 ←	14−6 ←	3+4 ←	16+5 ←	....
175	14−4 ↑	....	....	3+2 ↑	6−2 ↑	....	8−6 ↑	22−7 ↑	4−7 ←	18−8 ↑	5+6 ←	19+7 ↑	....
176	15−3 ←	1+12↑	....	9+1 ←	2−3 ←	21−4 ←	13−8 ←	....	8−9 ↑	22−10↑	8+8 ↑	21+9 ↑	1−12←
177	17−2 ↑	....	7+5 ↑	14+0 ↑	16−5 ↑	....	3−9 ←	17−10↑	13−11←	....	10+10←	23+11←	....
178	18−1 ←	....	....	20−1 ←	11−6 ↑	....	7−11↑	22−12←	3−12←	17+11↑	12+12←	....	....
179	20−0 ←	....	23+4 ↑	....	7−7 ←	....	12+11←	....	7+10↑	21+9 ↑	2−11↑	15−10↑	....
180	22+1 ↑	....	....	1−2 ↑	2−8 ←	21−9 ↑	2+10↑	16+9 ↑	12+8 ←	....	4−9 ↑	17−8 ←	....
181	23+2 ←	....	....	7−3 ←	16−10↑	....	7+8 ←	21+7 ←	2+7 ←	16+6 ↑	6−7 ←	19−6 ←	....
182	....	....	15+3 ←	12−4 ↑	12−11←	....	11+6 ↑	....	6+5 ↑	20+4 ↑	9−5 ↑	22−4 ↑	....
183	1+3 ↑	....	....	18−5 ←	7−12←	....	1+5 ↑	16+4 ←	11+3 ←	....	11−3 ↑	....	....
184	3+4 ↑	....	....	23−6 ↑	2+11←	21+10↑	6+3 ←	20+2 ↑	1+2 ←	15+1 ←	0−2 ←	13−1 ←	....
185	4+5 ←	....	6+2 ↑	....	16+9 ↑	....	10+1 ↑	....	5+0 ↑	19−1 ↑	2−0 ←	16+1 ↑	....
186	6+6 ↑	....	....	5−7 ←	12+8 ←	....	1+0 ←	15−1 ←	10−2 ←	....	5+2 ↑	18+3 ↑	....
187	7+7 ←	....	22+1 ←	10−8 ↑	7+7 ←	....	5−2 ↑	19−3 ↑	0−3 ←	14−4 ←	7+4 ←	20+5 ←	....
188	9+8 ←	....	....	16−9 ←	2+6 ↑	21+5 ↑	10−4 ←	....	4−5 ↑	18−6 ↑	9+6 ←	23+7 ↑	....
189	11+9 ↑	....	....	21−10↑	17+4 ←	....	0−5 ←	14−6 ↑	9−7 ←	23−8 ←	12+8 ↑	....	....
190	12+10←	....	14+0 ←	....	12+3 ←	....	4−7 ↑	19−8 ←	13−9 ←	....	1+9 ↑	14+10←	....
191	14+11↑	6−11←	....	3−11←	7+2 ↑	....	9−9 ←	23−10↑	3−10↑	17−11↑	3+11←	16+12←	6+11↑
192	16+12↑	....	....	9−12←	2+1 ↑	21+0 ↑	13−11↑	....	7−12↑	22+11←	6−11↑	19−10↑	....
193	17−11←	....	5−1 ↑	14+11↑	17−1 ←	....	4−12←	18+11←	12+10←	....	8−9 ↑	21−8 ←	....
194	19−10↑	....	....	20+10←	12−2 ←	....	8+10↑	22+9 ↑	2+9 ↑	16+8 ↑	10−7 ←	23−6 ←	....
195	20−9 ←	....	21−2 ←	....	7−3 ↑	....	13+8 ←	....	6+7 ↑	21+6 ←	13−5 ↑	....	....
196	22−8 ←	....	....	1+9 ↑	2−4 ↑	22−5 ←	3+7 ←	17+6 ↑	11+5 ←	....	2−4 ↑	15−3 ↑	....
197	....	....	....	7+8 ←	17−6 ←	....	8+5 ←	22+4 ←	1+4 ↑	15+3 ↑	4−2 ←	17−1 ←	....
198	0−7 ↑	....	12−3 ↑	12+7 ↑	12−7 ↑	....	12+3 ↑	....	5+2 ↑	20+1 ←	6−0 ←	20+1 ↑	....
199	1−6 ←	....	....	18+6 ←	7−8 ↑	....	2+2 ↑	17+1 ←	10+0 ←	....	9+2 ↑	22+3 ↑	....
200	3−5 ↑	....	....	23+5 ↑	3−9 ←	22−10←	7+0 ←	21−1 ↑	0−1 ←	14−2 ↑	11+4 ←	....	....
201	5−4 ↑	....	4−4 ↑	....	17−11←	....	11−2 ↑	....	4−3 ↑	19−4 ←	0+5 ←	14+6 ↑	....
202	6−3 ←	....	....	5+4 ←	12−12↑	....	2−3 ←	16−4 ←	9−5 ←	23−6 ←	3+7 ↑	16+8 ↑	....
203	8−2 ↑	....	20−5 ←	10+3 ↑	7+11↑	....	6−5 ↑	20−6 ↑	13−7 ↑	....	5+9 ←	18+10←	....
204	9−1 ←	....	....	16+2 ←	3+10←	22+9 ←	11−7 ←	....	3−8 ↑	18−9 ←	7+11←	21+12↑	....
205	11−0 ←	....	....	21+1 ↑	17+8 ↑	....	1−8 ←	15−9 ↑	8−10←	22−11←	10−11↑	23−10↑	....
206	13+1 ↑	11−10←	11−6 ↑	....	12+7 ↑	....	5−10↑	20−11←	12−12↑	....	12−9 ←	....	11+10↑
207	14+2 ←	....	....	3+0 ←	8+6 ←	....	10−12←	....	2+11↑	16+10↑	1−8 ←	14−7 ←	....
208	16+3 ↑	....	....	8−1 ↑	3+5 ←	22+4 ↑	0+11↑	14+10↑	7+9 ←	21+8 ←	4−6 ↑	17−5 ↑	....
209	18+4 ↑	....	3−7 ←	14−2 ←	17+3 ↑	....	5+9 ←	19+8 ←	11+7 ↑	....	6−4 ↑	19−3 ←	....
210	19+5 ←	....	....	19−3 ↑	12+2 ↑	....	9+7 ↑	23+6 ↑	1+6 ↑	15+5 ↑	8−2 ←	21−1 ←	....
211	21+6 ↑	....	19−8 ←	....	8+1 ←	....	14+5 ←	....	6+4 ←	20+3 ←	11−0 ↑	....	....
212	22+7 ←	....	....	1−4 ←	3+0 ←	22−1 ↑	4+4 ←	18+3 ↑	10+2 ←	....	0+1 ↑	13+2 ↑	....
213	....	....	....	7−5 ←	17−2 ↑	....	8+2 ↑	23+1 ←	0+1 ↑	14+0 ↑	2+3 ←	15+4 ←	....
214	0+8 ←	....	10−9 ↑	12−6 ↑	13−3 ←	....	13+0 ←	....	5−1 ←	19−2 ←	4+5 ←	18+6 ↑	....
215	2+9 ↑	....	....	18−7 ←	8−4 ←	....	3−1 ↑	17−2 ↑	9−3 ←	23−4 ↑	7+7 ↑	20+8 ↑	....
216	3+10←	....	....	23−8 ↑	3−5 ↑	22−6 ↑	8−3 ←	22−4 ←	13−5 ↑	....	9+9 ←	22+10←	....
217	5+11↑	....	2−10←	....	18−7 ←	....	12−5 ↑	....	4−6 ←	18−7 ←	11+11←	....	....
218	6+12←	....	....	5−9 ←	13−8 ←	....	2−6 ↑	17−7 ←	8−8 ←	22−9 ↑	1+12↑	14−11↑	....
219	8−11←	....	17−11↑	10−10↑	8−9 ←	....	7−8 ←	21−9 ↑	12−10↑	....	3−10↑	16−9 ←	....
220	10−10↑	....	....	16−11←	3−10↑	22−11↑	11−10↑	....	2−11↑	17−12←	5−8 ←	18−7 ←	....
221	11−9 ←	17−9 ↑	....	21−12↑	18−12←	....	2−11←	16−12←	7+11←	21+10↑	8−6 ↑	21−5 ↑	17+9 ←
222	13−8 ↑	....	9−12↑	....	13+11←	....	6+11↑	20+10↑	11+9 ↑	....	10−4 ↑	23−3 ←	....
223	15−7 ↑	....	....	3+11←	8+10↑	....	11+9 ←	....	1+8 ↑	16+7 ←	12−2 ←	....	....
224	16−6 ←	....	....	8+10↑	3+9 ↑	23+8 ←	1+8 ←	15+7 ↑	6+6 ←	20+5 ↑	1−1 ←	15−0 ↑	....
225	18−5 ↑	....	1+11←	14+9 ←	18+7 ←	....	5+6 ↑	20+5 ←	10+4 ↑	....	4+1 ↑	17+2 ↑	....
226	19−4 ←	....	....	19+8 ↑	13+6 ↑	....	10+4 ←	....	0+3 ↑	15+2 ←	6+3 ←	19+4 ←	....
227	21−3 ←	....	16+10↑	....	8+5 ↑	....	0+3 ↑	14+2 ↑	5+1 ←	19+0 ←	8+5 ←	22+6 ↑	....
228	23−2 ↑	....	....	1+7 ←	3+4 ↑	23+3 ←	5+1 ←	19+0 ←	9−1 ↑	23−2 ↑	11+7 ↑	....	....
229	....	....	....	6+6 ↑	18+2 ←	....	9−1 ↑	23−2 ↑	14−3 ←	....	0+8 ↑	13+9 ←	....
230	0−1 ←	....	8+9 ←	12+5 ←	13+1 ↑	....	14−3 ←	....	4−4 ←	18−5 ←	2+10←	15+11←	....
231	2−0 ↑	....	....	17+4 ↑	8+0 ↑	....	4−4 ←	18−5 ↑	8−6 ↑	22−7 ↑	5+12↑	18−11↑	....
232	4−1 ↑	....	....	23+3 ↑	4−1 ←	23−2 ←	9−6 ←	23−7 ←	13−8 ←	....	7−10↑	20−9 ←	....
233	5+2 ←	....	0+8 ←	....	18−3 ↑	....	13−8 ↑	....	3−9 ←	17−10←	9−8 ←	22−7 ←	....
234	7+3 ↑	....	....	5+2 ←	13−4 ↑	....	3−9 ↑	18−10←	7−11↑	21−12↑	12−6 ↑	....	....
235	8+4 ←	....	15+7 ↑	10+1 ↑	9−5 ←	....	8−11←	22−12↑	11+11↑	....	1−5 ↑	14−4 ↑	....
236	10+5 ←	22−8 ↑	....	16+0 ←	4−6 ←	23−7 ←	12+11↑	....	2+10←	16+9 ←	3−3 ←	16−2 ←	22+8 ←
237	12+6 ↑	....	....	21−1 ↑	18−8 ↑	....	3+10←	17+9 ←	6+8 ↑	20+7 ↑	5−1 ←	19−0 ↑	....
238	13+7 ←	....	7+6 ←	....	13−9 ↑	....	7+8 ↑	21+7 ↑	10+6 ↑	....	8+1 ↑	21+2 ↑	....
239	15+8 ↑	....	....	3−2 ←	9−10←	....	12+6 ←	....	1+5 ←	15+4 ←	10+3 ←	23+4 ←	....
240	17+9 ↑	....	22+5 ↑	8−3 ↑	4−11←	23−12↑	2+5 ←	16+4 ↑	5+3 ←	19+2 ↑	12+5 ←	....	....

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	J	K	L	M	N		2N		O		OO		P
241	18+10←	....	....	14−4 ←	18+11 ↑	....	6+3 ↑	21+2 ←	9+1 ↑	....	2+6 ↑	15+7 ↑	....
242	20+11 ↑	....	....	19−5 ↑	14+10←	....	11+1 ←	....	0+0 ←	14−1 ←	4+8 ↑	17+9 ←	....
243	21+12←	....	14+4 ↑	....	9+9 ←	....	1+0 ↑	15−1 ↑	4−2 ←	18−3 ↑	6+10←	19+11←	....
244	23−11←	....	....	1−6 ←	4+8 ↑	23+7 ↑	6−2 ←	20−3 ←	8−4 ↑	23−5 ←	9+12↑	22−11↑	....
245	....	....	....	6−7 ↑	18+6 ↑	....	10−4 ↑	....	13−6 ←	....	11−10↑	....	....
246	1−10↑	....	6+3 ←	12−8 ←	14+5 ←	....	0−5 ↑	15−6 ←	3−7 ←	17−8 ↑	0−9 ←	13−8 ←	....
247	2−9 ←	....	....	17−9 ↑	9+4 ←	....	5−7 ←	19−8 ↑	7−9 ↑	21−10↑	2−7 ←	16−6 ↑	....
248	4−8 ↑	....	21+2 ↑	23−10←	4+3 ↑	23+2 ↑	9−9 ↑	....	12−11←	....	5−5 ↑	18−4 ↑	....
249	6−7 ↑	....	....	....	19+1 ←	....	0−10←	14−11←	2−12←	16+11↑	7−3 ←	20−2 ←	....
250	7−6 ←	....	....	4−11↑	14+0 ←	....	4−12↑	18+11↑	6+10↑	20+9 ↑	9−1 ←	23−0 ↑	....
251	9−5 ↑	....	13+1 ←	10−12←	9−1 ↑	....	9+10←	23+9 ←	11+8 ←	....	12+1 ↑	....	....
252	10−4 ←	3−7 ←	....	16+11←	4−2 ↑	....	13+8 ↑	....	1+7 ←	15+6 ↑	1+2 ↑	14+3 ←	3+7 ↑
253	12−3 ←	....	....	21+10↑	0−3 ←	19−4 ←	3+7 ↑	18+6 ←	5+5 ↑	19+4 ↑	3+4 ←	16+5 ←	....
254	14−2 ↑	....	5+0 ←	....	14−5 ←	....	8+5 ←	22+4 ↑	10+3 ←	....	6+6 ↑	19+7 ↑	....
255	15−1 ←	....	....	3+9 ←	9−6 ↑	....	12+3 ↑	....	0+2 ←	14+1 ←	8+8 ↑	21+9 ←	....
256	17−0 ↑	....	20−1 ↑	8+8 ↑	4−7 ↑	....	3+2 ←	17+1 ←	4+0 ↑	18−1 ↑	10+10←	23+11←	....
257	18+1 ←	....	....	14+7 ←	0−8 ←	19−9 ←	7+0 ↑	21−1 ↑	9−2 ←	23−3 ←	13+12↑	....	....
258	20+2 ←	....	....	19+6 ↑	14−10↑	....	12−2 ←	....	13−4 ←	....	2−11↑	15−10←	....
259	22+3 ↑	....	12−2 ←	....	9−11↑	....	2−3 ←	16−4 ↑	3−5 ↑	17−6 ↑	4−9 ←	17−8 ←	....
260	23+4 ←	....	....	1+5 ←	5−12←	....	6−5 ↑	21−6 ←	8−7 ←	22−8 ←	7−7 ↑	20−6 ↑	....
261	....	....	....	6+4 ↑	0+11←	19+10↑	11−7 ←	....	12−9 ←	....	9−5 ↑	22−4 ←	....
262	1+5 ↑	....	3−3 ↑	12+3 ←	14+9 ↑	....	1−8 ↑	15−9 ↑	2−10↑	16−11↑	11−3 ←	....	....
263	3+6 ↑	....	....	17+2 ↑	9+8 ↑	....	6−10←	20−11←	6−12↑	21+11←	0−2 ←	14−1 ↑	....
264	4+7 ←	....	19−4 ↑	23+1 ←	5+7 ←	....	10−12↑	....	11+10←	....	3−0 ↑	16+1 ↑	....
265	6+8 ↑	....	....	....	0+6 ←	19+5 ↑	0+11↑	15+10←	1+9 ↑	15+8 ↑	5+2 ←	18+3 ←	....
266	7+9 ←	....	....	4+0 ↑	14+4 ↑	....	5+9 ←	19+8 ↑	5+7 ↑	20+6 ←	7+4 ←	21+5 ↑	....
267	9+10←	8−6 ←	11−5 ←	10−1 ←	10+3 ←	....	10+7 ←	....	10+5 ←	....	10+6 ↑	23+7 ↑	8+6 ↑
268	11+11↑	....	....	15−2 ↑	5+2 ←	....	0+6 ←	14+5 ↑	0+4 ↑	14+3 ↑	12+8 ←	....	....
269	12+12←	....	....	21−3 ←	0+1 ↑	19+0 ↑	4+4 ↑	19+3 ←	4+2 ↑	19+1 ←	1+9 ←	14+10←	....
270	14−11↑	....	2−6 ↑	....	15−1 ←	....	9+2 ←	23+1 ↑	9+0 ←	23−1 ←	4+11↑	17+12↑	....
271	16−10↑	....	....	2−4 ↑	10−2 ←	....	13+0 ↑	....	13−2 ↑	....	6−11↑	19−10←	....
272	17−9 ←	....	18−7 ←	8−5 ←	5−3 ←	....	4−1 ←	18−2 ←	3−3 ↑	18−4 ←	8−9 ←	21−8 ←	....
273	19−8 ↑	....	....	14−6 ←	0−4 ↑	19−5 ↑	8−3 ↑	22−4 ↑	8−5 ←	22−6 ←	11−7 ↑	....	....
274	20−7 ←	....	....	19−7 ↑	15−6 ←	....	13−5 ←	....	12−7 ↑	....	0−6 ↑	13−5 ↑	....
275	22−6 ←	....	10−8 ←	....	10−7 ←	....	3−6 ←	17−7 ↑	2−8 ↑	16−9 ↑	2−4 ←	15−3 ←	....
276	....	....	....	1−8 ←	5−8 ↑	....	7−8 ↑	22−9 ←	7−10←	21−11←	4−2 ←	18−1 ↑	....
277	0−5 ↑	....	....	6−9 ↑	0−9 ↑	20−10←	12−10←	....	11−12↑	....	7−0 ↑	20+1 ↑	....
278	1−4 ←	....	1−9 ↑	12−10←	15−11←	....	2−11↑	16−12↑	1+11↑	15+10↑	9+2 ←	22+3 ←	....
279	3−3 ↑	....	....	17−11↑	10−12↑	....	7+11←	21+10←	6+9 ←	20+8 ←	11+4 ←	....	....
280	5−2 ↑	....	17−10←	23−12←	5+11↑	....	11+9 ↑	....	10+7 ↑	....	1+5 ↑	14+6 ↑	....
281	6−1 ←	....	....	....	0+10↑	20+9 ←	1+8 ↑	16+7 ←	0+6 ↑	14+5 ↑	3+7 ↑	16+8 ←	....
282	8−0 ↑	13−5 ←	....	4+11↑	15+8 ←	....	6+6 ←	20+5 ↑	5+4 ←	19+3 ←	5+9 ←	18+10←	13+5 ↑
283	9+1 ←	....	8−11↑	10+10←	10+7 ↑	....	10+4 ↑	....	9+2 ←	23+1 ↑	8+11↑	21+12↑	....
284	11+2 ←	....	....	15+9 ↑	5+6 ↑	....	1+3 ←	15+2 ←	13+0 ↑	....	10−11↑	23−10←	....
285	13+3 ↑	....	....	21+8 ←	1+5 ←	20+4 ←	5+1 ↑	19+0 ↑	4−1 ←	18−2 ←	12−9 ←	....	....
286	14+4 ←	....	0−12↑	....	15+3 ↑	....	10−1 ←	....	8−3 ←	22−4 ↑	1−8 ←	15−7 ↑	....
287	16+5 ↑	....	....	2+7 ↑	10+2 ↑	....	0−2 ←	14−3 ↑	12−5 ↑	....	4−6 ↑	17−5 ↑	....
288	18+6 ↑	....	16+11←	8+6 ←	6+1 ←	....	4−4 ↑	19−5 ←	3−6 ←	17−7 ←	6−4 ←	19−3 ←	....
289	19+7 ←	....	....	13+5 ↑	1+0 ←	20−1 ←	9−6 ←	23−7 ↑	7−8 ←	21−9 ↑	8−2 ←	22−1 ↑	....
290	21+8 ↑	....	....	19+4 ←	15−2 ↑	....	13−8 ↑	....	11−10↑	....	11−0 ↑	....	....
291	22+9 ←	....	7+10↑	....	10−3 ↑	....	4−9 ←	18−10←	1−11↑	16−12←	0+1 ↑	13+2 ←	....
292	....	....	....	0+3 ↑	6−4 ←	....	8−11↑	22−12↑	6+11←	20+10↑	2+3 ←	15+4 ←	....
293	0+10←	....	23+9 ←	6+2 ↑	1−5 ←	20−6 ↑	13+11←	....	10+9 ↑	....	5+5 ↑	18+6 ↑	....
294	2+11↑	....	....	12+1 ←	15−7 ↑	....	3+10←	17+9 ↑	0+8 ↑	15+7 ←	7+7 ↑	20+8 ←	....
295	3+12←	....	....	17+0 ↑	11−8 ←	....	7+8 ↑	22+7 ←	5+6 ←	19+5 ↑	9+9 ←	22+10←	....
296	5−11↑	....	15+8 ←	23−1 ←	6−9 ←	....	12+6 ←	....	9+4 ↑	23+3 ↑	12+11↑	....	....
297	6−10←	19−4 ↑	....	....	1−10↑	20−11↑	2+5 ↑	16+4 ↑	14+2 ←	....	1+12↑	14−11↑	19+4 ←
298	8−9 ←	....	....	4−2 ↑	15−12↑	....	7+3 ←	21+2 ←	4+1 ←	18+0 ←	3−10←	16−9 ←	....
299	10−8 ↑	....	6+7 ↑	10−3 ←	11+11←	....	11+1 ↑	....	8−1 ↑	22−2 ↑	5−8 ←	19−7 ↑	....
300	11−7 ←	....	....	15−4 ↑	6+10←	....	1+0 ↑	16−1 ←	13−3 ←	....	8−6 ↑	21−5 ↑	....
301	13−6 ↑	....	22+6 ←	21−5 ←	1+9 ↑	20+8 ↑	6−2 ←	20−3 ↑	3−4 ←	17−5 ←	10−4 ←	23−3 ←	....
302	15−5 ↑	....	....	....	16+7 ←	....	10−4 ↑	....	7−6 ↑	21−7 ↑	12−2 ←	....	....
303	16−4 ←	....	....	2−6 ↑	11+6 ←	....	1−5 ←	15−6 ↑	11−8 ↑	....	2−1 ↑	15−0 ↑	....
304	18−3 ↑	....	13+5 ↑	8−7 ←	6+5 ↑	....	5−7 ↑	20−8 ←	2−9 ←	16−10←	4+1 ↑	17+2 ←	....
305	19−2 ←	....	....	13−8 ↑	1+4 ↑	21+3 ←	10−9 ←	....	6−11↑	20−12↑	6+3 ←	19+4 ←	....
306	21−1 ←	....	....	19−9 ←	16+2 ←	....	0−10↑	14−11↑	10+11↑	....	9+5 ↑	22+6 ↑	....
307	23−0 ↑	....	5+4 ↑	....	11+1 ←	....	5−12←	19+11←	1+10←	15+9 ←	11+7 ↑	....	....
308	....	....	....	0−10↑	6+0 ↑	....	9+10↑	23+9 ↑	5+8 ↑	19+7 ↑	0+8 ←	13+9 ←	....
309	0+1 ←	....	21+3 ←	6−11←	1−1 ↑	21−2 ←	14+8 ←	....	9+6 ↑	....	2+10←	16+11↑	....
310	2+2 ↑	....	....	11−12↑	16−3 ←	....	4+7 ←	18+6 ↑	0+5 ←	14+4 ←	5+12↑	18−11↑	....
311	4+3 ↑	....	....	17+11←	11−4 ↑	....	8+5 ↑	23+4 ←	4+3 ←	18+2 ↑	7−10←	20−9 ←	....
312	5+4 ←	....	12+2 ↑	23+10←	6−5 ↑	....	13+3 ←	....	8+1 ↑	23+0 ←	9−8 ←	23−7 ↑	....
313	7+5 ↑	0−3 ↑	....	....	2−6 ←	21−7 ←	3+2 ↑	17+1 ↑	13−1 ←	....	12−6 ↑	....	0+3 ←
314	8+6 ←	....	....	4+9 ↑	16−8 ↑	....	8+0 ←	22−1 ←	3−2 ←	17−3 ↑	1−5 ↑	14−4 ←	....
315	10+7 ←	....	4+1 ←	10+8 ←	11−9 ↑	....	12−2 ↑	....	7−4 ↑	22−5 ←	3−3 ←	17−2 ↑	....
316	12+8 ↑	....	....	15+7 ↑	6−10↑	....	2−3 ↑	17−4 ←	12−6 ←	....	6−1 ↑	19−0 ↑	....
317	13+9 ←	....	20+0 ←	21+6 ←	2−11←	21−12←	7−5 ←	21−6 ↑	2−7 ←	16−8 ↑	8+1 ←	21+2 ←	....
318	15+10↑	....	....	....	16+11↑	....	11−7 ↑	....	6−9 ↑	20−10↑	10+3 ←	....	....
319	17+11↑	....	....	2+5 ↑	11+10↑	....	2−8 ←	16−9 ←	11−11←	....	0+4 ↑	13+5 ↑	....
320	18+12←	....	11−1 ↑	8+4 ←	7+9 ←	....	6−10↑	20−11↑	1−12←	15+11↑	2+6 ↑	15+7 ←	....

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	J	K	L	M	N	2N	O	OO	P
321	20−11 ↑	....	....	13+3 ↑	2+8 ← 21+7 ↑	11−12 ←	5+10 ↑ 19+9 ↑	4+8 ← 17+9 ←	....
322	21−10 ←	....	....	19+2 ←	16+6 ↑	1+11 ← 15+10 ↑	10+8 ←	7+10 ↑ 20+11 ↑	....
323	23−9 ←	....	3−2 ←	....	12+5 ←	5+9 ↑ 20+8 ←	0+7 ← 14+6 ↑	9+12 ↑ 22−11 ←	....
324	....	....	....	0+1 ↑	7+4 ←	10+7 ←	4+5 ↑ 18+4 ↑	11−10 ←	....
325	1−8 ↑	....	18−3 ↑	6+0 ←	2+3 ↑ 21+2 ↑	0+6 ↑ 14+5 ↑	9+3 ← 23+2 ←	0−9 ← 14−8 ↑	....
326	2−7 ←	....	....	11−1 ↑	16+1 ↑	5+4 ← 19+3 ←	13+1 ←	3−7 ↑ 16−6 ↑	....
327	4−6 ↑	....	....	17−2 ←	12+0 ←	9+2 ↑ 23+1 ↑	3+0 ↑ 17−1 ↑	5−5 ← 18−4 ←	....
328	6−5 ↑	5−2 ←	10−4 ↑	22−3 ↑	7−1 ←	14+0 ←	8−2 ← 22−3 ←	7−3 ← 21−2 ↑	5+2 ↑
329	7−4 ←	....	....	....	2−2 ↑ 21−3 ↑	4−1 ← 18−2 ↑	12−4 ←	10−1 ↑ 23−0 ↑	....
330	9−3 ↑	....	....	4−4 ←	17−4 ←	8−3 ↑ 23−4 ←	2−5 ↑ 16−6 ↑	12+1 ←	....
331	10−2 ←	....	2−5 ←	9−5 ↑	12−5 ←	13−5 ←	7−7 ← 21−8 ←	1+2 ← 14+3 ←	....
332	12−1 ←	....	....	15−6 ←	7−6 ↑	3−6 ↑ 17−7 ↑	11−9 ←	4+4 ↑ 17+5 ↑	....
333	14−0 ↑	....	17−6 ↑	21−7 ←	2−7 ↑ 21−8 ↑	8−8 ← 22−9 ←	1−10 ↑ 15−11 ↑	6+6 ↑ 19+7 ←	....
334	15+1 ←	....	....	....	17−9 ←	12−10 ↑	5−12 ↑ 20+11 ←	8+8 ← 21+9 ←	....
335	17+2 ↑	....	....	2−8 ↑	12−10 ←	2−11 ↑ 17−12 ←	10+10 ←	11+10 ↑	....
336	18+3 ←	....	9−7 ←	8−9 ←	7−11 ↑	7+11 ← 21+10 ↑	0+9 ↑ 14+8 ↑	0+11 ↑ 13+12 ↑	....
337	20+4 ←	....	....	13−10 ↑	2−12 ↑ 22+11 ←	11+9 ↑	4+7 ↑ 19+6 ←	2−11 ← 15−10 ←	....
338	22+5 ↑	....	....	19−11 ←	17+10 ←	2+8 ← 16+7 ↑	9+5 ← 23+4 ←	4−9 ← 18−8 ↑	....
339	23+6 ←	....	1−8 ←	....	12+9 ↑	6+6 ↑ 21+5 ←	13+3 ↑	7−7 ↑ 20−6 ↑	....
340	....	....	....	0−12 ↑	7+8 ↑	11+4 ←	3+2 ↑ 18+1 ←	9−5 ← 22−4 ←	....
341	1+7 ↑	....	16−9 ↑	6+11 ←	3+7 ← 22+6 ←	1+3 ↑ 15+2 ↑	8+0 ← 22−1 ←	11−3 ←	....
342	3+8 ↑	....	....	11+10 ↑	17+5 ↑	6+1 ← 20+0 ←	12−2 ↑	1−2 ↑ 14−1 ↑	....
343	4+9 ←	10−1 ←	....	17+9 ←	12+4 ↑	10−1 ↑	2−3 ↑ 17−4 ←	3−0 ↑ 16+1 ←	10+1 ↑
344	6+10 ↑	....	8−10 ←	22+8 ↑	7+3 ↑	0−2 ↑ 15−3 ←	7−5 ← 21−6 ←	5+2 ← 18+3 ←	....
345	7+11 ←	....	....	....	3+2 ← 22+1 ←	5−4 ← 19−5 ↑	11−7 ↑	8+4 ↑ 21+5 ↑	....
346	9+12 ←	....	23−11 ↑	4+7 ←	17+0 ↑	9−6 ↑	1−8 ↑ 15−9 ↑	10+6 ↑ 23+7 ←	....
347	11−11 ↑	....	....	9+6 ↑	12−1 ↑	0−7 ← 14−8 ←	6−10 ← 20−11 ←	12+8 ←	....
348	12−10 ←	....	....	15+5 ←	8−2 ←	4−9 ↑ 18−10 ↑	10−12 ↑	1+9 ← 15+10 ↑	....
349	14−9 ↑	....	15−12 ↑	20+4 ↑	3−3 ← 22−4 ↑	9−11 ← 23−12 ←	0+11 ↑ 14+10 ↑	4+11 ↑ 17+12 ↑	....
350	16−8 ↑	....	....	....	17−5 ↑	13+11 ↑	5+9 ← 19+8 ←	6−11 ← 19−10 ←	....
351	17−7 ←	....	....	2+3 ←	13−6 ←	3+10 ↑ 18+9 ←	9+7 ↑ 23+6 ↑	8−9 ← 22−8 ↑	....
352	19−6 ↑	....	7+11 ←	7+2 ↑	8−7 ←	8+8 ← 22+7 ↑	13+5 ↑	11−7 ↑	....
353	20−5 ←	....	....	13+1 ↑	3−8 ← 22−9 ↑	12+6 ↑	4+4 ← 18+3 ←	0−6 ↑ 13−5 ←	....
354	22−4 ←	....	22+10 ↑	19+0 ←	17−10 ↑	3+5 ← 17+4 ←	8+2 ← 22+1 ↑	2−4 ← 15−3 ←	....
355	....	....	....	....	13−11 ←	7+3 ↑ 21+2 ↑	12+0 ↑	5−2 ↑ 18−1 ↑	....
356	0−3 ↑	....	....	0−1 ↑	8−12 ←	12+1 ←	3−1 ← 17−2 ←	7−0 ↑ 20+1 ←	....
357	1−2 ←	....	14+9 ←	6−2 ←	3+11 ↑ 22+10 ↑	2+0 ← 16−1 ↑	7−3 ← 21−4 ↑	9+2 ← 22+3 ←	....
358	3−1 ↑	16−0 ↑	....	11−3 ↑	18+9 ←	6−2 ↑ 21−3 ←	11−5 ↑	12+4 ↑	16+0 ←
359	5−0 ↑	....	....	17−4 ←	13+8 ←	11−4 ←	2−6 ← 16−7 ←	1+5 ↑ 14+6 ↑	....
360	6+1 ←	....	6+8 ←	22−5 ↑	8+7 ↑	1−5 ↑ 15−6 ↑	6−8 ← 20−9 ↑	3+7 ← 16+8 ←	....
361	8+2 ↑	....	....	....	3+6 ↑ 22+5 ↑	6−7 ← 20−8 ←	10−10 ↑	5+9 ← 19+10 ↑	....
362	9+3 ←	....	21+7 ↑	4−6 ←	18+4 ←	10−9 ↑	0−11 ↑ 15−12 ←	8+11 ↑ 21+12 ↑	....
363	11+4 ←	....	....	9−7 ↑	13+3 ←	0−10 ↑ 15−11 ←	5+11 ← 19+10 ↑	10−11 ← 23−10 ←	....
364	13+5 ↑	....	....	15−8 ←	8+2 ↑	5−12 ← 19+11 ↑	9+9 ↑ 23+8 ↑	12−9 ←	....
365	14+6 ←	....	13+6 ←	20−9 ↑	3+1 ↑ 23+0 ←	9+10 ↑	14+7 ←	2−8 ↑ 15−7 ↑	....
366	16+7 ↑	....	....	....	18−1 ←	0+9 ← 14+8 ←	4+6 ← 18+5 ←	4−6 ↑ 17−5 ←	....
367	18+8 ↑	....	....	2−10 ←	13−2 ↑	4+7 ↑ 18+6 ↑	8+4 ↑ 22+3 ↑	6−4 ← 19−3 ←	....
368	19+9 ←	....	4+5 ↑	7−11 ↑	8−3 ↑	9+5 ← 23+4 ←	13+2 ←	9−2 ↑ 22−1 ↑	....
369	21+10 ↑	....	....	13−12 ←	4−4 ← 23−5 ←	13+3 ↑	3+1 ← 17+0 ←	11−0 ↑	....
370	22+11 ←	....	20+4 ↑	18+11 ↑	18−6 ←	3+2 ↑ 18+1 ←	7−1 ↑ 21−2 ↑	0+1 ← 13+2 ←	....
371	....	....	....	....	13−7 ↑	8+0 ← 22−1 ↑	12−3 ←	2+3 ← 16+4 ↑	....

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series	Q			2 Q				R	T	λ	μ or 2 MS	
1	5−1 ↑	15−2 ←		4−1 ←	11−2 ←	18−3 ←					8−1 ←	23−2 ←
2	0−3 ←	9−4 ↑	19−5 ←	1−4 ←	8−5 ←	15−6 ←	22−7 ←			4−1 ↑	13−3 ↑	
3	4−6 ↑	13−7 ↑	23−8 ←	5−8 ←	12−9 ←	19−10 ←					4−4 ↑	19−5 ←
4	8−9 ↑	17−10 ↑		2−11 ←	9−12 ←	16+11 ←	23+10 ←			11−2 ↑	10−6 ←	
5	3−11 ←	12−12 ↑	22+11 ←	6+9 ←	13+8 ←	20+7 ←					0−7 ↑	15−8 ↑
6	7+10 ←	16+9 ↑		3+6 ←	10+5 ←	17+4 ←				18−3 ↑	6−9 ↑	21−10 ←
7	2+8 ←	11+7 ↑	20+6 ↑	0+3 ←	7+2 ←	14+1 ←	21+0 ←				12−11 ←	
8	6+5 ←	15+4 ↑		4−1 ←	11−2 ←	18−3 ←					2−12 ↑	17+11 ↑
9	1+3 ←	10+2 ←	19+1 ↑	1−4 ←	8−5 ←	15−6 ←	22−7 ←			1−4 ↑	8+10 ←	23+9 ←
10	5+0 ←	14−1 ←	23−2 ↑	5−8 ←	12−9 ←	19−10 ←					13+8 ↑	
11	9−3 ←	18−4 ↑		2−11 ←	9−12 ←	16+11 ←	23+10 ←			8−5 ↑	4+7 ↑	19+6 ←
12	3−5 ↑	13−6 ←	22−7 ↑	6+9 ←	13+8 ←	20+7 ←					10+5 ←	
13	8−8 ←	17−9 ←		3+6 ←	10+5 ←	17+4 ←				16−6 ←	0+4 ↑	15+3 ↑
14	2−10 ↑	12−11 ←	21−12 ↑	0+3 ←	7+2 ←	14+1 ←	21+0 ←				6+2 ←	21+1 ←
15	6+11 ↑	16+10 ←		4−1 ←	11−2 ←	18−3 ←				23−7 ←	11+0 ↑	
16	1+9 ↑	11+8 ←	20+7 ←	1−4 ←	8−5 ←	15−6 ←	21−7 ↑	6+1 ↑	6−1 ←		2−1 ↑	17−2 ↑
17	5+6 ↑	15+5 ←		4−8 ↑	11−9 ↑	18−10 ↑					8−3 ←	23−4 ←
18	0+4 ←	9+3 ↑	19+2 ←	1−11 ↑	8−12 ↑	15+11 ↑	22+10 ↑			6−8 ←	13−5 ↑	
19	4+1 ↑	13+0 ↑	23−1 ←	5+9 ↑	12+8 ↑	19+7 ↑					4−6 ↑	19−7 ←
20	8−2 ↑	18−3 ←		2+6 ↑	9+5 ↑	16+4 ↑	23+3 ↑			13−9 ←	10−8 ←	
21	3−4 ←	12−5 ↑	22−6 ←	6+2 ↑	13+1 ↑	20+0 ↑					0−9 ↑	15−10 ↑
22	7−7 ↑	16−8 ↑		3−1 ↑	10−2 ↑	17−3 ↑				20−10 ↑	6−11 ←	21−12 ←
23	2−9 ←	11−10 ↑	20−11 ↑	0−4 ↑	7−5 ↑	14−6 ↑	21−7 ↑				11+11 ↑	
24	6−12 ←	15+11 ↑		4−8 ↑	11−9 ↑	18−10 ↑					2+10 ↑	17+9 ←
25	1+10 ←	10+9 ←	19+8 ↑	1−11 ↑	8−12 ↑	15+11 ↑	22+10 ↑			3−11 ↑	8+8 ←	22+7 ↑
26	5+7 ←	14+6 ↑	23+5 ↑	5+9 ↑	12+8 ↑	19+7 ↑					13+6 ↑	
27	9+4 ←	18+3 ↑		2+6 ↑	9+5 ↑	16+4 ↑	23+3 ↑			10−12 ↑	4+5 ↑	19+4 ←
28	4+2 ←	13+1 ←	22+0 ↑	6+2 ↑	13+1 ↑	20+0 ↑					10+3 ←	
29	8−1 ←	17−2 ←		3−1 ↑	10−2 ↑	17−3 ↑				17+11 ↑	0+2 ↑	15+1 ↑
30	2−3 ↑	12−4 ←	21−5 ↑	0−4 ↑	7−5 ↑	14−6 ↑	21−7 ↑				6+0 ←	21−1 ←
31	6−6 ↑	16−7 ←		4−8 ↑	11−9 ↑	18−10 ↑					11−2 ↑	
32	1−8 ↑	11−9 ←	20−10 ←	1−11 ↑	8−12 ←	15+11 ←	22+10 ←			0+10 ↑	2−3 ↑	17−4 ←
33	5−11 ↑	15−12 ←		5+9 ←	12+8 ←	19+7 ←					8−5 ←	22−6 ↑
34	0+11 ↑	9+10 ↑	19+9 ←	2+6 ←	9+5 ←	16+4 ←	23+3 ←			8+9 ←	13−7 ↑	
35	4+8 ↑	13+7 ↑	23+6 ←	6+2 ←	13+1 ←	20+0 ←					4−8 ←	19−9 ←
36	8+5 ↑	18+4 ←		3−1 ←	10−2 ←	17−3 ←				15+8 ←	10−10 ←	
37	3+3 ←	12+2 ↑	22+1 ←	0−4 ←	7−5 ←	14−6 ←	21−7 ←				0−11 ↑	15−12 ↑
38	7+0 ↑	16−1 ↑		4−8 ←	11−9 ←	18−10 ←				22+7 ←	6+11 ←	21+10 ←
39	2−2 ←	11−3 ↑	21−4 ←	1−11 ←	8−12 ←	15+11 ←	22+10 ←				11+9 ↑	
40	6−5 ←	15−6 ↑		5+9 ←	12+8 ←	19+7 ←					2+8 ↑	17+7 ←
41	1−7 ←	10−8 ↑	19−9 ↑	2+6 ←	9+5 ←	16+4 ←	23+3 ←			5+6 ←	8+6 ←	22+5 ↑
42	5−10 ←	14−11 ↑	23−12 ↑	6+2 ←	13+1 ←	20+0 ←					13+4 ↑	
43	9+11 ←	18+10 ↑		3−1 ←	10−2 ←	17−3 ←				12+5 ↑	4+3 ←	19+2 ←
44	4+9 ←	13+8 ←	22+7 ↑	0−4 ←	7−5 ←	14−6 ←	21−7 ←				9+1 ↑	
45	8+6 ←	17+5 ↑		4−8 ←	11−9 ←	18−10 ←				19+4 ↑	0+0 ↑	15−1 ←
46	2+4 ↑	12+3 ←	21+2 ↑	1−11 ←	8−12 ←	15+11 ←	22+10 ←	16+2 ←	16−2 ↑		6−2 ←	21−3 ←
47	7+1 ←	16+0 ←		5+9 ←	12+8 ←	19+7 ←					11−4 ↑	
48	1−1 ↑	11−2 ←	20−3 ←	1+6 ↑	8+5 ↑	15+4 ↑	22+3 ↑			2+3 ↑	2−5 ↑	17−6 ←
49	5−4 ↑	15−5 ←		5+2 ↑	12+1 ↑	19+0 ↑					8−7 ←	22−8 ↑
50	0−6 ↑	9−7 ↑	19−8 ←	2−1 ↑	9−2 ↑	16−3 ↑	23−4 ↑			9+2 ↑	13−9 ↑	
51	4−9 ↑	14−10 ←	23−11 ←	6−5 ↑	13−6 ↑	20−7 ↑					4−10 ←	19−11 ←
52	8−12 ↑	18+11 ←		3−8 ↑	10−9 ↑	17−10 ↑				16+1 ↑	9−12 ↑	
53	3+10 ↑	12+9 ↑	22+8 ←	0−11 ↑	7−12 ↑	14+11 ↑	21+10 ↑				0+11 ↑	15+10 ←
54	7+7 ↑	16+6 ↑		4+9 ↑	11+8 ↑	18+7 ↑					6+9 ←	20+8 ↑
55	2+5 ←	11+4 ↑	21+3 ←	1+6 ↑	8+5 ↑	15+4 ↑	22+3 ↑			0+0 ←	11+7 ↑	
56	6+2 ←	15+1 ↑		5+2 ↑	12+1 ↑	19+0 ↑					2+6 ←	17+5 ←
57	1+0 ←	10−1 ↑	19−2 ↑	2−1 ↑	9−2 ↑	16−3 ↑	23−4 ↑			7−1 ←	8+4 ←	22+3 ↑
58	5−3 ←	14−4 ↑		6−5 ↑	13−6 ↑	20−7 ↑					13+2 ↑	
59	0−5 ←	9−6 ←	18−7 ↑	3−8 ↑	10−9 ↑	17−10 ↑				14−2 ←	4+1 ←	19+0 ←
60	4−8 ←	13−9 ←	22−10 ↑	0−11 ↑	7−12 ↑	14+11 ↑	21+10 ↑				9−1 ↑	
61	8−11 ←	17−12 ↑		4+9 ↑	11+8 ↑	18+7 ↑				21−3 ←	0−2 ↑	15−3 ←
62	2+11 ↑	12+10 ←	21+9 ↑	1+6 ↑	8+5 ↑	15+4 ↑	22+3 ↑				6−4 ←	20−5 ↑
63	7+8 ←	16+7 ←		5+2 ↑	12+1 ↑	19+0 ←					11−6 ↑	
64	1+6 ↑	11+5 ←	20+4 ↑	2−1 ←	9−2 ←	16−3 ←	23−4 ←			4−4 ↑	2−7 ←	17−8 ←
65	5+3 ↑	15+2 ←		6−5 ←	13−6 ←	20−7 ←					7−9 ↑	22−10 ↑
66	0+1 ↑	9+0 ↑	19−1 ←	3−8 ←	10−9 ←	17−10 ←				11−5 ↑	13−11 ↑	
67	4−2 ↑	14−3 ←	23−4 ←	0−11 ←	7−12 ←	14+11 ←	21+10 ←				4−12 ←	19+11 ←
68	8−5 ↑	18−6 ←		4+9 ←	11+8 ←	18+7 ←				18−6 ↑	9+10 ↑	
69	3−7 ↑	12−8 ↑	22−9 ←	1+6 ←	8+5 ←	15+4 ←	22+3 ←				0+9 ↑	15+8 ←
70	7−10 ↑	17−11 ←		5+2 ←	12+1 ←	19+0 ←					6+7 ←	20+6 ↑
71	2−12 ←	11+11 ↑	21+10 ←	2−1 ←	9−2 ←	16−3 ←	23−4 ←			1−7 ↑	11+5 ↑	
72	6+9 ↑	15+8 ↑		6−5 ←	13−6 ←	20−7 ←					2+4 ←	17+3 ←
73	1+7 ←	10+6 ↑	19+5 ↑	3−8 ←	10−9 ←	17−10 ←				8−8 ↑	7+2 ↑	22+1 ↑
74	5+4 ←	14+3 ↑		0−11 ←	7−12 ←	14+11 ←	21+10 ←				13+0 ←	
75	0+2 ←	9+1 ←	18+0 ↑	4+9 ←	11+8 ←	18+7 ←				16−9 ←	4−1 ←	18−2 ↑
76	4−1 ←	13−2 ↑	22−3 ↑	1+6 ←	8+5 ←	15+4 ←	22+3 ←				9−3 ↑	
77	8−4 ←	17−5 ↑		5+2 ←	12+1 ←	19+0 ←		3+3 ↑	3−3 ←	23−10 ←	0−4 ↑	15−5 ←
78	3−6 ←	12−7 ←	21−8 ↑	2−1 ←	9−2 ←	16−3 ←	23−4 ←				6−6 ←	20−7 ↑
79	7−9 ←	16−10 ←		5−5 ↑	12−6 ↑	19−7 ↑					11−8 ↑	
80	1−11 ↑	11−12 ←	20+11 ↑	2−8 ↑	9−9 ↑	16−10 ↑	23−11 ↑			6−11 ←	2−9 ←	17−10 ←

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series	Q			2 Q				R	T	λ	μ or 2 MS	
81	5+10 ↑	15+9 ←		6−12 ↑	13+11 ↑	20+10 ↑					7−11 ↑	22−12 ↑
82	0+8 ↑	10+7 ←	19+6 ←	3+9 ↑	10+8 ↑	17+7 ↑				13−12 ←	13+11 ←	
83	4+5 ↑	14+4 ←	23+3 ↑	0+6 ↑	7+5 ↑	14+4 ↑	21+3 ↑				4+10 ←	18+9 ↑
84	8+2 ↑	18+1 ←		4+2 ↑	11+1 ↑	18+0 ↑				20+11 ←	9+8 ↑	
85	3+0 ↑	12−1 ↑	22−2 ←	1−1 ↑	8−2 ↑	15−3 ↑	22−4 ↑				0+7 ←	15+6 ←
86	7−3 ↑	17−4 ←		5−5 ↑	12−6 ↑	19−7 ↑					5+5 ↑	20+4 ↑
87	2−5 ←	11−6 ↑	21−7 ←	2−8 ↑	9−9 ↑	16−10 ↑	23−11 ↑			3+10 ↑	11+3 ↑	
88	6−8 ↑	15−9 ↑		6−12 ↑	13+11 ↑	20+10 ↑					2+2 ←	17+1 ←
89	1−10 ←	10−11 ↑	20−12 ←	3+9 ↑	10+8 ↑	17+7 ↑				10+9 ↑	7+0 ↑	22−1 ↑
90	5+11 ←	14+10 ↑		0+6 ↑	7+5 ↑	14+4 ↑	21+3 ↑				13−2 ←	
91	0+9 ←	9+8 ←	18+7 ↑	4+2 ↑	11+1 ↑	18+0 ↑				17+8 ↑	4−3 ←	18−4 ↑
92	4+6 ←	13+5 ↑	22+4 ↑	1−1 ↑	8−2 ↑	15−3 ↑	22−4 ↑				9−5 ↑	
93	8+3 ←	17+2 ↑		5−5 ↑	12−6 ↑	19−7 ↑					0−6 ←	15−7 ←
94	3+1 ←	12+0 ←	21−1 ↑	2−8 ↑	9−9 ↑	16−10 ↑	23−11 ←			0+7 ↑	5−8 ↑	20−9 ↑
95	7−2 ←	16−3 ↑		6−12 ←	13+11 ←	20+10 ←					11−10 ←	
96	1−4 ↑	11−5 ←	20−6 ↑	3+9 ←	10+8 ←	17+7 ←				8+6 ←	2−11 ←	17−12 ←
97	6−7 ←	15−8 ←		0+6 ←	7+5 ←	14+4 ←	21+3 ←				7+11 ↑	22+10 ↑
98	0−9 ↑	10−10 ←1	9−11 ←	4+2 ←	11+1 ←	18+0 ←				15+5 ←	13+9 ←	
99	4−12 ↑	14+11 ←2	3+10 ↑	1−1 ←	8−2 ←	15−3 ←	22−4 ←				4+8 ←	18+7 ↑
100	8+9 ↑	18+8 ←		5−5 ←	12−6 ←	19−7 ←				22+4 ←	9+6 ↑	
101	3+7 ↑	13+6 ←	22+5 ←	2−8 ←	9−9 ←	16−10 ←	23−11 ←				0+5 ←	15+4 ←
102	7+4 ↑	17+3 ←		6−12 ←	13+11 ←	20+10 ←					5+3 ↑	20+2 ↑
103	2+2 ↑	11+1 ↑	21+0 ←	3+9 ←	10+8 ←	17+7 ←				5+3 ←	11+1 ←	
104	6−1 ↑	15−2 ↑		0+6 ←	7+5 ←	14+4 ←	21+3 ←				2+0 ←	16−1 ↑
105	1−3 ←	10−4 ↑	20−5 ←	4+2 ←	11+1 ←	18+0 ←				12+2 ←	7−2 ↑	22−3 ←
106	5−6 ←	14−7 ↑		1−1 ←	8−2 ←	15−3 ←	22−4 ←				13−4 ←	
107	0−8 ←	9−9 ↑	18−10 ↑	5−5 ←	12−6 ←	19−7 ←		13+4 ←	13−4 ↑	19+1 ↑	4−5 ←	18−6 ↑
108	4−11 ←	13−12 ↑	23+11 ←	2−8 ←	9−9 ←	16−10 ←	23−11 ←				9−7 ↑	
109	8+10 ←	17+9 ↑		6−12 ←	13+11 ←	20+10 ←					0−8 ←	15−9 ←
110	3+8 ←	12+7 ←	21+6 ↑	3+9 ←	10+8 ←	16+7 ↑	23+6 ↑			2+0 ↑	5−10 ↑	20−11 ↑
111	7+5 ←	16+4 ↑		6+5 ↑	13+4 ↑	20+3 ↑					11−12 ←	
112	1+3 ↑	11+2 ←	20+1 ↑	3+2 ↑	10+1 ↑	17+0 ↑				9−1 ↑	2+11 ←	16+10 ↑
113	6+0 ←	15−1 ←		0−1 ↑	7−2 ↑	14−3 ↑	21−4 ↑				7+9 ↑	22+8 ←
114	0−2 ↑	10−3 ←	19−4 ↑	4−5 ↑	11−6 ↑	18−7 ↑				16−2 ↑	13+7 ←	
115	4−5 ↑	14−6 ←	23−7 ↑	1−8 ↑	8−9 ↑	15−10 ↑	22−11 ↑				3+6 ↑	18+5 ↑
116	8−8 ↑	18−9 ←		5−12 ↑	12+11 ↑	19+10 ↑					9+4 ←	
117	3−10 ↑	13−11 ←	22−12 ←	2+9 ↑	9+8 ↑	16+7 ↑	23+6 ↑			0−3 ←	0+3 ←	15+2 ←
118	7+11 ↑	17+10 ←		6+5 ↑	13+4 ↑	20+3 ↑					5+1 ↑	20+0 ↑
119	2+9 ↑	11+8 ↑	21+7 ←	3+2 ↑	10+1 ↑	17+0 ↑				7−4 ←	11−1 ←	
120	6+6 ↑	16+5 ←		0−1 ↑	7−2 ↑	14−3 ↑	21−4 ↑				2−2 ←	16−3 ↑
121	1+4 ←	10+3 ↑	20+2 ←	4−5 ↑	11−6 ↑	18−7 ↑				14−5 ←	7−4 ↑	22−5 ←
122	5+1 ↑	14+0 ↑		1−8 ↑	8−9 ↑	15−10 ↑	22−11 ↑				13−6 ←	
123	0−1 ←	9−2 ↑	18−3 ↑	5−12 ↑	12+11 ↑	19+10 ↑				21−6 ←	3−7 ↑	18−8 ↑
124	4−4 ←	13−5 ↑	23−6 ←	2+9 ↑	9+8 ↑	16+7 ↑	23+6 ↑				9−9 ←	
125	8−7 ←	17−8 ↑		6+5 ↑	13+4 ↑	20+3 ↑					0−10 ←	14−11 ↑
126	3−9 ←	12−10 ↑	21−11 ↑	3+2 ←	10+1 ←	17+0 ←				4−7 ←	5−12 ↑	20+11 ↑
127	7−12 ←	16+11 ↑		0−1 ←	7−2 ←	14−3 ←	21−4 ←				11+10 ←	
128	2+10 ←	11+9 ←	20+8 ↑	4−5 ←	11−6 ←	18−7 ←				11−8 ↑	2+9 ←	16+8 ↑
129	6+7 ←	15+6 ←		1−8 ←	8−9 ←	15−10 ←	22−11 ←				7+7 ↑	22+6 ←
130	0+5 ↑	10+4 ←	19+3 ↑	5−12 ←	12+11 ←	19+10 ←				18−9 ↑	13+5 ←	
131	4+2 ↑	14+1 ←	23+0 ↑	2+9 ←	9+8 ←	16+7 ←	23+6 ←				3+4 ↑	18+3 ↑
132	9−1 ←	18−2 ←		6+5 ←	13+4 ←	20+3 ←					9+2 ←	
133	3−3 ↑	13−4 ←	22−5 ↑	3+2 ←	10+1 ←	17+0 ←				1−10 ↑	0+1 ←	14+0 ↑
134	7−6 ↑	17−7 ←		0−1 ←	7−2 ←	14−3 ←	21−4 ←				5−1 ↑	20−2 ←
135	2−8 ↑	11−9 ↑	21−10 ←	4−5 ←	11−6 ←	18−7 ←				8−11 ↑	11−3 ←	
136	6−11 ↑	16−12 ←		1−8 ←	8−9 ←	15−10 ←	22−11 ←				1−4 ↑	16−5 ↑
137	1+11 ←	10+10 ↑	20+9 ←	5−12 ←	12+11 ←	19+10 ←				15−12 ↑	7−6 ↑	22−7 ←
138	5+8 ↑	14+7 ↑		2+9 ←	9+8 ←	16+7 ←	23+6 ←	0+5 ↑	0−5 ←		13−8 ←	
139	0+6 ←	9+5 ↑	19+4 ←	6+5 ←	13+4 ←	20+13 ←				23+11 ←	3−9 ↑	18−10 ↑
140	4+3 ←	13+2 ↑	23+1 ←	3+2 ←	10+1 ←	17+0 ←					9−11 ←	
141	8+0 ←	17−1 ↑		0−1 ←	7−2 ←	14−3 ←	20−4 ↑				0−12 ←	14+11 ↑
142	3−2 ←	12−3 ↑	21−4 ↑	3−5 ↑	10−6 ↑	17−7 ↑				6+10 ←	5+10 ↑	20+9 ←
143	7−5 ←	16−6 ↑		0−8 ↑	7−9 ↑	14−10 ↑	21−11 ↑				11+8 ←	
144	2−7 ←	11−8 ←	20−9 ↑	4−12 ↑	11+11 ↑	18+10 ↑				13+9 ←	1+7 ↑	16+6 ↑
145	6−10 ←	15−11 ↑		1+9 ↑	8+8 ↑	15+7 ↑	22+6 ↑				7+5 ←	22+4 ←
146	0−12 ↑	10+11 ←	19+10 ↑	5+5 ↑	12+4 ↑	19+3 ↑				20+8 ←	12+3 ↑	
147	5+9 ←	14+8 ←	23+7 ↑	2+2 ↑	9+1 ↑	16+0 ↑	23−1 ↑				3+2 ↑	18+1 ↑
148	9+6 ←	18+5 ←		6−2 ↑	13−3 ↑	20−4 ↑					9+0 ←	
149	3+4 ↑	13+3 ←	22+2 ↑	3−5 ↑	10−6 ↑	17−7 ↑				3+7 ↑	0−1 ←	14−2 ↑
150	7+1 ↑	17+0 ←		0−8 ↑	7−9 ↑	14−10 ↑	21−11 ↑				5−3 ↑	20−4 ←
151	2−1 ↑	12−2 ←	21−3 ←	4−12 ↑	11+11 ↑	18+10 ↑				10+6 ↑	11−5 ←	
152	6−4 ↑	16−5 ←		1+9 ↑	8+8 ↑	15+7 ↑	22+6 ↑				1−6 ↑	16−7 ↑
153	1−6 ↑	10−7 ↑	20−8 ←	5+5 ↑	12+4 ↑	19+3 ↑				17+5 ↑	7−8 ←	22−9 ←
154	5−9 ↑	14−10 ↑		2+2 ↑	9+1 ↑	16+0 ↑	23−1 ↑				12−10 ↑	
155	0−11 ←	9−12 ↑	19+11 ←	6−2 ↑	13−3 ↑	20−4 ↑					3−11 ↑	18−12 ←
156	4+10 ←	13+9 ↑	23+8 ←	3−5 ↑	10−6 ↑	17−7 ↑				0+4 ↑	9+11 ←	
157	8+7 ↑	17+6 ↑		0−8 ↑	7−9 ↑	14−10 ←	21−11 ←				0+10 ←	14+9 ↑
158	3+5 ←	12+4 ↑	22+3 ←	4−12 ←	11+11 ←	18+10 ←				7+3 ↑	5+8 ↑	20+7 ←
159	7+2 ←	16+1 ↑		1+9 ←	8+8 ←	15+7 ←	22+6 ←				11+6 ←	
160	2+0 ←	11−1 ←	20−2 ↑	5+5 ←	12+4 ←	19+3 ←				15+2 ←	1+5 ↑	16+4 ↑

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	Q			2 Q				R	T	λ	μ or 2 MS	
161	6−3 ←	15−4 ↑	....	2+2 ←	9+1 ←	16+0 ←	23−1 ←	....	....	....	7+3 ←	22+2 ←
162	0−5 ↑	10−6 ←	19−7 ↑	6−2 ←	13−3 ←	20−4 ←	....	....	....	22+1 ←	12+1 ↑	....
163	5−8 ←	14−9 ←	23−10 ↑	3−5 ←	10−6 ←	17−7 ←	....	....	....	....	3+0 ↑	18−1 ←
164	9−11 ←	18−12 ↑	....	0−8 ←	7−9 ←	14−10 ←	21−11 ←	....	....	....	9−2 ←	23−3 ↑
165	3+11 ↑	13+10 ←	22+9 ↑	4−12 ←	11+11 ←	18+10 ←	....	....	....	5+0 ←	14−4 ↑	....
166	7+8 ↑	17+7 ←	....	1+9 ←	8+8 ←	15+7 ←	22+6 ←	....	....	....	5−5 ←	20−6 ←
167	2+6 ↑	12+5 ←	21+4 ←	5+5 ←	12+4 ←	19+3 ←	....	....	....	12−1 ←	11−7 ←	....
168	6+3 ↑	16+2 ←	....	2+2 ←	9+1 ←	16+0 ←	23−1 ←	10+6 ←	10−6 ↑	....	1−8 ↑	16−9 ↑
169	1+1 ↑	10+0 ↑	20−1 ←	6−2 ←	13−3 ←	20−4 ←	....	....	....	19−2 ↑	7−10 ←	22−11 ←
170	5−2 ↑	15−3 ←	....	3−5 ←	10−6 ←	17−7 ←	....	....	....	....	12−12 ↑	....
171	0−4 ←	9−5 ↑	19−6 ←	0−8 ←	7−9 ←	14−10 ←	21−11 ←	....	....	....	3+11 ↑	18+10 ←
172	4−7 ←	13−8 ↑	23−9 ←	4−12 ←	11+11 ←	18+10 ←	....	....	....	2−3 ↑	9+9 ←	23+8 ↑
173	8−10 ↑	17−11 ↑	....	0+9 ↑	7+8 ↑	14+7 ↑	21+6 ↑	....	....	....	14+7 ↑	....
174	3−12 ←	12+11 ↑	22+10 ←	4+5 ↑	11+4 ↑	18+3 ↑	....	....	....	9−4 ↑	5+6 ←	20+5 ←
175	7+9 ←	16+8 ↑	....	1+2 ↑	8+1 ↑	15+0 ↑	22−1 ↑	....	....	....	10+4 ↑	....
176	2+7 ←	11+6 ↑	20+5 ↑	5−2 ↑	12−3 ↑	19−4 ↑	....	....	....	16−5 ↑	1+3 ↑	16+2 ←
177	6+4 ←	15+3 ↑	....	2−5 ↑	9−6 ↑	16−7 ↑	23−8 ↑	....	....	....	7+1 ←	22+0 ←
178	1+2 ←	10+1 ←	19+0 ↑	6−9 ↑	13−10 ↑	20−11 ↑	....	....	....	23−6 ↑	12−1 ↑	....
179	5−1 ←	14−2 ←	23−3 ↑	3−12 ↑	10+11 ↑	17+10 ↑	....	....	....	....	3−2 ↑	18−3 ←
180	9−4 ←	18−5 ↑	....	0+9 ↑	7+8 ↑	14+7 ↑	21+6 ↑	....	....	....	9−4 ←	23−5 ↑
181	3−6 ↑	13−7 ←	22−8 ↑	4+5 ↑	11+4 ↑	18+3 ↑	....	....	....	7−7 ←	14−6 ↑	....
182	8−9 ←	17−10 ←	....	1+2 ↑	8+1 ↑	15+0 ↑	22−1 ↑	....	....	....	5−7 ←	20−8 ←
183	2−11 ↑	12−12 ←	21+11 ↑	5−2 ↑	12−3 ↑	19−4 ↑	....	....	....	14−8 ←	10−9 ↑	....
184	6+10 ↑	16+9 ←	....	2−5 ↑	9−6 ↑	16−7 ↑	23−8 ↑	....	....	....	1−10 ↑	16−11 ←
185	1+8 ↑	10+7 ↑	20+6 ←	6−9 ↑	13−10 ↑	20−11 ↑	....	....	....	21−9 ←	7−12 ←	21+11 ↑
186	5+5 ↑	15+4 ←	....	3−12 ↑	10+11 ↑	17+10 ↑	....	....	....	....	12+10 ↑	....
187	0+3 ←	9+2 ↑	19+1 ←	0+9 ↑	7+8 ↑	14+7 ↑	21+6 ↑	....	....	....	3+9 ↑	18+8 ←
188	4+0 ↑	13−1 ↑	23−2 ←	4+5 ↑	11+4 ↑	18+3 ←	....	....	....	4−10 ←	9+7 ←	23+6 ↑
189	8−3 ↑	18−4 ←	....	1+2 ←	8+1 ←	15+0 ←	22−1 ←	....	....	....	14+5 ↑	....
190	3−5 ←	12−6 ↑	22−7 ←	5−2 ←	12−3 ←	19−4 ←	....	....	....	11−11 ↑	5+4 ←	20+3 ←
191	7−8 ←	16−9 ↑	....	2−5 ←	9−6 ←	16−7 ←	23−8 ←	....	....	....	10+2 ↑	....
192	2−10 ←	11−11 ↑	20−12 ↑	6−9 ←	13−10 ←	20−11 ←	....	....	....	18−12 ↑	1+1 ↑	16+0 ←
193	6+11 ←	15+10 ↑	....	3−12 ←	10+11 ←	17+10 ←	....	....	....	....	7−1 ←	21−2 ↑
194	1+9 ←	10+8 ←	19+7 ↑	0+9 ←	7+8 ←	14+7 ←	21+6 ←	....	....	....	12−3 ↑	....
195	5+6 ←	14+5 ↑	23+4 ↑	4+5 ←	11+4 ←	18+3 ←	....	....	....	1+11 ↑	3−4 ←	18−5 ←
196	9+3 ←	18+2 ↑	....	1+2 ←	8+1 ←	15+0 ←	22−1 ←	....	....	....	8−6 ↑	23−7 ↑
197	3+1 ↑	13+0 ←	22−1 ↑	5−2 ←	12−3 ←	19−4 ←	....	....	....	8+10 ↑	14−8 ↑	....
198	8−2 ←	17−3 ←	....	2−5 ←	9−6 ←	16−7 ←	23−8 ←	21+7 ↑	21−7 ←	....	5−9 ←	20−10 ←
199	2−4 ↑	12−5 ←	21−6 ↑	6−9 ←	13−10 ←	20−11 ←	....	....	....	15+9 ↑	10−11 ↑	....
200	6−7 ↑	16−8 ←	....	3−12 ←	10+11 ←	17+10 ←	....	....	....	....	1−12 ↑	16+11 ←
201	1−9 ↑	11−10 ←	20−11 ←	0+9 ←	7+8 ←	14+7 ←	21+6 ←	....	....	23+8 ←	7+10 ←	21+9 ↑
202	5−12 ↑	15+11 ←	....	4+5 ←	11+4 ←	18+3 ←	....	....	....	....	12+8 ↑	....
203	0+10 ↑	9+9 ↑	19+8 ←	1+2 ←	8+1 ←	15+0 ←	22−1 ←	....	....	....	3+7 ←	18+6 ←
204	4+7 ↑	13+6 ↑	23+5 ←	4−2 ↑	11+3 ↑	18−4 ↑	....	....	....	6+7 ←	8+5 ↑	23+4 ↑
205	8+4 ↑	18+3 ←	....	1−5 ↑	8−6 ↑	15−7 ↑	22−8 ↑	....	....	....	14+3 ←	....
206	3+2 ←	12+1 ↑	22+0 ←	5−9 ↑	12−10 ↑	19−11 ↑	....	....	....	13+6 ←	5+2 ←	19+1 ↑
207	7−1 ↑	16−2 ↑	....	2−12 ↑	9+11 ↑	16+10 ↑	23+9 ↑	....	....	....	10+0 ↑	....
208	2−3 ←	11−4 ↑	21−5 ←	6+8 ↑	13+7 ↑	20+6 ↑	....	....	....	20+5 ←	1−1 ↑	16−2 ←
209	6−6 ←	15−7 ↑	....	3+5 ↑	10+4 ↑	17+3 ↑	....	....	....	....	7−3 ←	21−4 ↑
210	1−8 ←	10−9 ←	19−10 ←	0+2 ↑	7+1 ↑	14+0 ↑	21−1 ↑	....	....	....	12−5 ↑	....
211	5−11 ←	14−12 ↑	23+11 ↑	4−2 ↑	11−3 ↑	18−4 ↑	....	....	....	3+4 ←	3−6 ←	18−7 ←
212	9+10 ←	18+9 ↑	....	1−5 ↑	8−6 ↑	15−7 ↑	22−8 ↑	....	....	....	8−8 ↑	23−9 ↑
213	4+8 ←	13+7 ←	22+6 ↑	5−9 ↑	12−10 ↑	19−11 ↑	....	....	....	10+3 ↑	14−10 ←	....
214	8+5 ←	17+4 ↑	....	2−12 ↑	9+11 ↑	16+10 ↑	23+9 ↑	....	....	....	5−11 ←	19−12 ↑
215	2+3 ↑	12+2 ←	21+1 ↑	6+8 ↑	13+7 ↑	20+6 ↑	....	....	....	17+2 ↑	10+11 ↑	....
216	6+0 ↑	16−1 ←	....	3+5 ↑	10+4 ↑	17+3 ↑	....	....	....	....	1+10 ←	16+9 ←
217	1−2 ↑	11−3 ←	20−4 ←	0+2 ↑	7+1 ↑	14+0 ↑	21−1 ↑	....	....	....	7+8 ←	21+7 ↑
218	5−5 ↑	15−6 ←	....	4−2 ↑	11−3 ↑	18−4 ↑	....	....	....	0+1 ↑	12+6 ↑	....
219	0−7 ↑	9−8 ↑	19−9 ←	1−5 ↑	8−6 ↑	15−7 ↑	22−8 ←	....	....	....	3+5 ←	18+4 ←
220	4−10 ↑	14−11 ←	23−12 ←	5−9 ←	12−10 ←	19−11 ←	....	....	....	7+0 ↑	8+3 ↑	23+2 ↑
221	8+11 ↑	18+10 ←	....	2−12 ←	9+11 ←	16+10 ←	23+9 ←	....	....	....	14+1 ←	....
222	3+9 ←	12+8 ↑	22+7 ←	6+8 ←	13+7 ←	20+6 ←	....	....	....	15−1 ←	5+0 ←	19−1 ↑
223	7+6 ↑	16+5 ↑	....	3+5 ←	10+4 ←	17+3 ←	....	....	....	....	10−2 ↑	....
224	2+4 ←	11+3 ↑	21+2 ←	0+2 ←	7+1 ←	14+0 ←	21−1 ←	....	....	22−2 ←	1−3 ←	16−4 ←
225	6+1 ←	15+0 ↑	....	4−2 ←	11−3 ←	18−4 ←	....	....	....	....	6−5 ↑	21−6 ↑
226	1−1 ←	10−2 ↑	19−3 ↑	1−5 ←	8−6 ←	15−7 ←	22−8 ←	....	....	....	12−7 ←	....
227	5−4 ←	14−5 ↑	....	5−9 ←	12−10 ←	19−11 ←	....	....	....	5−3 ←	3−8 ←	18−9 ←
228	0−6 ←	9−7 ←	18−8 ↑	2−12 ←	9+11 ←	16+10 ←	23+9 ←	....	....	....	8−10 ↑	23−11 ↑
229	4−9 ←	13−10 ←	22−11 ↑	6+8 ←	13+7 ←	20+6 ←	....	7+8 ←	7−8 ↑	12−4 ←	14−12 ←	....
230	8−12 ←	17+11 ↑	....	3+5 ←	10+4 ←	17+3 ←	....	....	....	....	5+11 ←	19+10 ↑
231	2+10 ↑	12+9 ←	21+8 ↑	0+2 ←	7+1 ←	14+0 ←	21−1 ←	....	....	19−5 ←	10+9 ↑	....
232	7+7 ←	16+6 ←	....	4−2 ←	11−3 ←	18−4 ←	....	....	....	....	1+8 ←	16+7 ←
233	1+5 ↑	11+4 ←	20+3 ↑	1−5 ←	8−6 ←	15−7 ←	22−8 ←	....	....	....	6+6 ↑	21+5 ↑
234	5+2 ↑	15+1 ←	....	5−9 ←	12−10 ←	19−11 ←	....	....	....	2−6 ↑	12+4 ←	....
235	0+0 ↑	9−1 ↑	19−2 ←	2−12 ←	9+11 ←	15+10 ↑	22+9 ↑	....	....	....	3+3 ←	17+2 ↑
236	4−3 ↑	14−4 ←	23−5 ←	5+8 ↑	12+7 ↑	19+6 ↑	....	....	....	9−7 ↑	8+1 ↑	23+0 ←
237	8−6 ↑	18−7 ←	....	2+5 ↑	9+4 ↑	16+3 ↑	23+2 ↑	....	....	....	14−1 ←	....
238	3−8 ↑	12−9 ↑	22−10 ←	6+1 ↑	13+0 ↑	20−1 ↑	....	....	....	16−8 ↑	5−2 ←	19−3 ↑
239	7−11 ↑	17−12 ←	....	3−2 ↑	10−3 ↑	17−4 ↑	....	....	....	....	10−4 ↑	....
240	2+11 ←	11+10 ↑	21+9 ←	0−5 ↑	7−6 ↑	14−7 ↑	21−8 ↑	....	....	23−9 ↑	1−5 ←	16−6 ←

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	Q			2 Q				R	T	λ	μ or 2 MS	
241	6+8 ←	15+7 ↑	....	4−9 ↑	11−10 ↑	18−11 ↑	....	....	....	....	6−7 ↑	21−8 ↑
242	1+6 ←	10+5 ↑	19+4 ↑	1−12 ↑	8+11 ↑	15+10 ↑	22+9 ↑	....	....	....	12−9 ←	....
243	5+3 ←	14+2 ↑	....	5+8 ↑	12+7 ↑	19+6 ↑	....	....	....	7−10 ←	3−10 ←	17−11 ↑
244	0+1 ←	9+0 ←	18−1 ↑	2+5 ↑	9+4 ↑	16+3 ↑	23+2 ↑	....	....	....	8−12 ↑	23+11 ←
245	4−2 ←	13−3 ↑	22−4 ↑	6+1 ↑	13+0 ↑	20−1 ↑	....	....	....	14−11 ←	14+10 ←	....
246	8−5 ←	17−6 ↑	....	3−2 ↑	10−3 ↑	17−4 ↑	....	....	....	....	4+9 ↑	19+8 ↑
247	2−7 ↑	12−8 ←	21−9 ↑	0−5 ↑	7−6 ↑	14−7 ↑	21−8 ↑	....	....	21−12 ←	10+7 ←	....
248	7−10 ←	16−11 ←	....	4−9 ↑	11−10 ↑	18−11 ↑	....	....	....	....	1+6 ←	16+5 ←
249	1−12 ↑	11+11 ←	20+10 ↑	1−12 ↑	8+11 ↑	15+10 ↑	22+9 ↑	....	....	....	6+4 ↑	21+3 ↑
250	5+9 ↑	15+8 ←	....	5+8 ↑	12+7 ↑	19+6 ↑	....	....	....	4+11 ←	12+2 ←	....
251	0+7 ↑	10+6 ←	19+5 ←	2+5 ←	9+4 ←	16+3 ←	23+2 ←	....	....	....	3+1 ←	17+0 ↑
252	4+4 ↑	14+3 ←	23+2 ↑	6+1 ←	13+0 ←	20−1 ←	....	....	....	11+10 ←	8−1 ↑	23−2 ←
253	8+1 ↑	18+0 ←	....	3−2 ←	10−3 ←	17−4 ←	....	....	....	....	14−3 ←	....
254	3−1 ↑	12−2 ↑	22−3 ←	0−5 ←	7−6 ←	14−7 ←	21−8 ←	....	....	18+9 ↑	4−4 ↑	19−5 ↑
255	7−4 ↑	17−5 ←	....	4−9 ←	11−10 ←	18−11 ←	....	....	....	....	10−6 ←	....
256	2−6 ←	11−7 ↑	21−8 ←	1−12 ←	8+11 ←	15+10 ←	22+9 ←	....	....	....	1−7 ←	15−8 ↑
257	6−9 ↑	15−10 ↑	....	5+8 ←	12+7 ←	19+6 ←	....	....	....	1+8 ↑	6−9 ↑	21−10 ↑
258	1−11 ←	10−12 ↑	20+11 ←	2+5 ←	9+4 ←	16+3 ←	23+2 ←	....	....	....	12−11 ←	....
259	5+10 ←	14+9 ↑	....	6+1 ←	13+0 ←	20−1 ←	....	18+9 ↑	18−9 ←	8+7 ↑	3−12 ←	17+11 ↑
260	0+8 ←	9+7 ←	18+6 ↑	3−2 ←	10−3 ←	17−4 ←	....	....	....	....	8+10 ↑	23+9 ←
261	4+5 ←	13+4 ↑	22+3 ↑	0−5 ←	7−6 ←	14−7 ←	21−8 ←	....	....	15+6 ↑	14+8 ←	....
262	8+2 ←	17+1 ↑	....	4−9 ←	11−10 ←	18−11 ←	....	....	....	....	4+7 ↑	19+6 ↑
263	3+0 ←	12−1 ←	21−2 ↑	1−12 ←	8+11 ←	15+10 ←	22+9 ←	....	....	22+5 ↑	10+5 ←	....
264	7−3 ←	16−4 ↑	....	5+8 ←	12+7 ←	19+6 ←	....	....	....	....	1+4 ←	15+3 ↑
265	1−5 ↑	11−6 ←	20−7 ↑	2+5 ←	9+4 ←	16+3 ←	23+2 ←	....	....	....	6+2 ↑	21+1 ←
266	5−8 ↑	15−9 ←	....	6+1 ←	13+0 ←	19−1 ↑	....	....	....	6+4 ←	12+0 ←	....
267	0−10 ↑	10−11 ←	19−12 ←	2−2 ↑	9−3 ↑	16−4 ↑	23−5 ↑	....	....	....	2−1 ↑	17−2 ↑
268	4+11 ↑	14+10 ←	23+9 ↑	6−6 ↑	13−7 ↑	20−8 ↑	....	....	....	13+3 ←	8−3 ↑	23−4 ←
269	8+8 ↑	18+7 ←	....	3−9 ↑	10−10 ↑	17−11 ↑	....	....	....	....	14−5 ←	....
270	3+6 ↑	13+5 ←	22+4 ←	0−12 ↑	7+11 ↑	14+10 ↑	21+9 ↑	....	....	20+2 ←	4−6 ↑	19−7 ↑
271	7+3 ↑	17+2 ←	....	4+8 ↑	11+7 ↑	18+6 ↑	....	....	....	....	10−8 ←	....
272	2+1 ←	11+0 ↑	21−1 ←	1+5 ↑	8+4 ↑	15+3 ↑	22+2 ↑	....	....	....	1−9 ←	15−10 ↑
273	6−2 ↑	15−3 ↑	....	5+1 ↑	12+0 ↑	19−1 ↑	....	....	....	3+1 ←	6−11 ↑	21−12 ←
274	1−4 ←	10−5 ↑	20−6 ←	2−2 ↑	9−3 ↑	16−4 ↑	23−5 ↑	....	....	....	12+11 ←	....
275	5−7 ←	14−8 ↑	....	6−6 ↑	13−7 ↑	20−8 ↑	....	....	....	10+0 ↑	2+10 ↑	17+9 ↑
276	0−9 ←	9−10 ↑	18−11 ↑	3−9 ↑	10−10 ↑	17−11 ↑	....	....	....	....	8+8 ←	23+7 ←
277	4−12 ←	13+11 ↑	23+10 ←	0−12 ↑	7+11 ↑	14+10 ↑	21+9 ↑	....	....	17−1 ↑	13+6 ↑	....
278	8+9 ←	17+8 ↑	....	4+8 ↑	11+7 ↑	18+6 ↑	....	....	....	....	4+5 ↑	19+4 ↑
279	3+7 ←	12+6 ←	21+5 ↑	1+5 ↑	8+4 ↑	15+3 ↑	22+2 ↑	....	....	....	10+3 ←	....
280	7+4 ←	16+3 ↑	....	5+1 ↑	12+0 ↑	19−1 ↑	....	....	....	0−2 ↑	1+2 ←	15+1 ↑
281	1+2 ↑	11+1 ←	20+0 ↑	2−2 ↑	9−3 ↑	16−4 ↑	23−5 ↑	....	....	....	6+0 ↑	21−1 ←
282	6−1 ←	15−2 ←	....	6−6 ↑	13−7 ←	20−8 ←	....	....	....	7−3 ↑	12−2 ←	....
283	0−3 ↑	10−4 ←	19−5 ↑	3−9 ←	10−10 ←	17−11 ←	....	....	....	....	2−3 ↑	17−4 ↑
284	4−6 ↑	14−7 ←	23−8 ↑	0−12 ←	7+11 ←	14+10 ←	21+9 ←	....	....	14−4 ↑	8−5 ←	23−6 ←
285	8−9 ↑	18−10 ←	....	4+8 ←	11+7 ←	18+6 ←	....	....	....	....	13−7 ↑	....
286	3−11 ↑	13−12 ←	22+11 ←	1+5 ←	8+4 ←	15+3 ←	22+2 ←	....	....	22−5 ←	4−8 ↑	19−9 ←
287	7+10 ↑	17+9 ←	....	5+1 ←	12+0 ←	19−1 ←	....	....	....	....	10−10 ←	....
288	2+8 ↑	11+7 ↑	21+6 ←	2−2 ←	9−3 ←	16−4 ←	23−5 ←	....	....	....	1−11 ←	15−12 ↑
289	6+5 ↑	16+4 ←	....	6−6 ←	13−7 ←	20−8 ←	....	....	....	5−6 ←	6+11 ↑	21−10 ←
290	1+3 ←	10+2 ↑	20+1 ←	3−9 ←	10−10 ←	17−11 ←	....	4+10 ←	4−10 ↑	....	12+9 ←	....
291	5+0 ←	14−1 ↑	....	0−12 ←	7+11 ←	14+10 ←	21+9 ←	....	....	12−7 ←	2+8 ↑	17+7 ↑
292	0−2 ←	9−3 ↑	18−4 ↑	4+8 ←	11+7 ←	18+6 ←	....	....	....	....	8+6 ←	23+5 ←
293	4−5 ←	13−6 ↑	23−7 ←	1+5 ←	8+4 ←	15+3 ←	22+2 ←	....	....	19−8 ←	13+4 ↑	....
294	8−8 ←	17−9 ↑	....	5+1 ←	12+0 ←	19−1 ←	....	....	....	....	4+3 ↑	19+2 ←
295	3−10 ←	12−11 ↑	21−12 ↑	2−2 ←	9−3 ←	16−4 ←	23−5 ←	....	....	....	10+1 ←	....
296	7+11 ←	16+10 ↑	....	6−6 ←	13−7 ←	20−8 ←	....	....	....	2−9 ↑	0+0 ↑	15−1 ↑
297	1+9 ↑	11+8 ←	20+7 ↑	3−9 ←	10−10 ←	17−11 ←	23−12 ↑	....	....	....	6−2 ←	21−3 ←
298	6+6 ←	15+5 ←	....	6+11 ↑	13+10 ↑	20+9 ↑	....	....	....	9−10 ↑	12−4 ←	....
299	0+4 ↑	10+3 ←	19+2 ↑	3+8 ↑	10+7 ↑	17+6 ↑	....	....	....	....	2−5 ↑	17−6 ↑
300	4+1 ↑	14+0 ←	23−1 ↑	0+5 ↑	7+4 ↑	14+3 ↑	21+2 ↑	....	....	16−11 ↑	8−7 ←	23−8 ←
301	9−2 ←	18−3 ←	....	4+1 ↑	11+0 ↑	18−1 ↑	....	....	....	....	13−9 ↑	....
302	3−4 ↑	13−5 ←	22−6 ←	1−2 ↑	8−3 ↑	15−4 ↑	22−5 ↑	....	....	23−12 ↑	4−10 ↑	19−11 ←
303	7−7 ↑	17−8 ←	....	5−6 ↑	12−7 ↑	19−8 ↑	....	....	....	....	10−12 ←	....
304	2−9 ↑	11−10 ↑	21−11 ←	2−9 ↑	9−10 ↑	16−11 ↑	23−12 ↑	....	....	....	0+11 ↑	15+10 ↑
305	6−12 ↑	16+11 ←	....	6+11 ↑	13+10 ↑	20+9 ↑	....	....	....	6+11 ↑	6+9 ←	21+8 ←
306	1+10 ←	10+9 ↑	20+8 ←	3+8 ↑	10+7 ↑	17+6 ↑	....	....	....	....	11+7 ↑	....
307	5+7 ↑	14+6 ↑	....	0+5 ↑	7+4 ↑	14+3 ↑	21+2 ↑	....	....	14+10 ←	2+6 ↑	17+5 ←
308	0+5 ←	9+4 ↑	19+3 ←	4+1 ↑	11+0 ↑	18−1 ↑	....	....	....	....	8+4 ←	23+3 ←
309	4+2 ←	13+1 ↑	23+0 ←	1−2 ↑	8−3 ↑	15−4 ↑	22−5 ↑	....	....	21+9 ←	13+2 ↑	....
310	8−1 ←	17−2 ↑	....	5−6 ↑	12−7 ↑	19−8 ↑	....	....	....	....	4+1 ↑	19+0 ←
311	3−3 ←	12−4 ↑	21−5 ↑	2−9 ↑	9−10 ↑	16−11 ↑	23−12 ↑	....	....	....	10−1 ←	....
312	7−6 ←	16−7 ↑	....	6+11 ↑	13+10 ↑	20+9 ↑	....	....	....	4+8 ←	0−2 ↑	15−3 ↑
313	2−8 ←	11−9 ←	20−10 ↑	3+8 ↑	10+7 ↑	17+6 ←	....	....	....	....	6−4 ←	21−5 ←
314	6−11 ←	15−12 ↑	....	0+5 ←	7+4 ←	14+3 ←	21+2 ←	....	....	11+7 ←	11−6 ↑	....
315	0+11 ↑	10+10 ←	19+9 ↑	4+1 ←	11+0 ←	18−1 ←	....	....	....	....	2−7 ↑	17−8 ←
316	4+8 ↑	14+7 ←	23+6 ↑	1−2 ←	8−3 ←	15−4 ←	22−5 ←	....	....	18+6 ↑	8−9 ←	22−10 ↑
317	9+5 ←	18+4 ←	....	5−6 ←	12−7 ←	19−8 ←	....	....	....	....	13−11 ↑	....
318	3+3 ↑	13+2 ←	22+1 ↑	2−9 ←	9−10 ←	16−11 ←	23−12 ←	....	....	....	4−12 ↑	19+11 ←
319	7+0 ↑	17−1 ←	....	6+11 ←	13+10 ←	20+9 ←	....	....	....	1+5 ↑	10+10 ←	....
320	2−2 ↑	12−3 ←	21−4 ←	3+8 ←	10+7 ←	17+6 ←	....	15+11 ↑	15−11 ←	....	0+9 ↑	15+8 ↑

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series	Q			2 Q				R	T	λ	μ or 2 MS	
321	6−5 ↑	16−6 ←	....	0+5 ←	7+4 ←	14+3 ←	21+2 ←	....	....	8+4 ↑	6+7 ←	21+6 ←
322	1−7 ←	10−8 ↑	20−9 ←	4+1 ←	11+0 ←	18−1 ←	....	....	....	....	11+5 ↑	....
323	5−10 ↑	14−11 ↑	....	1−2 ←	8−3 ←	15−4 ←	22−5 ←	....	....	15+3 ↑	2+4 ↑	17+3 ←
324	0−12 ←	9+11 ↑	19+10 ←	5−6 ←	12−7 ←	19−8 ←	....	....	....	....	8+2 ←	22+1 ↑
325	4+9 ←	13+8 ↑	23+7 ←	2−9 ←	9−10 ←	16−11 ←	23−12 ←	....	....	22+2 ↑	13+0 ↑	....
326	8+6 ↑	17+5 ↑	....	6+11 ←	13+10 ←	20+9 ←	....	....	....	....	4−1 ←	19−2 ←
327	3+4 ←	12+3 ↑	21+2 ↑	3+8 ←	10+7 ←	17+6 ←	....	....	....	....	9−3 ↑	....
328	7+1 ←	16+0 ↑	....	0+5 ←	7+4 ←	14+3 ←	21+2 ←	....	....	6+1 ←	0−4 ↑	15−5 ↑
329	2−1 ←	11−2 ←	20−3 ↑	4+1 ←	10+0 ↑	17−1 ↑	....	....	....	....	6−6 ←	21−7 ←
330	6−4 ←	15−5 ↑	....	0−2 ↑	7−3 ↑	14−4 ↑	21−5 ↑	....	....	13+0 ←	11−8 ↑	....
331	0−6 ↑	10−7 ←	19−8 ↑	4−6 ↑	11−7 ↑	18−8 ↑	....	....	....	....	2−9 ↑	17−10 ←
332	5−9 ←	14−10 ←	23−11 ↑	1−9 ↑	8−10 ↑	15−11 ↑	22−12 ↑	....	....	20−1 ←	8−11 ←	22−12 ↑
333	9−12 ←	18+11 ↑	....	5+11 ↑	12+10 ↑	19+9 ↑	....	....	....	....	13+11 ↑	....
334	3+10 ↑	13+9 ←	22+8 ↑	2+8 ↑	9+7 ↑	16+6 ↑	23+5 ↑	....	....	....	4+10 ←	19+9 ←
335	7+7 ↑	17+6 ←	....	6+4 ↑	13+3 ↑	20+2 ↑	....	....	....	3−2 ←	9+8 ↑	....
336	2+5 ↑	12+4 ←	21+3 ←	3+1 ↑	10+0 ↑	17−1 ↑	....	....	....	....	0+7 ↑	15+6 ←
337	6+2 ↑	16+1 ←	....	0−2 ↑	7−3 ↑	14−4 ↑	21−5 ↑	....	....	10−3 ←	6+5 ←	20+4 ↑
338	1+0 ↑	10−1 ↑	20−2 ←	4−6 ↑	11−7 ↑	18−8 ↑	....	....	....	....	11+3 ↑	....
339	5−3 ↑	15−4 ←	....	1−9 ↑	8−10 ↑	15−11 ↑	22−12 ↑	....	....	17−4 ↑	2+2 ↑	17+1 ←
340	0−5 ←	9−6 ↑	19−7 ←	5+11 ↑	12+10 ↑	19+9 ↑	....	....	....	....	8+0 ←	22−1 ↑
341	4−8 ←	13−9 ↑	23−10 ←	2+8 ↑	9+7 ↑	16+6 ↑	23+5 ↑	....	....	....	13−2 ↑	....
342	8−11 ↑	17−12 ↑	....	6+4 ↑	13+3 ↑	20+2 ↑	....	....	....	0−5 ↑	4−3 ←	19−4 ←
343	3+11 ←	12+10 ↑	22+9 ←	3+1 ↑	10+0 ↑	17−1 ↑	....	....	....	....	9−5 ↑	....
344	7+8 ←	16+7 ↑	....	0−2 ↑	7−3 ↑	14−4 ↑	21−5 ←	....	....	7−6 ↑	0−6 ↑	15−7 ←
345	2+6 ←	11+5 ↑	20+4 ↑	4−6 ←	11−7 ←	18−8 ←	....	....	....	....	6−8 ←	20−9 ↑
346	6+3 ←	15+2 ↑	....	1−9 ←	8−10 ←	15−11 ←	22−12 ←	....	....	14−7 ↑	11−10 ↑	....
347	0+1 ↑	10+0 ←	19−1 ↑	5+11 ←	12+10 ←	19+9 ←	....	....	....	....	2−11 ←	17−12 ←
348	5−2 ←	14−3 ←	23−4 ↑	2+8 ←	9+7 ←	16+6 ←	23+5 ←	....	....	22−8 ←	8+11 ←	22+10 ↑
349	9−5 ←	18−6 ↑	....	6+4 ←	13+3 ←	20+2 ←	....	....	....	....	13+9 ↑	....
350	3−7 ↑	13−8 ←	22−9 ↑	3+1 ←	10+0 ←	17−1 ←	....	....	....	....	4+8 ←	19+7 ←
351	8−10 ←	17−11 ←	....	0−2 ←	7−3 ←	14−4 ←	21−5 ←	1+12 ←	1−12 ↑	5−9 ←	9+6 ↑	....
352	2−12 ↑	12+11 ←	21+10 ←	4−6 ←	11−7 ←	18−8 ←	....	....	....	....	0+5 ↑	15+4 ←
353	6+9 ↑	16+8 ←	....	1−9 ←	8−10 ←	15−11 ←	22−12 ←	....	....	12−10 ←	6+3 ←	20+2 ↑
354	1+7 ↑	10+6 ↑	20+5 ←	5+11 ←	12+10 ←	19+9 ←	....	....	....	....	11+1 ↑	....
355	5+4 ↑	15+3 ←	....	2+8 ←	9+7 ←	16+6 ←	23+5 ←	....	....	19−11 ←	2+0 ←	17−1 ←
356	0+2 ←	9+1 ↑	19+0 ←	6+4 ←	13+3 ←	20+2 ←	....	....	....	....	7−2 ↑	22−3 ↑
357	4−1 ↑	13−2 ↑	23−3 ←	3+1 ←	10+0 ←	17−1 ←	....	....	....	....	13−4 ←	....
358	8−4 ↑	18−5 ←	....	0−2 ←	7−3 ←	14−4 ←	21−5 ←	....	....	2−12 ←	4−5 ←	19−6 ←
359	3−6 ←	12−7 ↑	22−8 ←	4−6 ←	11−7 ←	18−8 ←	....	....	....	....	9−7 ↑	....
360	7−9 ←	16−10 ↑	....	1−9 ←	8−10 ←	14−11 ↑	21−12 ↑	....	....	9+11 ↑	0−8 ↑	15−9 ←
361	2−11 ←	11−12 ↑	20+11 ↑	4+11 ↑	11+10 ↑	18+9 ↑	....	....	....	....	6−10 ←	20−11 ↑
362	6+10 ←	15+9 ↑	....	1+8 ↑	8+7 ↑	15+6 ↑	22+5 ↑	....	....	16+10 ↑	11−12 ↑	....
363	1+8 ←	10+7 ←	19+6 ↑	5+4 ↑	12+3 ↑	19+2 ↑	....	....	....	....	2+11 ←	17+10 ←
364	5+5 ←	14+4 ↑	23+3 ↑	2+1 ↑	9+0 ↑	16−1 ↑	23−2 ↑	....	....	23+9 ↑	7+9 ↑	22+8 ↑
365	9+2 ←	18+1 ↑	....	6−3 ↑	13−4 ↑	20−5 ↑	....	....	....	....	13+7 ←	....
366	3+0 ↑	13−1 ←	22−2 ↑	3−6 ↑	10−7 ↑	17−8 ↑	....	....	....	....	4+6 ←	18+5 ↑
367	8−3 ←	17−4 ←	....	0−9 ↑	7−10 ↑	14−11 ↑	21−12 ↑	....	....	6+8 ↑	9+4 ↑	....
368	2−5 ↑	12−6 ←	21−7 ↑	4+11 ↑	11+10 ↑	18+9 ↑	....	....	....	....	0+3 ←	15+2 ←
369	6−8 ↑	16−9 ←	....	1+8 ↑	8+7 ↑	15+6 ↑	22+5 ↑	....	....	14+7 ←	6+1 ←	20+0 ↑
370	1−10 ↑	11−11 ←	20−12 ←	5+4 ↑	12+3 ↑	19+2 ↑	....	....	....	....	11−1 ↑	....
371	5+11 ↑	15+10 ←	....	2+1 ↑	9+0 ↑	16−1 ↑	23−2 ↑	....	....	21+6 ←	2−2 ←	17−3 ←

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	ν		ρ			MK	2MK		MN		MS	2SM
1	11-1 ←		5-1 ↑	15-2 ↑			11-1 ↑		12-1 ↑			15+1 ←
2	7-2 ←		1-3 ↑	11-4 ←	21-5 ←	0-1 ←	9-2 ↑		11-2 ↑		6-1 ↑	21+2 ↑
3	3-3 ←	23-4 ↑	6-6 ↑	16-7 ↑		22-2 ←	7-3 ←		11-3 ←			
4	19-5 ↑		2-8 ↑	12-9 ←	22-10 ←		4-4 ↑		10-4 ↑		17-2 ↑	2+3 ←
5	15-6 ↑		8-11 ←	17-12 ↑		20-3 ←	2-5 ↑		9-5 ↑			8+4 ↑
6	12-7 ←		3+11 ↑	13+10 ←	23+9 ←		0-6 ←	22-7 ←	9-6 ←			13+5 ←
7	8-8 ←		9+8 ←	18+7 ↑		18-4 ↑	19-8 ↑		8-7 ←		4-3 ↑	19+6 ↑
8	4-9 ←		4+6 ↑	14+5 ↑			17-9 ↑		7-8 ↑			
9	0-10 ↑	20-11 ↑	0+4 ←	10+3 ←	19+2 ↑	16-5 ↑	15-10 ←		6-9 ↑		15-4 ↑	0+7 ←
10	16-12 ↑		5+1 ↑	15+0 ↑			12-11 ↑		6-10 ←			6+8 ↑
11	13+11 ←		1-1 ←	11-2 ←	21-3 ←	14-6 ↑	10-12 ↑		5-11 ↑			12+9 ↑
12	9+10 ←		6-4 ↑	16-5 ↑			8+11 ←		4-12 ↑		2-5 ↑	17+10 ←
13	5+9 ←		2-6 ↑	12-7 ←	22-8 ←	13-7 ←	6+10 ←		4+11 ←			23+11 ↑
14	1+8 ↑	21+7 ↑	7-9 ↑	17-10 ↑			3+9 ↑		3+10 ←		13-6 ↑	
15	17+6 ↑		3-11 ↑	13-12 ←	23+11 ←	11-8 ←	1+8 ←	23+7 ←	2+9 ↑			4+12 ←
16	14+5 ←		9+10 ←	18+9 ↑			20+6 ↑		2+8 ←			10-11 ↑
17	10+4 ←		4+8 ↑	14+7 ←		9-9 ←	18+5 ↑		1+7 ←		0-7 ↑	15-10 ←
18	6+3 ←		0+6 ←	10+5 ←	19+4 ↑		16+4 ←		0+6 ↑	23+5 ↑		21-9 ↑
19	2+2 ↑	22+1 ↑	5+3 ↑	15+2 ↑		7-10 ↑	14+3 ←		23+4 ←		11-8 ↑	
20	18+0 ↑		1+1 ←	11+0 ←	20-1 ↑		11+2 ↑		22+3 ←			2-8 ←
21	15-1 ←		6-2 ↑	16-3 ↑		5-11 ↑	9+1 ←		21+2 ↑		23-9 ←	8-7 ↑
22	11-2 ←		2-4 ←	12-5 ←	22-6 ←		7+0 ←		21+1 ←			13-6 ←
23	7-3 ←		7-7 ↑	17-8 ↑		3-12 ↑	4-1 ↑		20+0 ←			19-5 ↑
24	3-4 ↑	23-5 ↑	3-9 ↑	13-10 ←	23-11 ←		2-2 ↑		19-1 ↑		10-10 ←	
25	19-6 ↑		8-12 ↑	18+11 ↑		2+11 ←	0-3 ←	22-4 ←	18-2 ↑			0-4 ←
26	16-7 ←		4+10 ↑	14+9 ←			19-5 ↑		18-3 ←		21-11 ←	6-3 ↑
27	12-8 ←		0+8 ←	9+7 ↑	19+6 ↑	0+10 ←	17-6 ←		17-4 ←			11-2 ←
28	8-9 ←		5+5 ↑	15+4 ←		22+9 ←	15-7 ←		16-5 ↑			17-1 ↑
29	4-10 ↑		1+3 ←	11+2 ←	20+1 ↑		12-8 ↑		16-6 ←		8-12 ←	22-0 ←
30	0-11 ↑	21-12 ←	6+0 ↑	16-1 ←		20+8 ↑	10-9 ↑		15-7 ←			
31	17+11 ←		2-2 ←	12-3 ←	21-4 ↑		8-10 ←		14-8 ↑		19+11 ←	4+1 ↑
32	13+10 ←		7-5 ↑	17-6 ↑		18+7 ↑	5-11 ↑		13-9 ↑			10+2 ↑
33	9+9 ↑		3-7 ←	13-8 ←	22-9 ↑		3-12 ↑		13-10 ←			15+3 ←
34	5+8 ↑		8-10 ↑	18-11 ↑		16+6 ↑	1+11 ←	23+10 ←	12-11 ↑		6+10 ←	21+4 ↑
35	1+7 ↑	22+6 ←	4-12 ←	14+11 ←			20+9 ↑		11-12 ↑			
36	18+5 ←		0+10 ←	9+9 ↑	19+8 ↑	14+5 ↑	18+8 ↑		11+11 ←		17+9 ←	2+5 ←
37	14+4 ←		5+7 ↑	15+6 ←			16+7 ←		10+10 ←			8+6 ↑
38	10+3 ↑		1+5 ←	10+4 ↑	20+3 ↑	13+4 ←	13+6 ↑		9+9 ↑			13+7 ←
39	6+2 ↑		6+2 ↑	16+1 ←			11+5 ↑		8+8 ↑		4+8 ←	19+8 ↑
40	2+1 ↑	23+0 ←	2+0 ←	12-1 ←	21-2 ↑	11+3 ←	9+4 ←		8+7 ←			
41	19-1 ←		7-3 ↑	17-4 ←			7+3 ←		7+6 ↑		15+7 ↑	0+9 ←
42	15-2 ←		3-5 ←	13-6 ←	22-7 ↑	9+2 ←	4+2 ↑		6+5 ↑			6+10 ↑
43	11-3 ↑		8-8 ↑	18-9 ↑			2+1 ←		6+4 ←			11+11 ←
44	7-4 ↑		4-10 ←	14-11 ←	23-12 ↑	7+1 ↑	0+0 ←	21-1 ↑	5+3 ←		2+6 ↑	17+12 ↑
45	3-5 ↑		9+11 ↑	19+10 ↑			19-2 ↑		4+2 ↑			22-11 ←
46	0-6 ←	20-7 ←	5+9 ←	15+8 ←		5+0 ↑	17-3 ←		4+1 ←		13+5 ↑	
47	16-8 ←		1+7 ←	10+6 ↑	20+5 ↑		15-4 ←		3+0 ←			4-10 ↑
48	12-9 ←		6+4 ←	16+3 ←		3-1 ↑	12-5 ↑		2-1 ↑			9-9 ←
49	8-10 ↑		2+2 ←	11+1 ↑	21+0 ↑		10-6 ←		1-2 ↑		0+4 ↑	15-8 ↑
50	4-11 ↑		7-1 ↑	17-2 ←		2-2 ←	8-7 ←		1-3 ←			20-7 ←
51	1-12 ←	21+11 ←	3-3 ←	12-4 ↑	22-5 ↑		5-8 ↑		0-4 ←	23-5 ↑	11+3 ↑	
52	17+10 ←		8-6 ↑	18-7 ←		0-3 ←	3-9 ↑		23-6 ←			2-6 ←
53	13+9 ↑		4-8 ←	14-9 ←	23-10 ↑	22-4 ←	1-10 ←	22-11 ↑	22-7 ←		22+2 ↑	8-5 ↑
54	9+8 ↑		9-11 ↑	19-12 ←			20-12 ↑		21-8 ↑			13-4 ←
55	5+7 ↑		5+11 ←	15+10 ←		20-5 ↑	18+11 ←		20-9 ↑			19-3 ↑
56	2+6 ←	22+5 ←	0+9 ↑	10+8 ↑	20+7 ↑		16+10 ←		20-10 ←		9+1 ↑	
57	18+4 ←		6+6 ←	16+5 ←		18-6 ↑	13+9 ↑		19-11 ←			0-2 ←
58	14+3 ↑		2+4 ←	11+3 ↑	21+2 ↑		11+8 ↑		18-12 ↑		20+0 ↑	6-1 ↑
59	10+2 ↑		7+1 ←	17+0 ←		16-7 ↑	9+7 ←		18+11 ←			11-0 ←
60	6+1 ↑		3-1 ←	12-2 ↑	22-3 ↑		6+6 ↑		17+10 ←			17+1 ↑
61	3+0 ←	23-1 ←	8-4 ↑	18-5 ←		15-8 ←	4+5 ↑		16+9 ↑		7-1 ↑	22+2 ←
62	19-2 ←		4-6 ←	13-7 ↑	23-8 ↑		2+4 ←		15+8 ↑			
63	15-3 ↑		9-9 ↑	19-10 ←		13-9 ←	0+3 ←	21+2 ↑	15+7 ←		19-2 ←	4+3 ↑
64	11-4 ↑		5-11 ←	15-12 ←			19+1 ←		14+6 ↑			9+4 ←
65	7-5 ↑		0+11 ↑	10+10 ↑	20+9 ←	11-10 ←	17+0 ←		13+5 ↑			15+5 ↑
66	4-6 ←		6+8 ←	16+7 ←			14-1 ↑		13+4 ←		6-3 ←	20+6 ←
67	0-7 ←	20-8 ←	1+6 ↑	11+5 ↑	21+4 ↑	9-11 ↑	12-2 ↑		12+3 ←			
68	16-9 ↑		7+3 ←	17+2 ←			10-3 ←		11+2 ↑		17-4 ←	2+7 ↑
69	12-10 ↑		2+1 ↑	12+0 ↑	22-1 ↑	7-12 ↑	8-4 ←		10+1 ↑			7+8 ←
70	8-11 ↑		8-2 ←	18-3 ←			5-5 ↑		10+0 ←			13+9 ↑
71	5-12 ←		4-4 ←	13-5 ↑	23-6 ↑	5+11 ↑	3-6 ←		9-1 ↑		4-5 ←	19+10 ↑
72	1+11 ←	21+10 ←	9-7 ←	19-8 ←			1-7 ←	22-8 ↑	8-2 ↑		15-6 ←	
73	17+9 ↑		5-9 ←	14-10 ↑		4+10 ←	20-9 ↑		8-3 ←			0+11 ←
74	13+8 ↑		0-11 ↑	10-12 ↑	20+11 ←		18-10 ←		7-4 ←			6+12 ↑
75	9+7 ↑		6+10 ←	15+9 ↑		2+9 ←	16-11 ←		6-5 ↑			11-11 ←
76	6+6 ←		1+8 ↑	11+7 ↑	21+6 ←		13-12 ↑		6-6 ←		2-7 ←	17-10 ↑
77	2+5 ←	22+4 ←	7+5 ←	17+4 ←		0+8 ←	11+11 ←		5-7 ←			22-9 ←
78	18+3 ↑		2+3 ↑	12+2 ↑	22+1 ←	22+7 ↑	9+10 ←		4-8 ↑		13-8 ←	
79	14+2 ↑		8+0 ←	18-1 ←			6+9 ↑		3-9 ↑			4-8 ↑
80	10+1 ↑		3-2 ↑	13-3 ↑	23-4 ↑	20+6 ↑	4+8 ↑		3-10 ←			9-7 ←

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	ν		ρ			MK	2MK		MN		MS	2SM
81	7+0 ←	....	9−5 ←	19−6 ←	....	....	2+7 ←	23+6 ↑	2−11←	....	0−9 ←	15−6 ↑
82	3−1 ←	23−2 ←	5−7 ←	14−8 ↑	....	18+5 ↑	21+5 ↑	....	1−12↑	....	....	20−5 ←
83	19−3 ↑	....	0−9 ↑	10−10←	20−11←	....	19+4 ←	....	1+11←	....	11−10↑	....
84	15−4 ↑	....	6−12←	15+11↑	....	17+4 ←	17+3 ←	....	0+10←	23+9 ↑	....	2−4 ↑
85	12−5 ←	....	1+10↑	11+9 ↑	21+8 ←	....	14+2 ↑	....	22+8 ↑	....	22−11↑	7−3 ←
86	8−6 ←	....	7+7 ←	16+6 ↑	....	15+3 ←	12+1 ↑	....	22+7 ←	....	....	13−2 ↑
87	4−7 ←	....	2+5 ↑	12+4 ↑	22+3 ←	....	10+0 ←	....	21+6 ←	....	....	18−1 ←
88	0−8 ↑	20−9 ↑	8+2 ←	18+1 ←	....	13+2 ←	7−1 ↑	....	20+5 ↑	....	9−12↑	....
89	16−10↑	....	3+0 ↑	13−1 ↑	23−2 ←	....	5−2 ↑	....	20+4 ←	....	....	0−0 ↑
90	13−11←	....	9−3 ←	19−4 ←	....	11+1 ↑	3−3 ←	....	19+3 ←	....	20+11↑	5−1 ←
91	9−12←	....	4−5 ↑	14−6 ↑	....	....	1−4 ←	22−5 ↑	18+2 ↑	....	....	11+2 ↑
92	5+11←	....	0−7 ↑	10−8 ←	20−9 ←	9+0 ↑	20−6 ←	....	17+1 →	....	....	17+3 ↑
93	1+10↑	21+9 ↑	5−10↑	15−11↑	....	....	18−7 ←	....	17+0 ←	....	7+10↑	22+4 ←
94	17+8 ↑	....	1−12↑	11+11←	21+10←	7−1 ↑	15−8 ↑	....	16−1 ↑	....	....	....
95	14+7 ←	....	7+9 ←	16+8 ↑	....	....	13−9 ↑	....	15−2 ↑	....	18+9 ↑	4+5 ↑
96	10+6 ←	....	2+7 ↑	12+6 ←	22+5 ←	6−2 ←	11−10←	....	15−3 ←	....	....	9+6 ←
97	6+5 ←	....	8+4 ←	17+3 ↑	....	....	9−11←	....	14−4 ←	....	....	15+7 ↑
98	2+4 ↑	22+3 ↑	3+2 ↑	13+1 ↑	23+0 ←	4−3 ←	6−12↑	....	13−5 ↑	....	5+8 ↑	20+8 ←
99	18+2 ↑	....	9−1 ←	18−2 ↑	....	....	4+11←	....	12−6 ↑	....	....	....
100	15+1 ←	....	4−3 ↑	14−4 ↑	....	2−4 ←	2+10←	23+9 ↑	12−7 ←	....	16+7 ↑	2+9 ↑
101	11+0 ←	....	0−5 ←	10−6 ←	20−7 ←	....	21+8 ↑	....	11−8 ↑	....	....	7+10←
102	7−1 ←	....	5−8 ↑	15−9 ↑	....	0−5 ↑	19+7 ←	....	10−9 ↑	....	....	13+11↑
103	3−2 ↑	23−3 ↑	1−10↑	11−11←	21−12←	22−6 ↑	16+6 ↑	....	10−10←	....	4+6 ←	18+12←
104	19−4 ↑	....	6+11↑	16+10↑	....	....	14+5 ↑	....	9−11←	....	....	....
105	16−5 ←	....	2+9 ↑	12+8 ←	22+7 ←	20−7 ↑	12+4 ←	....	8−12↑	....	15+5 ←	0−11↑
106	12−6 ←	....	8+6 ←	17+5 ↑	....	....	10+3 ←	....	8+11←	....	....	5−10←
107	8−7 ←	....	3+4 ↑	13+3 ←	23+2 ←	18−8 ↑	7+2 ↑	....	7+10←	....	....	11−9 ↑
108	4−8 ↑	....	9+1 ←	18+0 ↑	....	....	5+1 ↑	....	6+9 ↑	....	2+4 ←	16−8 ←
109	0−9 ↑	20−10↑	4−1 ↑	14−2 ↑	....	17−9 ←	3+0 ←	....	5+8 ↑	....	....	22−7 ↑
110	17−11←	....	0−3 ←	10−4 ←	19−5 ↑	....	0−1 ↑	22−2 ↑	5+7 ←	....	13+3 ←	....
111	13−12←	....	5−6 ↑	15−7 ↑	....	15−10←	20−3 ←	....	4+6 ←	....	....	3−6 ←
112	9+11←	....	1−8 ←	11−9 ←	21−10←	....	18−4 ←	....	3+5 ↑	....	....	9−5 ←
113	5+10↑	....	6−11↑	16−12↑	....	13−11←	15−5 ↑	....	3+4 ←	....	0+2 ←	15−4 ↑
114	1+9 ↑	21+8 ↑	2+11←	12+10←	22+9 ←	....	13−6 ←	....	2+3 ←	....	....	20−3 ←
115	18+7 ←	....	7+8 ↑	17+7 ↑	....	11−12↑	11−7 ←	....	1+2 ↑	....	11+1 ←	....
116	14+6 ←	....	3+6 ↑	13+5 ←	23+4 ←	....	8−8 ↑	....	0+1 ↑	....	....	2−2 ↑
117	10+5 ←	....	8+3 ↑	18+2 ↑	....	9+11↑	6−9 ↑	....	0+0 ←	23−1 ←	22+0 ←	7−1 ←
118	6+4 ↑	....	4+1 ↑	14+0 ←	....	....	4−10←	....	22−2 ↑	....	....	13−0 ↑
119	2+3 ↑	22+2 ↑	0−1 ←	10−2 ←	19−3 ↑	7+10↑	2−11←	23−12↑	22−3 ←	....	....	18+1 ←
120	19+1 ←	....	5−4 ↑	15−5 ←	....	....	21+11←	....	21−4 ←	....	9−1 ←	....
121	15+0 ←	....	1−6 ←	11−7 ←	20−8 ↑	6+9 ←	19+10←	....	20−5 ↑	....	....	0+2 ↑
122	11−1 ←	....	6−9 ↑	16−10↑	....	....	16+9 ↑	....	19−6 ↑	....	20−2 ↑	5+3 ←
123	7−2 ↑	....	2−11←	12−12←	21+11↑	4+8 ←	14+8 ↑	....	19−7 ←	....	....	11+4 ↑
124	3−3 ↑	23−4 ↑	7+10↑	17+9 ↑	....	....	12+7 ←	....	18−8 ↑	....	....	16+5 ←
125	20−5 ←	....	3+8 ←	13+7 ←	23+6 ←	2+7 ←	10+6 ←	....	17−9 ↑	....	7−3 ↑	22+6 ↑
126	16−6 ←	....	8+5 ↑	18+4 ↑	....	....	7+5 ↑	....	17−10←	....	....	....
127	12−7 ←	....	4+3 ↑	14+2 ←	....	0+6 ↑	5+4 ←	....	16−11←	....	18−4 ↑	3+7 ←
128	8−8 ↑	....	0+1 ←	9+0 ↑	19−1 ↑	22+5 ↑	3+3 ←	....	15−12↑	....	....	9+8 ↑
129	4−9 ↑	....	5−2 ↑	15−3 ←	....	....	0+2 ↑	22+1 ↑	14+11↑	....	....	14+9 ←
130	0−10↑	21−11←	1−4 ←	11−5 ←	20−6 ↑	20+4 ↑	20+0 ←	....	14+10←	....	5−5 ↑	20+10↑
131	17−12←	....	6−7 ↑	16−8 ←	....	....	17−1 ↑	....	13+9 ↑	....	....	....
132	13+11←	....	2−9 ←	12−10←	21−11↑	19+3 ←	15−2 ↑	....	12+8 ↑	....	16−6 ↑	2+11↑
133	9+10↑	....	7−12↑	17+11↑	....	....	13−3 ←	....	12+7 ←	....	....	7+12←
134	5+9 ↑	....	3+10←	13+9 ←	22+8 ↑	17+2 ←	11−4 ←	....	11+6 ←	....	....	13−11↑
135	1+8 ↑	22+7 ←	8+7 ↑	18+6 ↑	....	....	8−5 ↑	....	10+5 ↑	....	3−7 ↑	18−10←
136	18+6 ←	....	4+5 ←	14+4 ←	....	15+1 ←	6−6 ↑	....	10+4 ←	....	....	....
137	14+5 ←	....	0+3 ←	9+2 ↑	19+1 ↑	....	4−7 ←	....	9+3 ←	....	14−8 ↑	0−9 ↑
138	10+4 ↑	....	5+0 ←	15−1 ←	....	13+0 ↑	1−8 ↑	23−9 ↑	8+2 ↑	....	....	5−8 ←
139	6+3 ↑	....	1−2 ←	10−3 ↑	20−4 ↑	....	21−10←	....	7+1 ↑	....	....	11−7 ↑
140	3+2 ←	23+1 ←	6−5 ↑	16−6 ←	....	11−1 ↑	19−11←	....	7+0 ←	....	1−9 ↑	16−6 ←
141	19+0 ←	....	2−7 ←	11−8 ↑	21−9 ↑	....	16−12↑	....	6−1 ←	....	....	22−5 ↑
142	15−1 ↑	....	7−10↑	17−11←	....	9−2 ↑	14+11←	....	5−2 ↑	....	13−10←	....
143	11−2 ↑	....	3−12←	13+11←	22+10↑	....	12+10←	....	5−3 ←	....	....	3−4 ←
144	7−3 ↑	....	8+9 ↑	18+8 ←	....	8−3 ←	9+9 ↑	....	4−4 ←	....	....	9−3 ↑
145	4−4 ←	....	4+7 ←	14+6 ←	23+5 ↑	....	7+8 ↑	....	3−5 ↑	....	0−11←	14−2 ←
146	0−5 ←	20−6 ←	9+4 ↑	19+3 ↑	....	6−4 ←	5+7 ←	....	2−6 ↑	....	....	20−1 ↑
147	16−7 ↑	....	5+2 ←	15+1 ←	....	....	3+6 ←	....	2−7 ←	....	11−12←	....
148	12−8 ↑	....	1+0 ←	10−1 ↑	20−2 ↑	4−5 ←	0+5 ↑	22+4 ←	1−8 ←	....	....	1−0 ←
149	8−9 ↑	....	6−3 ←	16−4 ←	....	....	20+3 ←	....	0−9 ↑	....	22+11←	7+1 ↑
150	5−10←	....	2−5 ←	11−6 ↑	21−7 ↑	2−6 ↑	17+2 ↑	....	0−10←	23−11←	....	12+2 ←
151	1−11←	21−12←	7−8 ↑	17−9 ←	....	....	15+1 ↑	....	22−12↑	....	....	18+3 ↑
152	17+11↑	....	3−10←	12−11↑	22−12↑	0−7 ↑	13+0 ←	....	21+11↑	....	9+10←	....
153	13+10↑	....	8+11↑	18+10←	....	22−8 ↑	10−1 ↑	....	21+10←	....	....	0+4 ↑
154	9+9 ↑	....	4+9 ←	14+8 ←	23+7 ↑	....	8−2 ↑	....	20+9 ↑	....	20+9 ←	5+5 ←
155	6+8 ←	....	9+6 ↑	19+5 ←	....	21−9 ←	6−3 ←	....	19+8 ↑	....	....	11+6 ↑
156	2+7 ←	22+6 ←	5+4 ←	15+3 ←	....	....	4−4 ←	....	19+7 ←	....	....	16+7 ←
157	18+5 ↑	....	0+2 ↑	10+1 ↑	20+0 ↑	19−10←	1−5 ↑	23−6 ↑	18+6 ←	....	7+8 ←	22+8 ↑
158	14+4 ↑	....	6−1 ←	16−2 ←	....	....	21−7 ←	....	17+5 ↑	....	....	....
159	10+3 ↑	....	1−3 ↑	11−4 ↑	21−5 ↑	17−11←	18−8 ↑	....	16+4 ↑	....	18+7 ←	3+9 ←
160	7+2 ←	....	7−6 ←	17−7 ←	....	....	16−9 ↑	....	16+3 ←	....	....	9+10↑

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series	ν		ρ			MK	2 M K		M N		MS	2 S M
161	3+1 ←	23+0 ←	3−8 ←	12−9 ↑	22−10↑	15−12↑	14−10←	....	15+2 ↑	....	....	14+11←
162	19−1 ↑	....	8−11←	18−12←	....	....	12−11←	....	14+1 ↑	....	5+6 ↑	20+12↑
163	15−2 ↑	....	4+11←	13+10↑	23+9 ↑	13+11↑	9−12↑	....	14+0 ←	....	....	....
164	11−3 ↑	....	9+8 ↑	19+7 ←	....	....	7+11←	....	13−1 ←	....	16+5 ↑	1−11←
165	8−4 ←	....	5+6 ←	14+5 ↑	....	11+10↑	5+10←	....	12−2 ↑	....	....	7−10↑
166	4−5 ←	....	0+4 ↑	10+3 ↑	20+2 ←	....	2+9 ↑	....	12−3 ←	....	....	12−9 ←
167	0−6 ←	20−7 ↑	6+1 ←	16+0 ←	....	10+9 ←	0+8 ↑	22+7 ←	11−4 ←	....	3+4 ↑	18−8 ↑
168	16−8 ↑	....	1−1 ↑	11−2 ↑	21−3 ←	....	20+6 ←	....	10−5 ↑	....	....	23−7 ←
169	12−9 ↑	....	7−4 ←	17−5 ←	....	8+8 ←	17+5 ↑	....	9−6 ↑	....	14+3 ↑	....
170	9−10←	....	2−6 ↑	12−7 ↑	22−8 ↑	....	15+4 ←	....	9−7 ←	....	....	5−6 ↑
171	5−11←	....	8−9 ←	18−10←	....	6+7 ←	13+3 ←	....	8−8 ←	....	....	10−5 ←
172	1−12←	21+11↑	4−11←	13−12↑	23+11↑	....	10+2 ↑	....	7−9 ↑	....	1+2 ↑	16−4 ←
173	17+10↑	....	9+10←	19+9 ←	....	4+6 ↑	8+1 ↑	....	7−10←	....	....	22−3 ↑
174	13+9 ↑	....	5+8 ←	14+7 ↑	....	....	6+0 ←	....	6−11←	....	12+1 ↑	....
175	10+8 ←	....	0+6 ↑	10+5 ↑	20+4 ←	2+5 ↑	4−1 ←	....	5−12↑	....	....	3−2 ←
176	6+7 ←	....	6+3 ←	15+2 ↑	....	....	1−2 ↑	23−3 ←	4+11↑	....	23+0 ↑	9−1 ↑
177	2+6 ←	22+5 ↑	1+1 ↑	11+0 ↑	21−1 ←	0+4 ↑	21−4 ←	....	4+10←	....	....	14−0 ←
178	18+4 ↑	....	7−2 ←	17−3 ←	....	22+3 ↑	18−5 ↑	....	3+9 ←	....	....	20+1 ↑
179	14+3 ↑	....	2−4 ↑	12−5 ↑	22−6 ←	....	16−6 ↑	....	2+8 ↑	....	10−1 ↑	....
180	11+2 ←	....	8−7 ←	18−8 ←	....	21+2 ←	14−7 ←	....	2+7 ←	....	....	1+2 ←
181	7+1 ←	....	3−9 ↑	13−10↑	23−11↑	....	11−8 ↑	....	1+6 ←	....	21−2 ↑	7+3 ↑
182	3+0 ←	23−1 ↑	9−12←	19+11←	....	19+1 ←	9−9 ↑	....	0+5 ↑	23+4 ↑	....	12+4 ←
183	19−2 ↑	....	4+10↑	14+9 ↑	....	....	7−10←	....	23+3 ←	....	....	18+5 ↑
184	15−3 ↑	....	0+8 ↑	10+7 ←	20+6 ←	17+0 ←	5−11←	....	22+2 ↑	....	9−3 ←	23+6 ←
185	12−4 ←	....	6+5 ←	15+4 ↑	....	....	2−12↑	....	21+1 ↑	....	....	....
186	8−5 ←	....	1+3 ↑	11+2 ←	21+1 ←	15−1 ↑	0+11↑	22+10←	21+0 ←	....	20−4 ←	5+7 ↑
187	4−6 ←	....	7+0 ←	16−1 ↑	....	....	19+9 ↑	....	20−1 ←	....	....	10+8 ←
188	0−7 ↑	20−8 ↑	2−2 ↑	12−3 ↑	22−4 ←	13−2 ↑	17+8 ↑	....	19−2 ↑	....	....	16+9 ↑
189	16−9 ↑	....	8−5 ←	17−6 ↑	....	....	15+7 ←	....	18−3 ↑	....	7−5 ←	21+10←
190	13−10←	....	3−7 ↑	13−8 ↑	23−9 ←	11−3 ↑	13+6 ←	....	18−4 ←	....	....	....
191	9−11←	....	9−10←	19−11←	....	....	10+5 ↑	....	17−5 ↑	....	18−6 ←	3+11↑
192	5−12←	....	4−12↑	14+11↑	....	10−4 ←	8+4 ←	....	16−6 ↑	....	....	9−12↑
193	1+11↑	21+10↑	0+10↑	10+9 ←	20+8 ←	....	6+3 ←	....	16−7 ←	....	....	14−11←
194	17+9 ↑	....	5+7 ↑	15+6 ↑	....	8−5 ←	3+2 ↑	....	15−8 ←	....	5−7 ←	20−10↑
195	14+8 ←	....	1+5 ↑	11+4 ←	21+3 ←	....	1+1 ↑	23+0 ←	14−9 ↑	....	....	....
196	10+7 ←	....	7+2 ←	16+1 ↑	....	6−6 ←	21−1 ←	....	14−10←	....	16−8 ←	1−9 ←
197	6+6 ↑	....	2+0 ↑	12−1 ←	22−2 ←	....	18−2 ↑	....	13−11←	....	....	7−8 ↑
198	2+5 ↑	22+4 ↑	8−3 ←	17−4 ↑	....	4−7 ↑	16−3 ←	....	12−12↑	....	....	12−7 ←
199	19+3 ←	....	3−5 ↑	13−6 ↑	23−7 ←	....	14−4 ←	....	11+11↑	....	3−9 ←	18−6 ↑
200	15+2 ←	....	9−8 ←	18−9 ↑	....	2−8 ↑	11−5 ↑	....	11+10←	....	....	23−5 ←
201	11+1 ←	....	4−10↑	14−11↑	....	....	9−6 ↑	....	10+9 ←	....	14−10←	....
202	7+0 ↑	....	0−12←	10+11←	20+10←	0−9 ↑	7−7 ←	....	9+8 ↑	....	....	5−4 ↑
203	3−1 ↑	23−2 ↑	5+9 ↑	15+8 ↑	....	23−10←	4−8 ↑	....	9+7 ←	....	....	10−3 ←
204	20−3 ←	....	1+7 ←	11+6 ←	21+5 ←	....	2−9 ↑	....	8+6 ←	....	1−11↑	16−2 ↑
205	16−4 ←	....	6+4 ↑	16+3 ↑	....	21−11←	0−10←	22−11←	7+5 ↑	....	....	21−1 ←
206	12−5 ←	....	2+2 ↑	12+1 ←	22+0 ←	....	19−12↑	....	6+4 ↑	....	12−12↑	....
207	8−6 ↑	....	7−1 ↑	17−2 ↑	....	19−12←	17+11↑	....	6+3 ←	....	....	3−0 ↑
208	4−7 ↑	....	3−3 ↑	13−4 ←	23−5 ←	....	15+10←	....	5+2 ↑	....	23+11↑	8+1 ←
209	0−8 ↑	21−9 ←	9−6 ←	18−7 ↑	....	17+11↑	12+9 ↑	....	4+1 ↑	....	....	14+2 ↑
210	17−10←	....	4−8 ↑	14−9 ←	....	....	10+8 ↑	....	4+0 ←	....	....	19+3 ←
211	13−11←	....	0−10←	10−11←	19−12↑	15+10↑	8+7 ←	....	3−1 ←	....	10+10↑	....
212	9−12↑	....	5+11↑	15+10↑	....	....	6+6 ←	....	2−2 ↑	....	....	1+4 ↑
213	5+11↑	....	1+9 ←	11+8 ←	20+7 ↑	13+9 ↑	3+5 ↑	....	1−3 ↑	....	21+9 ↑	7+5 ↑
214	1+10↑	22+9 ←	6+6 ↑	16+5 ↑	....	....	1+4 ←	23+3 ←	1−4 ←	....	....	12+6 ←
215	18+8 ←	....	2+4 ←	12+3 ←	22+2 ←	12+8 ←	20+2 ↑	....	0−5 ↑	23−6 ↑	....	18+7 ↑
216	14+7 ←	....	7+1 ↑	17+0 ↑	....	....	18+1 ↑	....	23−7 ←	....	8+8 ↑	23+8 ←
217	10+6 ↑	....	3−1 ↑	13−2 ←	23−3 ←	10+7 ←	16+0 ←	....	22−8 ←	....	....	....
218	6+5 ↑	....	8−4 ↑	18−5 ↑	....	....	14−1 ←	....	21−9 ↑	....	19+7 ↑	5+9 ↑
219	2+4 ↑	23+3 ←	4−6 ↑	14−7 ←	....	8+6 ←	11−2 ↑	....	20−10↑	....	....	10+10←
220	19+2 ←	....	0−8 ←	10−9 ←	19−10↑	....	9−3 ←	....	20−11←	....	....	16+11↑
221	15+1 ←	....	5−11↑	15−12←	....	6+5 ↑	7−4 ←	....	19−12↑	....	6+6 ↑	21+12←
222	11+0 ↑	....	1+11←	11+10←	20+9 ↑	....	4−5 ↑	....	18+11↑	....	....	....
223	7−1 ↑	....	6+8 ↑	16+7 ↑	....	4+4 ↑	2−6 ↑	....	18+10←	....	18+5 ←	3−11↑
224	3−2 ↑	....	2+6 ←	12+5 ←	21+4 ↑	....	0−7 ←	22−8 ←	17+9 ←	....	....	8−10←
225	0−3 ←	20−4 ←	7+3 ↑	17+2 ↑	....	2+3 ↑	19−9 ↑	....	16+8 ↑	....	....	14−9 ↑
226	16−5 ←	....	3+1 ←	13+0 ←	23−1 ←	....	17−10←	....	16+7 ←	....	5+4 ←	19−8 ←
227	12−6 ↑	....	8−2 ↑	18−3 ↑	....	1+2 ←	15−11←	....	15+6 ←	....	....	....
228	8−7 ↑	....	4−4 ←	14−5 ←	....	23+1 ←	12−12↑	....	14+5 ↑	....	16+3 ←	1−7 ↑
229	4−8 ↑	....	0−6 ←	9−7 ↑	19−8 ↑	....	10+11↑	....	13+4 ↑	....	....	6−6 ←
230	1−9 ←	21−10←	5−9 ↑	15−10←	....	21+0 ←	8+10←	....	13+3 ←	....	....	12−5 ↑
231	17−11←	....	1−11←	10−12↑	20+11↑	....	5+9 ↑	....	12+2 ←	....	3+2 ←	17−4 ←
232	13−12↑	....	6+10↑	16+9 ←	....	19−1 ↑	3+8 ↑	....	11+1 ↑	....	....	23−3 ←
233	9+11↑	....	2+8 ←	12+7 ←	21+6 ↑	....	1+7 ←	23+6 ←	11+0 ←	....	14+1 ←	....
234	5+10↑	....	7+5 ↑	17+4 ←	....	17−2 ↑	20+5 ↑	....	10−1 ←	....	....	5−2 ↑
235	2+9 ←	22+8 ←	3+3 ←	13+2 ←	22+1 ↑	....	18+4 ↑	....	9−2 ↑	....	....	10−1 ←
236	18+7 ←	....	8+0 ↑	18−1 ↑	....	15−3 ↑	16+3 ←	....	8−3 ↑	....	1+0 ←	16−0 ↑
237	14+6 ↑	....	4−2 ←	14−3 ←	....	....	13+2 ↑	....	8−4 ←	....	....	21+1 ←
238	10+5 ↑	....	0−4 ←	9−5 ↑	19−6 ↑	14−4 ←	11+1 ↑	....	7−5 ↑	....	12−1 ←	....
239	6+4 ↑	....	5−7 ←	15−8 ←	....	....	9+0 ←	....	6−6 ↑	....	....	3+2 ↑
240	3+3 ←	23+2 ←	1−9 ←	10−10↑	20−11↑	12−5 ←	7−1 ←	....	6−7 ←	....	23−2 ←	8+3 ←

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	ν		ρ			MK	2MK		MN		MS	2SM
241	19+1 ←		6−12↑	16+11←			4−2 ↑		5−8 ←			14+4 ↑
242	15+0 ↑		2+10←	11+9 ↑	21+8 ↑	10−6 ←	2−3 ←		4−9 ↑			19+5 ←
243	11−1 ↑		7+7 ↑	17+6 ←			0−4 ←	21−5 ↑	3−10↑		10−3 ↑	
244	7−2 ↑		3+5 ←	13+4 ←	22+3 ↑	8−7 ←	19−6 ↑		3−11←			1+6 ↑
245	4−3 ←		8+2 ↑	18+1 ←			17−7 ←		2−12↑		21−4 ↑	6+7 ←
246	0−4 ←	20−5 ←	4+0 ←	14−1 ←	23−2 ↑	6−8 ↑	15−8 ←		1+11↑			12+8 ↑
247	16−6 ↑		9−3 ↑	19−4 ↑			12−9 ↑		1+10←			17+9 ←
248	12−7 ↑		5−5 ←	15−6 ←		4−9 ↑	10−10←		0+9 ←	23+8 ↑	8−5 ↑	23+10↑
249	8−8 ↑		0−7 ↑	10−8 ↑	20−9 ↑		8−11←		22+7 ↑			
250	5−9 ←		6−10←	16−11←		2−10↑	5−12↑		22+6 ←		19−6 ↑	4+11←
251	1−10←	21−11↑	2−12←	11+11↑	21+10↑		3+11↑		21+5 ↑			10+12↑
252	17−12↑		7+9 ←	17+8 ←		1−11←	1+10←	22+9 ↑	20+4 ↑			16−11↑
253	13+11↑		3+7 ←	12+6 ↑	22+5 ↑	23−12←	20+8 ↑		20+3 ←		6−7 ↑	21−10←
254	10+10←		8+4 ↑	18+3 ←			18+7 ←		19+2 ←			
255	6+9 ←		4+2 ←	13+1 ↑	23+0 ↑	21+11←	16+6 ←		18+1 ↑		17−8 ↑	3−9 ↑
256	2+8 ←	22+7 ↑	9−1 ↑	19−2 ←			13+5 ↑		18+0 ←			8−8 ←
257	18+6 ↑		5−3 ←	15−4 ←		19+10↑	11+4 ↑		17−1 ←			14−7 ↑
258	14+5 ↑		0−5 ↑	10−6 ↑	20−7 ←		9+3 ←		16−2 ↑		4−9 ↑	19−6 ←
259	11+4 ←		6−8 ←	16−9 ←		17+9 ↑	6+2 ↑		15−3 ↑			
260	7+3 ←		1−10↑	11−11↑	21−12↑		4+1 ↑		15−4 ←		15−10↑	1−5 ↑
261	3+2 ←	23+1 ↑	7+11←	17+10←		15+8 ↑	2+0 ←		14−5 ←			6−4 ←
262	19+0 ↑		3+9 ←	12+8 ↑	22+7 ↑		0−1 ←	21−2 ↑	13−6 ↑			12−3 ↑
263	15−1 ↑		8+6 ←	18+5 ←		14+7 ←	19−3 ↑		13−7 ←		3−11←	17−2 ←
264	12−2 ←		4+4 ←	13+3 ↑	23+2 ↑		17−4 ←		12−8 ←			23−1 ↑
265	8−3 ←		9+1 ↑	19+0 ←		12+6 ←	14−5 ↑		11−0 ↑		14−12←	
266	4−4 ←		5−1 ←	14−2 ↑			12−6 ↑		10−10↑			4−0 ←
267	0−5 ↑	20−6 ↑	0−3 ↑	10−4 ↑	20−5 ←	10+5 ←	10−7 ←		10−11←			10+1 ↑
268	16−7 ↑		6−6 ←	16−7 ←			8−8 ←		9−12↑		1+11←	15+2 ←
269	13−8 ←		1−8 ↑	11−9 ↑	21−10←	8+4 ↑	5−9 ↑		8+11↑			21+3 ↑
270	9−9 ←		7−11←	17−12←			3−10←		8+10←		12+10←	
271	5−10←		2+11↑	12+10↑	22+9 ↑	6+3 ↑	1−11←	22−12↑	7+9 ←			2+4 ←
272	1−11↑	21−12↑	8+8 ←	18+7 ←			20+11↑		6+8 ↑		23+9 ←	8+5 ↑
273	17+11↑		3+6 ↑	13+5 ↑	23+4 ↑	4+2 ↑	18+10←		5+7 ↑			14+6 ↑
274	14+10←		9+3 ←	19+2 ←			16+9 ←		5+6 ←			19+7 ←
275	10+9 ←		5+1 ←	14+0 ↑		3+1 ←	13+8 ↑		4+5 ↑		10+8 ←	
276	6+8 ←		0−1 ↑	10−2 ←	20−3 ←		11+7 ←		3+4 ↑			1+8 ↑
277	2+7 ↑	22+6 ↑	6−4 ←	15−5 ↑		1+0 ←	9+6 ←		3+3 ←		21+7 ←	6+9 ←
278	18+5 ↑		1−6 ↑	11−7 ↑	21−8 ←	23−1 ←	6+5 ↑		2+2 ←			12+10↑
279	15+4 ←		7−9 ←	16−10↑			4+4 ↑		1+1 ↑			17+11←
280	11+3 ←		2−11↑	12−12↑	22+11←	21−2 ↑	2+3 ←	23+2 ↑	0+0 ↑		8+6 ←	23+12↑
281	7+2 ←		8+10←	18+9 ←			21+1 ↑		0−1 ←	23−2 ↑		
282	3+1 ↑	23+0 ↑	3+8 ↑	13+7 ↑	23+6 ↑	19−3 ↑	19+0 ←		22−3 ↑		19+5 ↑	4−11←
283	19−1 ↑		9+5 ←	19+4 ←			17−1 ←		22−4 ←			10−10↑
284	16−2 ←		4+3 ↑	14+2 ↑		17−4 ↑	14−2 ↑		21−5 ←			15−9 ←
285	12−3 ←		0+1 ↑	10+0 ←	20−1 ←		12−3 ↑		20−6 ↑		6+4 ↑	21−8 ↑
286	8−4 ←		6−2 ←	15−3 ↑		16−5 ←	10−4 ←		20−7 ←			
287	4−5 ↑		1−4 ↑	11−5 ←	21−6 ←		7−5 ↑		19−8 ←		17+3 ↑	2−7 ←
288	0−6 ↑	20−7 ↑	7−7 ←	16−8 ↑		14−6 ←	5−6 ↑		18−9 ↑			8−6 ↑
289	17−8 ←		2−9 ↑	12−10↑	22−11←		3−7 ←		17−10↑			13−5 ←
290	13−9 ←		8−12←	17+11↑		12−7 ←	1−8 ←	22−9 ↑	17−11←		4+2 ↑	19−4 ↑
291	9−10←		3+10↑	13+9 ↑	23+8 ←		20−10←		16−12←			
292	5−11↑		9+7 ←	19+6 ←		10−8 ↑	18−11←		15+11↑		15+1 ↑	0−3 ←
293	1−12↑	21+11↑	4+5 ↑	14+4 ↑			15−12↑		15+10←			6−2 ←
294	18+10←		0+3 ←	10+2 ←	20+1 ←	8−9 ↑	13+11↑		14+9 ←			12−1 ↑
295	14+9 ←		5+0 ↑	15−1 ↑			11+10←		13+8 ↑		2+0 ↑	17−0 ←
296	10+8 ←		1−2 ↑	11−3 ←	21−4 ←	6−10↑	9+9 ←		12+7 ↑			23+1 ↑
297	6+7 ↑		6−5 ↑	16−6 ↑			6+8 ↑		12+6 ←		13−1 ↑	
298	2+6 ↑	22+5 ↑	2−7 ↑	12−8 ←	22−9 ←	5−11←	4+7 ←		11+5 ↑			4+2 ←
299	19+4 ←		8−10←	17−11↑			2+6 ←	23+5 ↑	10+4 ↑			10+3 ↑
300	15+3 ←		3−12↑	13+11←	23+10←	3−12←	21+4 ↑		10+3 ←		0−2 ↑	15+4 ←
301	11+2 ←		9+9 ←	18+8 ↑			19+3 ←		9+2 ←			21+5 ↑
302	7+1 ↑		4+7 ↑	14+6 ↑		1+11←	16+2 ↑		8+1 ↑		11−3 ↑	
303	3+0 ↑	23−1 ↑	0+5 ←	10+4 ←	19+3 ↑	23+10↑	14+1 ↑		7+0 ↑			2+6 ←
304	20−2 ←		5+2 ↑	15+1 ↑			12+0 ←		7−1 ←		23−4 ←	8+7 ↑
305	16−3 ←		1+0 ←	11−1 ←	21−2 ←	21+9 ↑	10−1 ←		6−2 ↑			13+8 ←
306	12−4 ↑		6−3 ↑	16−4 ↑			7−2 ↑		5−3 ↑			19+9 ↑
307	8−5 ↑		2−5 ↑	12−6 ←	22−7 ←	19+8 ↑	5−3 ↑		5−4 ←		10−5 ←	
308	4−6 ↑		7−8 ↑	17−9 ↑			3−4 ←		4−5 ←			0+10←
309	1−7 ←	21−8 ←	3−10↑	13−11←	23−12←	18+7 ←	0−5 ↑	22−6 ↑	3−6 ↑		21−6 ←	6+11↑
310	17−9 ←		9+11←	18+10↑			20−7 ←		2−7 ↑			11+12←
311	13−10↑		4+9 ↑	14+8 ←		16+6 ←	18−8 ←		2−8 ←			17−11↑
312	9−11↑		0+7 ←	10+6 ←	19+5 ↑		15−9 ↑		1−9 ↑		8−7 ←	23−10↑
313	5−12↑		5+4 ↑	15+3 ↑		14+5 ←	13−10↑		0−10↑			
314	2+11←	22+10←	1+2 ←	11+1 ←	20+0 ↑		11−11←		0−11←	23−12←	19−8 ←	4−9 ←
315	18+9 ←		6−1 ↑	16−2 ↑		12+4 ←	8−12↑		22+11↑			10−8 ↑
316	14+8 ↑		2−3 ←	12−4 ←	22−5 ←		6+11↑		22+10←			15−7 ←
317	10+7 ↑		7−6 ↑	17−7 ↑		10+3 ↑	4+10←		21+9 ←		6−9 ←	21−6 ↑
318	6+6 ↑		3−8 ←	13−9 ←	23−10←		2+9 ←	23+8 ↑	20+8 ↑			
319	3+5 ←	23+4 ←	8−11↑	18−12↑		8+2 ↑	21+7 ←		19+7 ↑		17−10←	2−5 ←
320	19+3 ←		4+11↑	14+10←			19+6 ←		19+6 ←			8−4 ↑

TABLE 42.—*Component hours derived from solar hours*—Continued.

Day of series.	ν		ρ			MK	2MK		MN		MS	2SM
321	15+2 ↑	....	0+9 ←	9+8 ↑	19+7 ↑	6+1 ↑	16+5 ↑	....	18+5 ←	....	....	13−3 ←
322	11+1 ↑	....	5+6 ↑	15+5 ←	....	....	14+4 ↑	....	17+4 ↑	....	4−11 ←	19−2 ↑
323	7+0 ↑	....	1+4 ←	11+3 ←	20+2 ↑	5+0 ←	12+3 ←	....	17+3 ←	....	....	....
324	4−1 ←	....	6+1 ↑	16+0 ←	....	....	10+2 ←	....	16+2 ←	....	15−12 ↑	0−1 ←
325	0−2 ←	20−3 ←	2−1 ←	12−2 ←	21−3 ↑	3−1 ←	7+1 ↑	....	15+1 ↑	....	....	6−0 ↑
326	16−4 ↑	....	7−4 ↑	17−5 ↑	....	....	5+0 ←	....	14+0 ↑	....	....	11+1 ←
327	12−5 ↑	....	3−6 ←	13−7 ←	23−8 ←	1−2 ←	3−1 ←	....	14−1 ←	....	2+11 ↑	17+2 ↑
328	8−6 ↑	....	8−9 ↑	18−10 ↑	....	23−3 ↑	0−2 ↑	22−3 ↑	13−2 ↑	....	....	22+3 ←
329	5−7 ←	....	4−11 ←	14−12 ←	....	....	20−4 ←	....	12−3 ↑	....	13+10 ↑	....
330	1−8 ←	21−9 ←	0+11 ←	9+10 ↑	19+9 ↑	21−4 ↑	17−5 ↑	....	12−4 ←	....	....	4+4 ↑
331	17−10 ↑	....	5+8 ↑	15+7 ←	....	....	15−6 ↑	....	11−5 ←	....	....	9+5 ←
332	13−11 ↑	....	1+6 ←	10+5 ↑	20+4 ↑	19−5 ↑	13−7 ←	....	10−6 ↑	....	0+9 ↑	15+6 ↑
333	9−12 ↑	....	6+3 ↑	16+2 ←	....	....	11−8 ←	....	9−7 ↑	....	....	21+7 ↑
334	6+11 ←	....	2+1 ←	12+0 ←	21−1 ↑	18−6 ←	8−9 ↑	....	9−8 ←	....	11+8 ↑	....
335	2+10 ←	22+9 ←	7−2 ↑	17−3 ←	....	....	6−10 ↑	....	8−9 ↑	....	....	2+8 ←
336	18+8 ↑	....	3−4 ←	13−5 ←	22−6 ↑	16−7 ←	4−11 ←	....	7−10 ↑	....	22+7 ↑	8+9 ↑
337	14+7 ↑	....	8−7 ↑	18−8 ↑	....	....	1−12 ↑	23+11 ↑	7−11 ←	....	....	13+10 ←
338	10+6 ↑	....	4−9 ←	14−10 ←	23−11 ↑	14−8 ←	21+10 ←	....	6−12 ←	....	....	19+11 ↑
339	7+5 ←	....	9−12 ↑	19+11 ↑	....	....	19+9 ←	....	5+11 ↑	....	9+6 ↑	....
340	3+4 ←	23+3 ←	5+10 ←	15+9 ←	....	12−9 ↑	16+8 ↑	....	5+10 ←	....	....	0+12 ←
341	19+2 ↑	....	1+8 ←	10+7 ↑	20+6 ↑	....	14+7 ←	....	4+9 ←	....	20+5 ↑	6−11 ↑
342	15+1 ↑	....	6+5 ←	16+4 ←	....	10−10 ↑	12+6 ←	....	3+8 ↑	....	....	11−10 ←
343	11+0 ↑	....	2+3 ←	11+2 ↑	21+1 ↑	....	9+5 ↑	....	2+7 ↑	....	....	17−9 ↑
344	8−1 ←	....	7+0 ↑	17−1 ←	....	8−11 ↑	7+4 ↑	....	2+6 ←	....	8+4 ←	22−8 ←
345	4−2 ←	....	3−2 ←	12−3 ↑	22−4 ↑	....	5+3 ←	....	1+5 ←	....	....	....
346	0−3 ←	20−4 ↑	8−5 ↑	18−6 ←	....	7−12 ←	3+2 ←	....	0+4 ↑	....	19+3 ←	4−7 ↑
347	16−5 ↑	....	4−7 ←	14−8 ←	23−9 ↑	....	0+1 ↑	22+0 ←	0+3 ←	23+2 ←	....	9−6 ←
348	12−6 ↑	....	9−10 ↑	19−11 ←	....	5+11 ←	20−1 ←	....	22+1 ↑	....	....	15−5 ↑
349	9−7 ←	....	5−12 ←	15+11 ←	....	....	17−2 ↑	....	21+0 ↑	....	6+2 ←	20−4 ←
350	5−8 ←	....	0+10 ↑	10+9 ↑	20+8 ↑	3+10 ←	15−3 ↑	....	21−1 ←	....	....	....
351	1−9 ←	21−10 ↑	6+7 ←	16+6 ←	....	....	13−4 ←	....	20−2 ←	....	17+1 ←	2−3 ↑
352	17−11 ↑	....	2+5 ←	11+4 ↑	21+3 ↑	1+9 ↑	10−5 ↑	....	19−3 ↑	....	....	7−2 ←
353	13−12 ↑	....	7+2 ←	17+1 ←	....	23+8 ↑	8−6 ↑	....	19−4 ←	....	....	13−1 ←
354	10+11 ←	....	3+0 ←	12−1 ↑	22−2 ↑	....	6−7 ←	....	18−5 ←	....	4+0 ←	19−0 ↑
355	6+10 ←	....	8−3 ↑	18−4 ←	....	21+7 ↑	4−8 ←	....	17−6 ↑	....	....	....
356	2+9 ←	22+8 ↑	4−5 ←	13−6 ↑	23−7 ↑	....	1−9 ↑	23−10 ↑	16−7 ↑	....	15−1 ←	0+1 ←
357	18+7 ↑	....	9−8 ↑	19−9 ←	....	20+6 ←	21−11 ←	....	16−8 ←	....	....	6+2 ↑
358	14+6 ↑	....	5−10 ←	15−11 ←	....	....	18−12 ↑	....	15−9 ↑	....	....	11+3 ←
359	11+5 ←	....	0−12 ↑	10+11 ↑	20+10 ←	18+5 ←	16+11 ↑	....	14−10 ↑	....	2−2 ←	17+4 ↑
360	7+4 ←	....	6+9 ←	16+8 ←	....	....	14+10 ←	....	14−11 ←	....	....	22+5 ←
361	3+3 ←	23+2 ↑	1+7 ↑	11+6 ↑	21+5 ↑	16+4 ←	12+9 ←	....	13−12 ←	....	13−3 ←	....
362	19+1 ↑	....	7+4 ←	17+3 ←	....	....	9+8 ↑	....	12+11 ↑	....	....	4+6 ↑
363	16+0 ←	....	2+2 ↑	12+1 ↑	22+0 ↑	14+3 ↑	7+7 ↑	....	11+10 ↑	....	....	9+7 ←
364	12−1 ←	....	8−1 ←	18−2 ←	....	....	5+6 ←	....	11+9 ←	....	0−4 ↑	15+8 ↑
365	8−2 ←	....	4−3 ←	13−4 ↑	23−5 ↑	12+2 ↑	2+5 ↑	....	10+8 ↑	....	....	20+9 ←
366	4−3 ↑	....	9−6 ←	19−7 ←	....	....	0+4 ↑	22+3 ←	9+7 ↑	....	11−5 ↑	....
367	0−4 ↑	20−5 ↑	5−8 ←	14−9 ↑	....	10+1 ↑	20+2 ←	....	9+6 ←	....	....	2+10 ↑
368	17−6 ←	....	0−10 ↑	10−11 ↑	20−12 ←	....	17+1 ↑	....	8+5 ←	....	22−6 ↑	7+11 ←
369	13−7 ←	....	6+11 ←	15+10 ↑	....	9+0 ←	15+0 ←	....	7+4 ↑	....	....	13+12 ↑
370	9−8 ←	....	1+9 ↑	11+8 ↑	21+7 ←	....	13−1 ←	....	7+3 ←	....	....	18−11 ←
371	5−9 ↑	....	7+6 ←	17+5 ←	....	7−1 ←	10−2 ↑	....	6+2 ←	....	9−7 ↑	....

Where one, and only one, hourly height is to go on each component hour, the arrow is used to indicate which hourly height to use. A horizontal arrow indicates that the hourly height belonging to the solar hour written is the one to be taken; an arrow pointing upward indicates that the hourly height belonging to the solar hour next preceding the solar hour written is the one to be taken. For the components J, K, OO, R, and 2 SM the value thus indicated is to be used twice. The group covered is obviously a solar and not a component hour. See §§ 53, 57, Part II.

Rules for constructing or verifying this table.

$$\text{Left-hand part of tabular value} = 1 + \left(d - \tfrac{1}{2}\right) \frac{15}{15 \sim c_1},$$

discarding the decimal even if it exceed 0.5; d is an integer such that

$$d = 24 \left[(\text{day of series} - 1) \div \frac{15}{15 \sim c_1} \right] \mp \text{right-hand part of tabular value, including sign.}$$

The quotient in the brackets is taken to the nearest integer generally. The upper sign is used when $c_1 < 15$; the lower, when $c_1 > 15$.

$c_1 < 15$. If the decimal of solar hour in the first equation above fall between 0.0 and 0.5, the arrow should be horizontal; if between 0.5 and 1.0, vertical.

$c_1 > 15$. Reverse this rule.

The speeds used are those derived from the mean motions given in § 13.

Instead of the above rules, the following may be used:

Suppose all solar hours of the series to have been converted into component hours; in each doubtful case mark that solar hour which lies nearest the component hour thus considered.

TABLE 43.—*For the summation of long-period tides.*

Day of series.	Mf	MSf	Mm	Sa	Day of series.	Mf	MSf	Mm	Sa	Day of series.	Mf	MSf	Mm	Sa
1	0	0	0	0	76	18	13←	18		151	12	2	11	
2	1	1	1		77	19	14	19←		152	13	3	12	
3	2	2	2		78	20	15	19		153	14	4	13	
4	3	3	3		79	21	16	20		154	15	5	14	
5	4	4←	4		80	22	17←	21		155	16	6	15	
6	5	4	5		81	23	17	22		156	17←	6←	15←	
7	6	5	6		82	0←	18	23		157	17	7	16	
8	7←	6	7	0	83	0	19	0	5	158	18	8	17	
9	7	7	7←	1	84	1	20	1	6	159	19	9	18	10
10	8	8	8		85	2	21←	2←		160	20	10←	19	11
11	9	9	9		86	3	21	2		161	21	10	20	
12	10	9←	10		87	4	22	3		162	22	11	21	
13	11	10	11		88	5	23	4		163	23	12	22	
14	12	11	12		89	6	0	5		164	0←	13	22←	
15	13	12	13←		90	7←	1	6		165	0	14←	23	
16	14←	13	13		91	7	2	7		166	1	14	0	
17	14	13←	14		92	8	2←	8		167	2	15	1	
18	15	14	15		93	9	3	9		168	3	16	2	
19	16	15	16		94	10	4	9←		169	4	17	3	
20	17	16	17		95	11	5	10		170	5	18	4←	
21	18	17←	18		96	12	6←	11		171	6	19	4	
22	19	17	19		97	13	6	12		172	7	19←	5	
23	20	18	20←	1	98	14	7	13	6	173	8	20	6	
24	21	19	20	2	99	15	8	14	7	174	8←	21	7	11
25	22	20	21		100	15←	9	15		175	9	22	8	12
26	22←	21	22		101	16	10←	16		176	10	23←	9	
27	23	22	23		102	17	10	16←		177	11	23	10	
28	0	22←	0		103	18	11	17		178	12	0	11←	
29	1	23	1		104	19	12	18		179	13	1	11	
30	2	0	2		105	20	13	19		180	14	2	12	
31	3	1	3		106	21	14	20		181	15	3←	13	
32	4	2	3←		107	22	15	21		182	15←	3	14	
33	5	2←	4		108	22←	15←	22←		183	16	4	15	
34	5←	3	5		109	23	16	22		184	17	5	16	
35	6	4	6		110	0	17	23		185	18	6	17	
36	7	5	7		111	1	18	0		186	19	7	18	
37	8	6←	8		112	2	19←	1		187	20	8	18←	
38	9	6	9	2	113	3	19	2	7	188	21	8←	19	
39	10	7	10	3	114	4	20	3	8	189	22←	9	20	12
40	11	8	10←		115	5←	21	4		190	22	10	21	13
41	12	9	11		116	5	22	5←		191	23	11	22	
42	12←	10	12		117	6	23←	5		192	0	12←	23	
43	13	11	13		118	7	23	6		193	1	12	0	
44	14	11←	14		119	8	0	7		194	2	13	1	
45	15	12	15		120	9	1	8		195	3	14	1←	
46	16	13	16←		121	10	2	9		196	4	15	2	
47	17	14	16		122	11	3	10		197	5←	16←	3	
48	18	15	17		123	12←	4	11		198	5	16	4	
49	19←	15←	18		124	12	4←	12		199	6	17	5	
50	19	16	19		125	13	5	12←		200	7	18	6	
51	20	17	20		126	14	6	13		201	8	19	7←	
52	21	18	21	3	127	15	7	14		202	9	20	7	
53	22	19←	22	4	128	16	8←	15	8	203	10	21	8	
54	23	19	23←		129	17	8	16	9	204	11	21←	9	13
55	0	20	23		130	18	9	17		205	12←	22	10	14
56	1	21	0		131	19←	10	18		206	12	23	11	
57	2←	22	1		132	19	11	19		207	13	0	12	
58	2	23	2		133	20	12←	19←		208	14	1←	13	
59	3	0	3		134	21	12	20		209	15	1	14←	
60	4	0←	4		135	22	13	21		210	16	2	14	
61	5	1	5		136	23	14	22		211	17	3	15	
62	6	2	6		137	0	15	23		212	18	4	16	
63	7	3	6←		138	1	16	0		213	19	5←	17	
64	8	4	7		139	2	17	1←		214	20	5	18	
65	9	4←	8		140	3	17←	1		215	20←	6	19	
66	10	5	9		141	3←	18	2		216	21	7	20	
67	10←	6	10		142	4	19	3		217	22	8	21	
68	11	7	11	4	143	5	20	4		218	23	9	21←	
69	12	8←	12	5	144	6	21←	5	9	219	0	10	22	
70	13	8	13		145	7	21	6	10	220	1	10←	23	14
71	14	9	13←		146	8	22	7		221	2	11	0	15
72	15	10	14		147	9	23	8←		222	3	12	1	
73	16	11	15		148	10	0	8		223	3←	13	2	
74	17	12	16		149	10←	1←	9		224	4	14←	3	
75	17←	13	17		150	11	1	10		225	5	14	4	

This table gives the nearest component "hour" (i. e., 24th of monthly or yearly period) for each day (11:30 a. m.) of the series.

In Mf, MSf, and Mm two days sometimes fall upon the same "hour." The arrow is used to indicate the one making the closer coincidence. Consequently the one so marked, or rather the corresponding daily height, is the one to be taken in preference to the other. See note given below Table 38.

TABLE 43.—*For the summation of long-period tides*—Continued.

Day of series.	Mf	MSf	Mm	Sa	Day of series.	Mf	MSf	Mm	Sa	Day of series.	Mf	MSf	Mm	Sa
226	6	15	4←		311	9	12←	6←	20	396	11←	9	8	
227	7	16	5		312	10←	13	7	21	397	12	10	9	
228	8	17	6		313	10	14	8		398	13	11	10	
229	9	18	7		314	11	15	9		399	14	12	11	
230	10←	19	8		315	12	16	10		400	15	13←	12	
231	10	19←	9		316	13	16←	11		401	16	13	13	
232	11	20	10←		317	14	17	12		402	17	14	14	2
233	12	21	10		318	15	18	13		403	18	15	15	3
234	13	22	11		319	16	19	13←		404	18←	16	15←	
235	14	23	12	15	320	17	20←	14		405	19	17	16	
236	15	23←	13	16	321	18	20	15		406	20	18	17	
237	16	0	14		322	18←	21	16		407	21	18←	18	
238	17←	1	15		323	19	22	17		408	22	19	19	
239	17	2	16		324	20	23	18		409	23	20	20	
240	18	3←	17←		325	21	0	19←		410	0	21	21	
241	19	3	17		326	22	1	19	21	411	1←	22←	22	
242	20	4	18		327	23	1←	20	22	412	1	22	22←	
243	21	5	19		328	0	2	21		413	2	23	23	
244	22	6	20		329	1	3	22		414	3	0	0	
245	23	7	21		330	1←	4	23		415	4	1	1	
246	0	8	22		331	2	5	0		416	5	2←	2	
247	1	8←	23		332	3	5←	1		417	6	2	3	
248	1←	9	0		333	4	6	2←		418	7	3	4←	3
249	2	10	0←		334	5	7	2		419	8←	4	4	4
250	3	11	1	16	335	6	8	3		420	8	5	5	
251	4	12	2	17	336	7	9←	4		421	9	6	6	
252	5	12←	3		337	8←	9	5		422	10	7	7	
253	6	13	4		338	8	10	6		423	11	7←	8	
254	7	14	5		339	9	11	7		424	12	8	9	
255	8	15	6		340	10	12	8		425	13	9	10	
256	8←	16←	7		341	11	13	9	22	426	14	10	11←	
257	9	16	7←		342	12	14	9←	23	427	15	11←	11	
258	10	17	8		343	13	14←	10		428	16	11	12	
259	11	18	9		344	14	15	11		429	16←	12	13	
260	12	19	10		345	15←	16	12		430	17	13	14	
261	13	20	11		346	15	17	13		431	18	14	15	
262	14	21	12		347	16	18	14		432	19	15←	16	
263	15←	21←	13←		348	17	18←	15		433	20	15	17	4
264	15	22	13		349	18	19	16		434	21	16	18	5
265	16	23	14	17	350	19	20	16←		435	22	17	18←	
266	17	0	15	18	351	20	21	17		436	23	18	19	
267	18	1	16		352	21	22←	18		437	23←	19	20	
268	19	1←	17		353	22	22	19		438	0	20	21	
269	20	2	18		354	23	23	20		439	1	20←	22	
270	21	3	19		355	23←	0	21		440	2	21	23	
271	22←	4	20←		356	0	1	22←		441	3	22	0	
272	22	5←	20		357	1	2	22	23	442	4	23	1	
273	23	5	21		358	2	3	23	0	443	5	0←	1←	
274	0	6	22		359	3	3←	0		444	6←	0	2	
275	1	7	23		360	4	4	1		445	6	1	3	
276	2	8	0		361	5	5	2		446	7	2	4	
277	3	9	1		362	6	6	3		447	8	3	5	
278	4	10	2		363	6←	7	4		448	9	4←	6	5
279	5	10←	3		364	7	7←	5←		449	10	4	7←	6
280	6	11	3←		365	8	8	5		450	11	5	7	
281	6←	12	4	18	366	9	9	6		451	12	6	8	
282	7	13	5	19	367	10	10	7		452	13←	7	9	
283	8	14	6		368	11	11←	8		453	13	8	10	
284	9	14←	7		369	12	11	9		454	14	9	11	
285	10	15	8		370	13	12	10		455	15	9←	12	
286	11	16	9		371	13←	13	11		456	16	10	13	
287	12	17	10		372	14	14	12	0	457	17	11	14←	
288	13	18←	10←		373	15	15	12←	1	458	18	12	14	
289	13←	18	11		374	16	16	13		459	19	13←	15	
290	14	19	12		375	17	16←	14		460	20←	13	16	
291	15	20	13		376	18	17	15		461	20	14	17	
292	16	21	14		377	19	18	16		462	21	15	18	
293	17	22	15		378	20←	19	17		463	22	16	19	6
294	18	23	16←		379	20	20←	18		464	23	17←	20	7
295	19	23←	16		380	21	20	19		465	0	17	21	
296	20	0	17	19	381	22	21	19←		466	1	18	21←	
297	20←	1	18	20	382	23	22	20		467	2	19	22	
298	21	2	19		383	0	23	21		468	3	20	23	
299	22	3	20		384	1	0←	22		469	4	21	0	
300	23	3←	21		385	2	0	23		470	4←	22	1	
301	0	4	22		386	3←	1	0		471	5	22←	2	
302	1	5	23←		387	3	2	1←	1	472	6	23	3	
303	2	6	23		388	4	3	1	2	473	7	0	4	
304	3←	7←	0		389	5	4	2		474	8	1	4←	
305	3	7	1		390	6	5	3		475	9	2←	5	
306	4	8	2		391	7	5←	4		476	10	2	6	
307	5	9	3		392	8	6	5		477	11	3	7	
308	6	10	4		393	9	7	6		478	11←	4	8	7
309	7	11	5		394	10	8	7		479	12	5	9	8
310	8	12	6		395	11	9←	8←		480	13	6←	10←	

TABLE 43.—*For the summation of long-period tides*—Continued.

Day of series.	Mf	Msf	Mm	Sa	Day of series.	Mf	MSf	Mm	Sa	Day of series.	Mf	MSf	Mm	Sa
481	14	6	10		566	17	4	13		651	19←	1←	15←	
482	15	7	11		567	18←	4←	13←		652	20	1	15	
483	16	8	12		568	18	5	14		653	21	2	16	
484	17	9	13		569	19	6	15		654	22	3	17	
485	18←	10	14		570	20	7	16	13	655	23	4	18	
486	18	11	15		571	21	8←	17	14	656	0	5	19	
487	19	11←	16		572	22	8	18		657	1	6	20	
488	20	12	17←		573	23	9	19		658	2	6←	21	
489	21	13	17		574	0	10	20		659	2←	7	22	
490	22	14	18		575	1	11	20←		660	3	8	22←	
491	23	15←	19		576	2	12	21		661	4	9	23	19
492	0	15	20		577	2←	13	22		662	5	10←	0	20
493	1←	16	21		578	3	13←	23		663	6	10	1	
494	1	17	22	8	579	4	14	0		664	7	11	2	
495	2	18	23	9	580	5	15	1		665	8	12	3	
496	3	19←	0		581	6	16	2←		666	9←	13	4	
497	4	19	0←		582	7	17	2		667	9	14←	5	
498	5	20	1		583	8	17←	3		668	10	14	5←	
499	6	21	2		584	9	18	4		669	11	15	6	
500	7	22	3		585	9←	19	5	14	670	12	16	7	
501	8	23	4		586	10	20	6	15	671	13	17	8	
502	9	0	5		587	11	21←	7		672	14	18	9	
503	9←	0←	6		588	12	21	8		673	15	19	10	
504	10	1	7		589	13	22	9←		674	16←	19←	11←	
505	11	2	7←		590	14	23	9		675	16	20	11	
506	12	3	8		591	15	0	10		676	17	21	12	20
507	13	4←	9		592	16←	1	11		677	18	22	13	21
508	14	4	10		593	16	2	12		678	19	23←	14	
509	15	5	11	9	594	17	2←	13		679	20	23	15	
510	16	6	12	10	595	18	3	14		680	21	0	16	
511	16←	7	13←		596	19	4	15		681	22	1	17	
512	17	8	13		597	20	5	16		682	23	2	18←	
513	18	9	14		598	21	6	16←		683	0	3←	18	
514	19	9←	15		599	22	6←	17		684	0←	3	19	
515	20	10	16		600	23←	7	18	15	685	1	4	20	
516	21	11	17		601	23	8	19	16	686	2	5	21	
517	22	12	18		602	0	9	20		687	3	6	22	
518	23←	13	19		603	1	10←	21		688	4	7	23	
519	23	13←	20←		604	2	10	22		689	5	8	0	
520	0	14	20		605	3	11	23		690	6	8←	1	
521	1	15	21		606	4	12	23←		691	7	9	1←	21
522	2	16	22		607	5	13	0		692	7←	10	2	22
523	3	17←	23		608	6	14	1		693	8	11	3	
524	4	17	0	10	609	7	15	2		694	9	12←	4	
525	5	18	1	11	610	7←	15←	3		695	10	12	5	
526	6←	19	2		611	8	16	4		696	11	13	6	
527	6	20	3		612	9	17	5←		697	12	14	7	
528	7	21	3←		613	10	18	5		698	13	15	8	
529	8	22	4		614	11	19	6		699	14←	16←	8←	
530	9	22←	5		615	12	19←	7	16	700	14	16	9	
531	10	23	6		616	13	20	8	17	701	15	17	10	
532	11	0	7		617	14	21	9		702	16	18	11	
533	12	1	8		618	14←	22	10		703	17	19	12	
534	13	2	9		619	15	23←	11		704	18	20	13	
535	14	2←	10		620	16	23	12←		705	19	21	14←	
536	14←	3	10←		621	17	0	12		706	20	21←	14	22
537	15	4	11		622	18	1	13		707	21←	22	15	23
538	16	5	12		623	19	2	14		708	21	23	16	
539	17	6←	13	11	624	20	3	15		709	22	0	17	
540	18	6	14	12	625	21	4	16		710	23	1←	18	
541	19	7	15		626	21←	4←	17		711	0	1	19	
542	20	8	16←		627	22	5	18		712	1	2	20	
543	21	9	16		628	23	6	19		713	2	3	21←	
544	21←	10	17		629	0	7	19←		714	3	4	21	
545	22	11	18		630	1	8	20		715	4←	5←	22	
546	23	11←	19		631	2	8←	21	17	716	4	5	23	
547	0	12	20		632	3	9	22	18	717	5	6	0	
548	1	13	21		633	4←	10	23		718	6	7	1	
549	2	14	22		634	4	11	0		719	7	8	2	
550	3	15	23←		635	5	12←	1		720	8	9	3	
551	4	15←	23		636	6	12	2		721	9	10	4	23
552	4←	16	0		637	7	13	2←		722	10	10←	4←	0
553	5	17	1		638	8	14	3		723	11	11	5	
554	6	18	2	12	639	9	15	4		724	12	12	6	
555	7	19←	3	13	640	10	16	5		725	12←	13	7	
556	8	19	4		641	11←	17	6		726	13	14←	8	
557	9	20	5		642	11	17←	7		727	14	14	9	
558	10	21	6		643	12	18	8←		728	15	15	10	
559	11←	22	6←		644	13	19	8		729	16	16	11	
560	11	23	7		645	14	20	9		730	17	17	11←	
561	12	0	8		646	15	21	10	18	731	18	18←	12	
562	13	0←	9		647	16	21←	11	19	732	19	18	13	
563	14	1	10		648	17	22	12		733	19←	19	14	
564	15	2	11		649	18	23	13		734	20	20	15	
565	16	3	12		650	19	0	14		735	21	21	16	

TABLE 43.—*For the summation of long-period tides*—Contiuued.

Day of series.	Mf	MSf	Mm	Sa	Day of series.	Mf	MSf	Mm	Sa	Day of series.	Mf	MSf	Mm	Sa
736	22	22	17←		821	1	19	19		906	3←	16	21	12
737	23	23	17	0	822	2←	20←	20		907	4	17	22	
738	0	23←	18	1	823	2	20	20←		908	5	18	22←	
739	1	0	19		824	3	21	21		909	6	18←	23	
740	2←	1	20		825	4	22	22		910	7	19	0	
741	2	2	21		826	5	23	23		911	8	20	1	
742	3	3←	22		827	6	0	0		912	9	21	2	
743	4	3	23		828	7	1	1	6	913	10	22	3	
744	5	4	0←		829	8	1←	2←	7	914	10←	22←	4	
745	6	5	0		830	9	2	2		915	11	23	5	
746	7	6	1		831	10	3	3		916	12	0	5←	
747	8	7←	2		832	10←	4	4		917	13	1	6	
748	9←	7	3		833	11	5	5		918	14	2←	7	
749	9	8	4		834	12	5←	6		919	15	2	8	
750	10	9	5		835	13	6	7		920	16	3	9	12
751	11	10	6		836	14	7	8		921	17←	4	10	13
752	12	11	7	1	837	15	8	9←		922	17	5	11←	
753	13	12	7←	2	838	16	9←	9		923	18	6	11	
754	14	12←	8		839	17	9	10		924	19	7	12	
755	15	13	9		840	17←	10	11		925	20	7←	13	
756	16	14	10		841	18	11	12		926	21	8	14	
757	17	15	11		842	19	12	13		927	22	9	15	
758	17←	16←	12		843	20	13	14		928	23	10	16	
759	18	16	13		844	21	14	15	7	929	0←	11	17	
760	19	17	14		845	22	14←	16	8	930	0	11←	18←	
761	20	18	14←		846	23	15	16←		931	1	12	18	
762	21	19	15		847	0←	16	17		932	2	13	19	
763	22	20←	16		848	0	17	18		933	3	14	20	
764	23	20	17		849	1	18	19		934	4	15←	21	
765	0	21	18		850	2	18←	20		935	5	15	22	13
766	0←	22	19		851	3	19	21		936	6	16	23	14
767	1	23	20←		852	4	20	22		937	7	17	0	
768	2	0	20	2	853	5	21	23		938	8	18	1	
769	3	1	21	3	854	6	22←	23←		939	8←	19	1←	
770	4	1←	22		855	7←	22	0		940	9	20	2	
771	5	2	23		856	7	23	1		941	10	20←	3	
772	6	3	0		857	8	0	2		942	11	21	4	
773	7←	4	1		858	9	1	3		943	12	22	5	
774	7	5←	2		859	10	2	4	8	944	13	23	6	
775	8	5	3←		860	11	3	5←	9	945	14	0←	7	
776	9	6	3		861	12	3←	5		946	15	0	8	
777	10	7	4		862	13	4	6		947	15←	1	8←	
778	11	8	5		863	14	5	7		948	16	2	9	
779	12	9←	6		864	15	6	8		949	17	3	10	
780	13	9	7		865	15←	7	9		950	18	4←	11	14
781	14←	10	8		866	16	7←	10		951	19	4	12	15
782	14	11	9		867	17	8	11		952	20	5	13	
783	15	12	10	3	868	18	9	12←		953	21	6	14←	
784	16	13	10←	4	869	19	10	12		954	22←	7	14	
785	17	14	11		870	20	11←	13		955	22	8	15	
786	18	14←	12		871	21	11	14		956	23	9	16	
787	19	15	13		872	22	12	15		957	0	9←	17	
788	20	16	14		873	22←	13	16		958	1	10	18	
789	21←	17	15		874	23	14	17	9	959	2	11	19	
790	21	18←	16		875	0	15	18	10	960	3	12	20	
791	22	18	17		876	1	16	19		961	4	13←	21←	
792	23	19	17←		877	2	16←	19←		962	5←	13	21	
793	0	20	18		878	3	17	20		963	5	14	22	
794	1	21	19		879	4	18	21		964	6	15	23	
795	2	22←	20		880	5	19	22		965	7	16	0	15
796	3	22	21		881	5←	20	23		966	8	17←	1	16
797	4	23	22		882	6	20←	0		967	9	17	2	
798	5	0	23←	4	883	7	21	1		968	10	18	3	
799	5←	1	23	5	884	8	22	2		969	11	19	4	
800	6	2	0		885	9	23	2←		970	12←	20	4←	
801	7	3	1		886	10	0←	3		971	12	21	5	
802	8	3←	2		887	11	0	4		972	13	22	6	
803	9	4	3		888	12←	1	5		973	14	22←	7	
804	10	5	4		889	12	2	6	10	974	15	23	8	
805	11	6	5		890	13	3	7	11	975	16	0	9	
806	12	7←	6←		891	14	4	8←		976	17	1	10	
807	12←	7	6		892	15	5	8		977	18	2←	11	
808	13	8	7		893	16	5←	9		978	19	2	11←	
809	14	9	8		894	17	6	10		979	20	3	12	
810	15	10	9		895	18	7	11		980	20←	4	13	
811	16	11	10		896	19←	8	12		981	21	5	14	16
812	17	12	11		897	19	9	13		982	22	6←	15	17
813	18	12←	12	5	898	20	9←	14		983	23	6	16	
814	19←	13	13	6	899	21	10	15←		984	0	7	17←	
815	19	14	13←		900	22	11	15		985	1	8	17	
816	20	15	14		901	23	12	16		986	2	9	18	
817	21	16	15		902	0	13←	17		987	3	10	19	
818	22	16←	16		903	1	13	18		988	3←	11	20	
819	23	17	17		904	2	14	19		989	4	11←	21	
820	0	18	18		905	3	15	20	11	990	5	12	22	

TABLE 43.—*For the summation of long-period tides*—Continued.

Day of series	Mf	MSf	Mm	Sa	Day of series	Mf	MSf	Mm	Sa	Day of series	Mf	MSf	Mm	Sa
991	6	13	23		1071	4	6	20←		1151	3←	23	18	
992	7	14	0←		1072	5	7	21	22	1152	3	0	19	
993	8	15←	0		1073	6	8←	22	23	1153	4	1←	20	
994	9	15	1		1074	7	8	23		1154	5	1	21	
995	10←	16	2		1075	8	9	0		1155	6	2	22	
996	10	17	3	17	1076	9	10	1		1156	7	3	22←	
997	11	18	4	18	1077	10←	11	2←		1157	8	4	23	
998	12	19←	5		1078	10	12←	2		1158	9	5	0	
999	13	19	6		1079	11	12	3		1159	10	6	1	
1000	14	20	7		1080	12	13	4		1160	11	6←	2	
1001	15	21	7←		1081	13	14	5		1161	11←	7	3	
1002	16	22	8		1082	14	15	6		1162	12	8	4	
1003	17←	23	9		1083	15	16	7		1163	13	9	5	4
1004	17	0	10		1084	16	17	8		1164	14	10	5←	5
1005	18	0←	11		1085	17	17←	9←		1165	15	10←	6	
1006	19	1	12		1086	18	18	9		1166	16	11	7	
1007	20	2	13		1087	18←	19	10	23	1167	17	12	8	
1008	21	3	14		1088	19	20	11	0	1168	18	13	9	
1009	22	4←	14←		1089	20	21←	12		1169	18←	14←	10	
1010	23	4	15		1090	21	21	13		1170	19	14	11←	
1011	0	5	16	18	1091	22	22	14		1171	20	15	11	
1012	1	6	17	19	1092	23	23	15		1172	21	16	12	
1013	1←	7	18		1093	0	0	16		1173	22	17	13	
1014	2	8←	19		1094	1	1	16←		1174	23	18	14	
1015	3	8	20←		1095	1←	2	17		1175	0	19	15	
1016	4	9	20		1096	2	2←	18		1176	1←	19←	16	
1017	5	10	21		1097	3	3	19		1177	1	20	17	
1018	6	11	22		1098	4	4	20		1178	2	21	18←	5
1019	7	12	23		1099	5	5	21		1179	3	22	18	6
1020	8	13	0		1100	6	6	22		1180	4	23	19	
1021	8←	13←	1		1101	7	6←	23		1181	5	23←	20	
1022	9	14	2		1102	8←	7	23←	0	1182	6	0	21	
1023	10	15	3←		1103	8	8	0	1	1183	7	1	22	
1024	11	16	3		1104	9	9	1		1184	8←	2	23	
1025	12	17←	4		1105	10	10←	2		1185	8	3←	0	
1026	13	17	5	19	1106	11	10	3		1186	9	3	1	
1027	14	18	6	20	1107	12	11	4		1187	10	4	1←	
1028	15←	19	7		1108	13	12	5←		1188	11	5	2	
1029	15	20	8		1109	14	13	5		1189	12	6	3	
1030	16	21←	9		1110	15←	14	6		1190	13	7	4	
1031	17	21	10		1111	15	15	7		1191	14	8	5	
1032	18	22	10←		1112	16	15←	8		1192	15	8←	6	
1033	19	23	11		1113	17	16	9		1193	16	9	7	6
1034	20	0	12		1114	18	17	10		1194	16←	10	8	7
1035	21	1	13		1115	19	18	11		1195	17	11	8←	
1036	22←	2	14		1116	20	19	12←		1196	18	12	9	
1037	22	2←	15		1117	21	19←	12		1197	19	12←	10	
1038	23	3	16		1118	22	20	13	1	1198	20	13	11	
1039	0	4	17		1119	23	21	14	2	1199	21	14	12	
1040	1	5	17←		1120	23←	22	15		1200	22	15	13	
1041	2	6←	18	20	1121	0	23←	16		1201	23	16←	14←	
1042	3	6	19	21	1122	1	23	17		1202	23←	16	14	
1043	4	7	20		1123	2	0	18		1203	0	17	15	
1044	5←	8	21		1124	3	1	19		1204	1	18	16	
1045	5	9	22		1125	4	2	19←		1205	2	19	17	
1046	6	10←	23←		1126	5	3	20		1206	3	20	18	
1047	7	10	23		1127	6	4	21		1207	4	21	19	
1048	8	11	0		1128	6←	4←	22		1208	5	21←	20	7
1049	9	12	1		1129	7	5	23		1209	6	22	21←	8
1050	10	13	2		1130	8	6	0		1210	6←	23	21	
1051	11	14	3		1131	9	7	1		1211	7	0	22	
1052	12	15	4		1132	10	8	2		1212	8	1	23	
1053	13	15←	5		1133	11	8←	2←	2	1213	9	1←	0	
1054	13←	16	6←		1134	12	9	3	3	1214	10	2	1	
1055	14	17	6		1135	13	10	4		1215	11	3	2	
1056	15	18	7		1136	13←	11	5		1216	12	4	3	
1057	16	19←	8	21	1137	14	12←	6		1217	13←	5←	4	
1058	17	19	9	22	1138	15	12	7		1218	13	5	4←	
1059	18	20	10		1139	16	13	8←		1219	14	6	5	
1060	19	21	11		1140	17	14	8		1220	15	7	6	
1061	20	22	12		1141	18	15	9		1221	16	8	7	
1062	20←	23←	13		1142	19	16	10		1222	17	9	8	
1063	21	23	13←		1143	20←	17	11		1223	18	10	9	8
1064	22	0	14		1144	20	17←	12		1224	19	10←	10	9
1065	23	1	15		1145	21	18	13		1225	20←	11	11	
1066	0	2	16		1146	22	19	14		1226	20	12	11←	
1067	1	3	17		1147	23	20	15←		1227	21	13	12	
1068	2	4	18		1148	0	21	15	3	1228	22	14	13	
1069	3←	4←	19		1149	1	21←	16	4	1229	23	14←	14	
1070	3	5	20		1150	2	22	17		1230	0	15	15	

TABLE 43.—*For the summation of long-period tides*—Continued.

Day of series	Mf	MSf	Mm	Sa
1231	1	16	16	
1232	2	17	17←	
1233	3	18←	17	
1234	4	18	18	
1235	4←	19	19	
1236	5	20	20	
1237	6	21	21	
1238	7	22	22	9
1239	8	23	23	10
1240	9	23←	0←	
1241	10	0	0	
1242	11	1	1	
1243	11←	2	2	
1244	12	3←	3	
1245	13	3	4	
1246	14	4	5	
1247	15	5	6	
1248	16	6	7	
1249	17	7←	7←	
1250	18←	7	8	
1251	18	8	9	
1252	19	9	10	
1253	20	10	11	
1254	21	11	12	10
1255	22	12	13	11
1256	23	12←	14	
1257	0	13	14←	
1258	1←	14	15	
1259	1	15	16	
1260	2	16←	17	
1261	3	16	18	
1262	4	17	19	
1263	5	18	20←	
1264	6	19	20	
1265	7	20←	21	
1266	8	20	22	
1267	9	21	23	
1268	9←	22	0	
1269	10	23	1	11
1270	11	0	2	12
1271	12	1	3←	
1272	13	1←	3	
1273	14	2	4	
1274	15	3	5	
1275	16	4	6	
1276	16←	5←	7	
1277	17	5	8	
1278	18	6	9	
1279	19	7	10	
1280	20	8	10←	
1281	21	9←	11	
1282	22	9	12	
1283	23←	10	13	
1284	23	11	14	12
1285	0	12	15	13
1286	1	13	16	
1287	2	14	17	
1288	3	14←	17←	
1289	4	15	18	
1290	5	16	19	
1291	6←	17	20	
1292	6	18←	21	
1293	7	18	22	
1294	8	19	23←	
1295	9	20	23	
1296	10	21	0	
1297	11	22←	1	
1298	12	22	2	
1299	13←	23	3	13
1300	13	0	4	14
1301	14	1	5	
1302	15	2	6←	
1303	16	3	6	
1304	17	3←	7	
1305	18	4	8	
1306	19	5	9	
1307	20	6	10	
1308	21	7←	11	
1309	21←	7	12	
1310	22	8	13	

Day of series	Mf	MSf	Mm	Sa
1311	23	9	13←	
1312	0	10	14	
1313	1	11←	15	
1314	2	11	16	14
1315	3	12	17	15
1316	4	13	18	
1317	4←	14	19	
1318	5	15	20	
1319	6	16	20←	
1320	7	16←	21	
1321	8	17	22	
1322	9	18	23	
1323	10	19	0	
1324	11←	20←	1	
1325	11	20	2←	
1326	12	21	2	
1327	13	22	3	
1328	14	23	4	
1329	15	0←	5	
1330	16	0	6	15
1331	17	1	7	16
1332	18	2	8	
1333	19	3	9←	
1334	19←	4	9	
1335	20	5	10	
1336	21	5←	11	
1337	22	6	12	
1338	23	7	13	
1339	0	8	14	
1340	1	9←	15	
1341	2	9	16	
1342	2←	10	16←	
1343	3	11	17	
1344	4	12	18	
1345	5	13←	19	16
1346	6	13	20	17
1347	7	14	21	
1348	8	15	22	
1349	9	16	23	
1350	9←	17	23←	
1351	10	18	0	
1352	11	18←	1	
1353	12	19	2	
1354	13	20	3	
1355	14	21	4	
1356	15	22←	5←	
1357	16←	22	5	
1358	16	23	6	
1359	17	0	7	
1360	18	1	8	17
1361	19	2←	9	18
1362	20	2	10	
1363	21	3	11	
1364	22	4	12←	
1365	23←	5	12	
1366	23	6	13	
1367	0	7	14	
1368	1	7←	15	
1369	2	8	16	
1370	3	9	17	
1371	4	10	18	
1372	5	11←	19	
1373	6	11	19←	
1374	7	12	20	
1375	7←	13	21	18
1376	8	14	22	19
1377	9	15	23	
1378	10	16	0	
1379	11	16←	1	
1380	12	17	2	
1381	13	18	2←	
1382	14	19	3	
1383	14←	20	4	
1384	15	20←	5	
1385	16	21	6	
1386	17	22	7	
1387	18	23	8←	
1388	19	0←	8	
1389	20	0	9	
1390	21	1	10	

Day of series	Mf	MSf	Mm	Sa
1391	21←	2	11	19
1392	22	3	12	20
1393	23	4	13	
1394	0	5	14	
1395	1	5←	15←	
1396	2	6	15	
1397	3	7	16	
1398	4←	8	17	
1399	4	9	18	
1400	5	9←	19	
1401	6	10	20	
1402	7	11	21	
1403	8	12	22	
1404	9	13←	22←	
1405	10	13	23	
1406	11←	14	0	20
1407	11	15	1	21
1408	12	16	2	
1409	13	17	3	
1410	14	18	4	
1411	15	18←	5	
1412	16	19	5←	
1413	17	20	6	
1414	18	21	7	
1415	19	22	8	
1416	19←	22←	9	
1417	20	23	10	
1418	21	0	11←	
1419	22	1	11	
1420	23	2←	12	
1421	0	2	13	21
1422	1	3	14	22
1423	2	4	15	
1424	2←	5	16	
1425	3	6	17	
1426	4	7	18←	
1427	5	7←	18	
1428	6	8	19	
1429	7	9	20	
1430	8	10	21	
1431	9←	11	22	
1432	9	11←	23	
1433	10	12	0	
1434	11	13	1	
1435	12	14	1←	
1436	13	15←	2	22
1437	14	15	3	23
1438	15	16	4	
1439	16←	17	5	
1440	16	18	6	
1441	17	19	7	
1442	18	20	8	
1443	19	20←	8←	
1444	20	21	9	
1445	21	22	10	
1446	22	23	11	
1447	23	0	12	
1448	0	0←	13	
1449	0←	1	14←	
1450	1	2	14	
1451	2	3	15	23
1452	3	4←	16	0
1453	4	4	17	
1454	5	5	18	
1455	6	6	19	
1456	7	7	20	
1457	7←	8	21←	
1458	8	9	21	
1459	9	9←	22	
1460	10	10	23	
1461	11	11	0	

TABLE 44.—*Acceleration in* HW *and* LW *of*

[The amplitude of the semidiurnal wave is taken as unity.]

HW phase.* / LW phase.*	0° / 180	10° / 190	20° / 200	30° / 210	40° / 220	50° / 230	60° / 240	70° / 250	80° / 260	90° / 270
Amplitude of diurnal wave.	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
0·0	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 00
0·1	0 00	0 29	0 57	1 24	1 48	2 10	2 27	2 40	2 49	2 52
0·2	0 00	0 57	1 52	2 45	3 33	4 15	4 50	5 19	5 36	5 44
0·3	0 00	1 23	2 45	4 02	5 14	6 17	7 11	7 53	8 22	8 36
0·4	0 00	1 49	3 35	5 16	6 51	8 15	9 28	10 26	11 07	11 29
0·5	0 00	2 13	4 23	6 28	8 24	10 10	11 41	12 56	13 50	14 22
0·6	0 00	2 36	5 09	7 37	9 55	12 03	13 52	15 24	16 33	17 15
0·7	0 00	2 58	5 53	8 43	11 23	13 50	16 00	17 50	19 15	20 09
0·8	0 00	3 19	6 36	9 46	12 48	15 35	18 06	20 14	21 56	23 04
0·9	0 00	3 40	7 17	10 48	14 09	17 18	20 09	22 37	24 36	26 00
1·0	0 00	4 00	7 56	11 47	15 29	18 58	22 09	24 57	27 16	28 57
1·1	0 00	4 18	8 34	12 44	16 46	20 35	24 06	27 15	29 55	31 55
1·2	0 00	4 36	9 10	13 39	18 00	22 09	26 01	29 32	32 33	34 55
1·3	0 00	4 54	9 46	14 33	19 12	23 41	27 54	31 46	35 11	37 56
1·4	0 00	5 11	10 20	15 24	20 22	25 10	29 44	34 00	37 48	40 58
1·5	0 00	5 27	10 52	16 14	21 30	26 37	31 33	36 11	40 24	44 03
1·6	0 00	5 43	11 24	17 02	22 36	28 02	33 18	38 20	43 00	47 09
1·7	0 00	5 58	11 53	17 49	23 40	29 25	35 02	40 28	45 36	50 18
1·8	0 00	6 12	12 24	18 34	24 41	30 45	36 43	42 34	48 11	53 29
1·9	0 00	6 26	12 52	19 18	25 42	32 04	38 23	44 38	50 46	56 43
2·0	0 00	6 40	13 20	20 00	26 40	33 20	40 00	46 40	53 20	60 00
2·1	0 00	6 53	13 47	20 41	27 37	34 34	41 35	48 40	55 54	63 20
2·2	0 00	7 06	14 13	21 21	28 32	35 47	43 08	50 39	58 27	66 44
2·3	0 00	7 18	14 38	22 00	29 25	36 57	44 39	52 36	60 59	70 12
2·4	0 00	7 30	15 02	22 37	30 17	38 06	46 08	54 32	63 31	73 44
2·5	0 00	7 42	15 26	23 13	31 08	39 13	47 36	56 25	66 03	77 22
2·6	0 00	7 53	15 48	23 49	31 57	40 18	49 01	58 16	68 33	81 05
2·7	0 00	8 04	16 11	24 23	32 45	41 22	50 24	60 08	71 03	84 54
2·8	0 00	8 15	16 32	24 56	33 32	42 22	51 45	61 54	73 32	88 51
2·9	0 00	8 25	16 53	25 29	34 17	43 25	53 05	63 39	76 00	92 56
3·0	0 00	8 35	17 14	26 00	35 01	44 22	54 22	65 23	78 26	97 11
3·1	0 00	8 44	17 33	26 31	35 44	45 21	55 38	67 05	80 52	101 36
3·2	0 00	8 54	17 52	27 01	36 26	46 17	56 53	68 45	83 16	106 15
3·3	0 00	9 03	18 11	27 30	37 06	47 12	58 05	70 23	85 39	111 11
3·4	0 00	9 12	18 30	27 56	37 46	48 05	59 16	71 58	87 59	116 25
3·5	0 00	9 21	18 47	28 26	38 25	48 57	60 25	73 32	90 19	122 05
3·6	0 00	9 29	19 05	28 53	39 01	49 47	61 30	75 04	92 35	128 19
3·7	0 00	9 38	19 22	29 19	39 39	50 37	62 38	76 34	94 49	135 21
3·8	0 00	9 46	19 38	29 44	40 14	51 25	63 42	78 02	97 01	143 37
3·9	0 00	9 53	19 54	30 09	40 49	52 12	64 44	79 27	99 10	154 19
4·0	0 00	10 01	20 09	30 33	41 23	52 58	65 45	80 50	101 16	180 00
HW Phase. / LW Phase.	360° / 180	350° / 170	340° / 160	330° / 150	320° / 140	310° / 130	300° / 120	290° / 110	280° / 100	270° / 90

*I. e., the phase of the diurnal wave (B) at the time of HW or LW of the semidiurnal wave (A).

HW phase = (time of HW of A − time of HW of B) b,
LW phase = (time of LW of A − time of HW of B) b.

If one of the speeds be somewhat variable, the resultant times and heights will be given more accurately by keeping the phases between −90° and +90°. If they do not fall within these limits, this and the following table may be entered with the phases:

HW phase = (time of LW of A − time of LW of B) b,
LW phase = (time of HW of A − time of LW of B) b:

the resultant heights must, however, have their signs changed. For tropic tides

$$\text{HW phase} = n\pi + \tfrac{1}{2}A^\circ - B^\circ, \quad \text{LW phase} = n\pi \pm \frac{\pi}{2} + \tfrac{1}{2}A^\circ - B^\circ,$$

n being an integer. (See §§ 5, 20, Part III.)

a semidiurnal wave due to a diurnal wave.

[The amplitude of the semidiurnal wave is taken as unity.]

HW Phase. LW Phase.	100°, 280	110° 290	120° 300	130° 310	140° 320	150° 330	160° 340	170° 350	180° 360
Amplitude of diurnal wave.	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
0·0	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 00
0·1	2 50	2 43	2 31	2 14	1 53	1 28	1 00	0 31	0 00
0·2	5 42	5 29	5 05	4 32	3 50	3 00	2 03	1 03	0 00
0·3	8 35	8 18	7 44	6 55	5 52	4 36	3 09	1 37	0 00
0·4	11 39	11 10	10 28	9 24	7 59	6 17	4 20	2 12	0 00
0·5	14 28	14 06	13 16	11 58	10 12	8 02	5 33	2 50	0 00
0·6	17 27	17 06	16 10	14 38	12 31	9 54	6 51	3 30	0 00
0·7	20 29	20 10	19 09	17 24	14 57	11 51	8 13	4 13	0 00
0·8	23 34	23 19	22 14	20 18	17 30	13 55	9 40	4 58	0 00
0·9	26 42	26 33	25 27	23 20	20 12	16 06	11 13	5 45	0 00
1·0	29 53	29 52	28 47	26 31	23 03	18 26	12 52	6 36	0 00
1·1	33 07	33 18	32 16	29 53	26 04	20 54	14 37	7 31	0 00
1·2	36 26	36 51	35 56	33 27	29 18	23 34	16 30	8 29	0 00
1·3	39 49	40 32	39 47	37 15	32 46	26 26	18 32	9 32	0 00
1·4	43 17	44 23	43 52	41 21	36 33	29 32	20 43	10 40	0 00
1·5	46 51	48 24	48 15	45 48	40 40	32 56	23 07	11 53	0 00
1·6	50 31	52 40	52 59	50 43	45 17	36 43	25 44	13 13	0 00
1·7	54 19	57 11	58 12	56 17	50 32	40 59	28 38	14 40	0 00
1·8	58 15	62 03	64 05	62 53	56 48	45 55	31 53	16 16	0 00
1·9	62 21	67 22	71 01	71 19	64 57	51 56	35 39	18 02	0 00
2·0	66 40	73 20	80 00	86 40	80 00	60 00	40 00	20 00	0 00
2·1	71 14	80 18					45 24	22 13	0 00
2·2	76 07	89 14					52 48	24 45	0 00
2·3	81 25							27 44	0 00
2·4	87 18							31 17	0 00
2·5	94 08							35 47	0 00
2·6	102 48							42 08	0 00
2·7									0 00
2·8									0 00
2·9									0 00
3·0									0 00
3·1									0 00
3·2									0 00
3·3									0 00
3·4									0 00
3·5									0 00
3·6									0 00
3·7									0 00
3·8									0 00
3·9									0 00
4·0									0 00
HW Phase. LW Phase.	260° 80	250° 70	240° 60	230° 50	220° 40	210° 30	200° 20	190° 10	180° 0

When the top argument is used the tabular values are positive; when the bottom argument, they are negative.
To express the acceleration in time divide by a, the speed of the semidiurnal component.
To find the acceleration when b is not exactly equal to ½ a, multiply the tabular values by $\frac{2b}{a}$.

This acceleration is directly expressed in time by multiplying the tabular values by $\frac{2b}{a^2}$.

Table 17 is a graphic form of this table.

Rollet de l'Isle has given in the Annales Hydrographique for 1896 (p. 248) a graphic table serving the purpose of Tables 17, 18, or Tables 44, 45, and which he calls an abacus. It is really the inverse of Tables 17, 18.

TABLE 45.—*Height of* HW *and* LW *for a tide*

[The amplitude of the semidiurnal wave is taken as unity.]

HW Phase.* LW Phase.*	0° 180	10° 190	20° 200	30° 210	40° 220	50° 230	60° 240	70° 250	80° 260	90° 270
Amplitude of diurnal wave.										
0·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
0·1	1·1000	1·0985	1·0941	1·0869	1·0771	1·0650	1·0509	1·0353	1·0186	1·0012
0·2	1·2000	1·1971	1·1885	1·1744	1·1552	1·1314	1·1039	1·0727	1·0395	1·0050
0·3	1·3000	1·2958	1·2831	1·2624	1·2342	1·1991	1·1581	1·1123	1·0629	1·0112
0·4	1·4000	1·3945	1·3780	1·3510	1·3141	1·2681	1·2143	1·1539	1·0885	1·0200
0·5	1·5000	1·4932	1·4731	1·4401	1·3948	1·3384	1·2721	1·1975	1·1165	1·0312
0·6	1·6000	1·5921	1·5684	1·5296	1·4763	1·4098	1·3314	1·2430	1·1467	1·0450
0·7	1·7000	1·6908	1·6639	1·6195	1·5585	1·4822	1·3922	1·2904	1·1792	1·0612
0·8	1·8000	1·7899	1·7596	1·7099	1·6415	1·5558	1·4545	1·3397	1·2139	1·0800
0·9	1·9000	1·8888	1·8555	1·8006	1·7253	1·6304	1·5182	1·3908	1·2508	1·1012
1·0	2·0000	1·9878	1·9515	1·8917	1·8094	1·7059	1·5832	1·4436	1·2898	1·1250
1·1	2·1000	2·0869	2·0477	1·9832	1·8942	1·7824	1·6496	1·4982	1·3309	1·1512
1·2	2·2000	2·1860	2·1440	2·0749	1·9797	1·8598	1·7172	1·5544	1·3741	1·1800
1·3	2·3000	2·2851	2·2405	2·1670	2·0656	1·9380	1·7860	1·6122	1·4194	1·2112
1·4	2·4000	2·3842	2·3371	2·2594	2·1522	2·0170	1·8560	1·6716	1·4668	1·2450
1·5	2·5000	2·4834	2·4339	2·3520	2·2392	2·0970	1·9271	1·7325	1·5161	1·2812
1·6	2·6000	2·5826	2·5307	2·4450	2·3266	2·1774	1·9993	1·7949	1·5673	1·3200
1·7	2·7000	2·6818	2·6276	2·5382	2·4145	2·2586	2·0725	1·8588	1·6206	1·3612
1·8	2·8000	2·7811	2·7247	2·6316	2·5029	2·3406	2·1468	1·9241	1·6757	1·4050
1·9	2·9000	2·8804	2·8219	2·7252	2·5917	2·4232	2·2220	1·9908	1·7327	1·4512
2·0	3·0000	2·9797	2·9191	2·8191	2·6809	2·5065	2·2981	2·0587	1·7915	1·5000
2·1	3·1000	3·0790	3·0165	2·9132	2·7704	2·5903	2·3752	2·1280	1·8521	1·5512
2·2	3·2000	3·1784	3·1139	3·0074	2·8604	2·6747	2·4531	2·1985	1·9144	1·6050
2·3	3·3000	3·2778	3·2114	3·1019	2·9506	2·7597	2·5318	2·2702	1·9785	1·6612
2·4	3·4000	3·3772	3·3090	3·1965	3·0412	2·8452	2·6114	2·3431	2·0443	1·7200
2·5	3·5000	3·4766	3·4067	3·2913	3·1321	2·9312	2·6917	2·4171	2·1117	1·7812
2·6	3·6000	3·5760	3·5044	3·3863	3·2233	3·0177	2·7728	2·4922	2·1808	1·8450
2·7	3·7000	3·6755	3·6023	3·4814	3·3148	3·1047	2·8545	2·5683	2·2514	1·9112
2·8	3·8000	3·7749	3·7002	3·5767	3·4065	3·1921	2·9370	2·6455	2·3234	1·9800
2·9	3·9000	3·8744	3·7981	3·6721	3·4985	3·2800	3·0201	2·7236	2·3970	2·0512
3·0	4·0000	3·9739	3·8961	3·7677	3·5908	3·3682	3·1038	2·8027	2·4721	2·1250
3·1	4·1000	4·0734	3·9941	3·8634	3·6833	3·4569	3·1882	2·8827	2·5485	2·2012
3·2	4·2000	4·1730	4·0922	3·9592	3·7760	3·5459	3·2731	2·9636	2·6262	2·2800
3·3	4·3000	4·2725	4·1904	4·0552	3·8690	3·6353	3·3586	3·0452	2·7053	2·3612
3·4	4·4000	4·3720	4·2886	4·1512	3·9622	3·7250	3·4446	3·1277	2·7856	2·4450
3·5	4·5000	4·4716	4·3869	4·2474	4·0556	3·8151	3·5311	3·2110	2·8671	2·5312
3·6	4·6000	4·5712	4·4852	4·3437	4·1492	3·9055	3·6181	3·2950	2·9500	2·6200
3·7	4·7000	4·6708	4·5836	4·4401	4·2429	3·9962	3·7056	3·3797	3·0334	2·7112
3·8	4·8000	4·7704	4·6820	4·5366	4·3369	4·0872	3·7936	3·4652	3·1182	2·8050
3·9	4·9000	4·8699	4·7804	4·6331	4·4310	4·1785	3·8820	3·5512	3·2039	2·9012
4·0	5·0000	4·9695	4·8789	4·7298	4·5253	4·2701	3·9708	3·6379	3·2906	3·0000
HW Phase. LW Phase.	360° 180°	350° 170°	340° 160°	330° 150°	320° 140°	310° 130°	300° 120°	290° 110°	280° 100°	270° 90°

* See footnote, preceding table.

For high waters use the tabular values as given; but for low waters, alter their signs.

composed of a diurnal and semidiurnal wave.

[The amplitude of the semidiurnal wave is taken as unity.]

HW Phase. LW Phase.	100° 280	110° 290	120° 300	130° 310	140° 320	150° 330	160° 340	170° 350	180° 360	Mean value.†
Amplitude of diurnal wave. 0·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
0·1	0·9839	0·9669	0·9510	0·9365	0·9238	0·9137	0·9062	0·9016	0·9000	1·0006
0·2	0·9702	0·9361	0·9038	0·8745	0·8489	0·8281	0·8127	0·8032	0·8000	1·0025
0·3	0·9590	0·9076	0·8587	0·8141	0·7751	0·7432	0·7195	0·7049	0·7000	1·0056
0·4	0·9503	0·8815	0·8158	0·7554	0·7025	0·6591	0·6267	0·6067	0·6000	1·0100
0·5	0·9442	0·8578	0·7750	0·6986	0·6313	0·5758	0·5342	0·5087	0·5000	1·0157
0·6	0·9406	0·8367	0·7365	0·6436	0·5614	0·4933	0·4423	0·4107	0·4000	1·0226
0·7	0·9397	0·8181	0·7004	0·5906	0·4932	0·4118	0·3508	0·3129	0·3000	1·0308
0·8	0·9415	0·8023	0·6667	0·5397	0·4262	0·3314	0·2598	0·2152	0·2000	1·0404
0·9	0·9460	0·7891	0·6357	0·4911	0·3612	0·2521	0·1693	0·1174	0·1000	1·0513
1·0	0·9532	0·7789	0·6074	0·4448	0·2980	0·1739	0·0794	0·0202	0·0000	1·0635
1·1	0·9632	0·7715	0·5819	0·4012	0·2368	0·0971	−0·0098	−0·0770	−0·1000	1·0772
1·2	0·9760	0·7672	0·5595	0·3602	0·1778	0·0213	−0·0982	−0·1740	−0·2000	1·0921
1·3	0·9919	0·7661	0·5403	0·3223	0·1212	−0·0520	−0·1859	−0·2709	−0·3000	1·1088
1·4	1·0105	0·7682	0·5245	0·2874	0·0672	−0·1239	−0·2727	−0·3674	−0·4000	1·1268
1·5	1·0322	0·7738	0·5123	0·2561	0·0161	−0·1939	−0·3584	−0·4637	−0·5000	1·1465
1·6	1·0569	0·7830	0·5041	0·2287	−0·0326	−0·2616	−0·4429	−0·5597	−0·6000	1·1677
1·7	1·0848	0·7959	0·5002	0·2058	−0·0756	−0·3266	−0·5264	−0·6554	−0·7000	1·1909
1·8	1·1159	0·8129	0·5012	0·1879	−0·1151	−0·3886	−0·6082	−0·7506	−0·8000	1·2160
1·9	1·1504	0·8343	0·5077	0·1765	−0·1487	−0·4468	−0·6882	−0·8454	−0·9000	1·2433
2·0	1·1882	0·8604	0·5210	0·1744	−0·1736	−0·5000	−0·7660	−0·9397	−1·0000	1·2732
2·1	1·2296	0·8919					−0·8412	−1·0333	−1·1000	
2·2	1·2748	0·9298					−0·9126	−1·1262	−1·2000	
2·3	1·3238							−1·2182	−1·3000	
2·4	1·3770							−1·3090	−1·4000	
2·5	1·4348							−1·3984	−1·5000	
2·6	1·4979							−1·4855	−1·6000	
2·7									−1·7000	
2·8									−1·8000	
2·9									−1·9000	
3·0									−2·0000	
3·1									−2·1000	
3·2									−2·2000	
3·3									−2·3000	
3·4									−2·4000	
3·5									−2·5000	
3·6									−2·6000	
3·7									−2·7000	
3·8									−2·8000	
3·9									−2·9000	
4·0									−3·0000	
HW Phase. LW Phase.	260° 80°	250° 70°	240° 60°	230° 50°	220° 40°	210° 30°	200° 20°	190° 10°	180° 0°	

† When b is not exactly equal to $\frac{1}{2}a$, mean value $= 1 + (\text{tabular value} - 1)\,\frac{4b^2}{a^2}$.

Table 18 is a graphic form of this table.

The above column of mean values may be compared with expression (29), Part III, and with the last column of Table 21.

TABLE 46.—*Hyperbolic functions.*

u	u In degrees.	v	v In degrees.	θ	$\sinh u$ $= \tan v$	$\cosh u$ $= \sec v$	$\tanh u$ $= \sin v$	e^u	e^{-u}
	°		°	°					
0·00	0·000000	0·0000	0·0000	0·000	0·0000	1·0000	0·0000	1·0000	1·0000
0·02	1·1459156	0·0200	1·1458	1·145	0·0200	1·0002	0·0200	1·0202	0·9802
0·04	2·291831	0·0400	2·2912	2·288	0·0400	1·0008	0·0400	1·0408	0.9608
0·06	3·437747	0·0600	3·4357	3·428	0·0600	1·0018	0·0599	1·0618	0·9418
0·08	4·58366	0·0799	4·5788	4·561	0·0801	1·0032	0·0798	1·0833	0·9231
0·10	5·72958	0·0998	5·720	5·693	0.1002	1·0050	0·0997	1·1052	0·9048
0·12	6·87549	0·1197	6·859	6·811	0·1203	1·0072	0·1194	1·1275	0·8869
0·14	8·02141	0·1395	7·995	7·917	0·1405	1·0098	0·1391	1·1503	0·8694
0·16	9·16732	0·1593	9·128	9·011	0·1607	1·0128	0·1586	1·1735	0·8521
0·18	10·3132	0·1790	10·258	10·100	0·1810	1·0162	0·1781	1·1972	0·8353
0·20	11·4592	0·1987	11·384	11·167	0·2013	1·0201	0·1974	1·2214	0·8187
0·22	12·6051	0·2183	12·505	12·216	0·2218	1·0243	0·2165	1·2461	0·8025
0·24	13·7510	0·2377	13·621	13·254	0·2423	1·0289	0·2355	1·2712	0·7866
0·26	14·8969	0·2571	14·732	14·271	0·2629	1·0340	0·2543	1·2969	0·7711
0·28	16·0428	0·2764	15·837	15·265	0·2837	1·0395	0·2729	1·3231	0·7558
0·30	17·1887	0·2956	16·937	16·245	0·3045	1·0453	0·2913	1·3499	0·7408
0·32	18·3346	0·3147	18·030	17·197	0·3255	1·0516	0·3095	1·3771	0·7261
0·34	19·4806	0·3336	19·116	18·134	0·3466	1·0584	0·3275	1·4049	0·7118
0·36	20·6265	0·3525	20·195	19·045	0·3678	1·0655	0·3452	1·4333	0·6977
0·38	21·7724	0·3712	21·267	19·935	0·3892	1·0731	0·3627	1·4623	0·6839
0·40	22·9183	0·3894	22·331	20·801	0·4108	1·0811	0·3799	1·4918	0·6703
0·42	24·0642	0·4082	23·386	21·648	0·4325	1·0895	0·3969	1·5220	0·6570
0·44	25·2101	0·4264	24·434	22·470	0·4543	1·0984	0·4136	1·5527	0·6440
0·46	26·3561	0·4446	25·473	23·275	0·4764	1·1077	0·4301	1·5841	0·6313
0·48	27·5020	0·4626	26·503	24·045	0·4986	1·1174	0·4462	1·6161	0·6188
0·50	28·6479	0·4804	27·524	24·803	0·5211	1·1276	0·4621	1·6487	0·6065
0·52	29·7938	0·4980	28·535	25·533	0·5438	1·1383	0·4777	1·6820	0·5945
0·54	30·9397	0·5155	29·537	26·245	0·5666	1·1494	0·4930	1·7160	0·5827
0·56	32·0856	0·5328	30·529	26·930	0·5897	1·1609	0·5080	1·7507	0·5712
0·58	33·2316	0·5500	31·511	27·595	0·6131	1·1730	0·5227	1·7860	0·5599
0·60	34·3775	0·5669	32·483	28·237	0·6367	1·1855	0·5370	1·8221	0·5488
0·62	35·5234	0·5837	33·444	28·861	0·6605	1·1984	0·5511	1·8589	0·5379
0·64	36·6693	0·6003	34·395	29·462	0·6846	1·2119	0·5649	1·8965	0·5273
0·66	37·8152	0·6167	35·336	30·045	0·7090	1·2258	0·5784	1·9348	0·5169
0·68	38·9611	0·6329	36·265	30·604	0·7336	1·2402	0·5915	1·9739	0·5066
0·70	40·1070	0·6489	37·183	31·149	0·7586	1·2552	0·6044	2·0138	0·4966
0·72	41·2530	0·6648	38·091	31·670	0·7838	1·2706	0·6169	2·0544	0·4868
0·74	42·3989	0·6804	38·987	32·174	0·8094	1·2865	0·6291	2·0959	0·4771
0·76	43·5448	0·6958	39·872	32·663	0·8353	1·3030	0·6411	2·1383	0·4677
0·78	44·6907	0·7111	40·746	33·132	0·8615	1·3199	0·6527	2·1815	0·4584
0·80	45·8366	0·7261	41·608	33·587	0·8881	1·3374	0·6640	2·2255	0·4493
0·82	46·9825	0·7412	42·460	34·025	0·9150	1·3555	0·6751	2·2705	0·4404
0·84	48·1285	0·7557	43·299	34·446	0·9423	1·3740	0·6858	2·3164	0·4317
0·86	49·2744	0.7702	44·128	34·848	0·9700	1·3932	0·6963	2·3632	0·4232
0·88	50·4203	0·7844	44·944	35·238	0·9981	1·4128	0·7064	2·4109	0·4148
0·90	51·5662	0·7985	45·750	35·613	1·0265	1·4331	0·7163	2·4596	0·4066
0·92	52·7121	0·8123	46·544	35·976	1·0554	1·4539	0·7259	2·5093	0·3985
0·94	53·8580	0·8260	47·326	36·323	1·0847	1·4753	0·7352	2·5600	0·3906
0·96	55·0039	0·8394	48·097	36·660	1·1144	1·4973	0·7443	2·6117	0·3829
0·98	56·1499	0·8528	48·857	36·983	1·1446	1·5199	0·7531	2·6645	0·3753
1·00	57·2958	0·8658	49·605	37·293	1·1752	1·5431	0·7616	2·7183	0·3679
1·02	58·4417	0·8787	50·343	37·593	1·2063	1·5669	0·7699	2·7732	0·36059
1·04	59·5876	0·8913	51·069	37·880	1·2379	1·5913	0·7779	2·8292	0·35345
1·06	60·7335	0·9038	51·783	38·158	1·2700	1·6164	0·7857	2·8864	0·34646
1·08	61·8794	0·9160	52·485	38·423	1·3025	1·6421	0·7932	2·9447	0·33960
1·10	63·0254	0·9281	53·178	38·677	1·3356	1·6685	0·8005	3·0042	0·33287
1·12	64·1713	0·9400	53·860	38·924	1·3693	1·6956	0·8076	3·0649	0·32628
1·14	65·3172	0·9518	54·531	39·160	1·4035	1·7233	0·8144	3·1268	0·31982
1·16	66·4631	0·9632	55·189	39·387	1·4382	1·7517	0·8210	3·1899	0·31349
1·18	67·6090	0·9745	55·837	39·607	1·4735	1·7808	0·8275	3·2544	0·30728
1·20	68·7549	0·9857	56·476	59·817	1·5095	1·8107	0·8337	3·3201	0·30119

$\theta =$ the angle at the center of the hyperbola made by any secant line and the transverse axis of the hyperbola.
$u =$ twice the area of the hyperbolic rector thus determined, the length of the semiaxis being unity.
$\tan \theta = \tanh u$.
$v =$ an auxiliary angle called the *gudermanian*,* such that the equations of the hyperbola are $x = \sec v$, $y = \tan v$.

* For representations of this angle and for further particulars concerning hyperbolic functions see Chapter IV, by James McMahon, in Merriman and Woodward's Higher Mathematics; and Hoüel, Recueil de Formules et de Tables numérique. Newman and Glaisher have tabulated e^{-x} and e^{x} in the Transactions of the Cambridge Phil. Soc., Vol. 13 (1883), III.

TABLE 46.—*Hyperbolic functions*—Continued.

u	u In degrees.	v	v In degrees.	θ	$\sinh u$ $=\tan v$	$\cosh u$ $=\sec v$	$\tanh u$ $=\sin v$	e^u	e^{-u}
	°		°	°					
1·22	69·9009	0·9967	57·103	40·023	1·5460	1·8412	0·8397	3·3872	0·29523
1·24	71·0468	1·0074	57·721	40·215	1·5831	1·8725	0·8455	3·4556	0·28938
1·26	72·1927	1·0180	58·328	40·401	1·6209	1·9045	0·8511	3·5254	0·28365
1·28	73·3386	1·0284	58·925	40·582	1·6593	1·9373	0·8565	3·5966	0·27804
1·30	74·4845	1·0387	59·511	40·753	1·6984	1·9709	0·8617	3·6693	0·27253
1·32	75·6304	1·0490	60·087	40·920	1·7381	2·0053	0·8668	3·7434	0·26714
1·34	76·7763	1·0586	60·654	41·080	1·7786	2·0404	0·8717	3·8190	0·26185
1·36	77·9223	1·0684	61·212	41·232	1·8198	2·0764	0·8764	3·8962	0·25666
1·38	79·0682	1·0779	61·758	41·380	1·8617	2·1132	0·8810	3·9749	0·25158
1·40	80·2141	1·0873	62·295	41·523	1·9043	2·1509	0·8854	4·0552	0·24660
1·42	81·3600	1·0965	62·823	41·657	1·9477	2·1894	0·8896	4·1371	0·24171
1·44	82·5059	1·1055	63·343	41·788	1·9919	2·2288	0·8937	4·2207	0·23693
1·46	83·6518	1·1145	63·851	41·915	2·0369	2·2691	0·8977	4·3060	0·23224
1·48	84·7978	1·1231	64·351	42·034	2·0827	2·3103	0·9015	4 3929	0·22764
1·50	85·9437	1·1317	64·843	42·148	2·1293	2·3524	0·9051	4·4817	0·22313
1·52	87·0896	1·1402	65·327	42·261	2·1768	2·3955	0·9087	4·5722	0.21871
1·54	88·2355	1·1484	65·800	42·370	2·2251	2·4395	0·9121	4·6646	0·21438
1·56	89·3814	1·1566	66·265	42·473	2·2743	2·4845	0·9154	4·7588	0·21014
1·58	90·5273	1·1646	66·728	42·571	2·3245	2·5305	0·9186	4·8550	0·20598
1·60	91·6732	1·1724	67·171	42·668	2·3756	2·5775	0·9217	4·9530	0·20190
1·62	92·8192	1·1800	67·612	42·756	2·4276	2·6255	0·9246	5·0531	0·19790
1·64	93·9651	1·1876	68·045	42·846	2·4806	2·6746	0·9275	5·1552	0·19398
1·66	95·1110	1·1953	68·469	42·930	2·5346	2·7247	0·9302	5·2593	0·19014
1·68	96·2569	1·2023	68·885	43·013	2·5896	2·7760	0·9329	5·3656	0·18637
1·70	97·4028	1·2094	69·294	43·090	2·6456	2·8283	0·9354	5·4739	0·18268
1·72	98·5487	1·2164	69·696	43·166	2·7027	2·8818	0·9379	5·5845	0·17907
1·74	99·6947	1·2233	70·091	43·233	2·7609	2·9364	0·9402	5·6973	0·17552
1·76	100·8406	1·2300	70·476	43·303	2·8202	2·9922	0·9425	5·8124	0·17204
1·78	101·9865	1·2366	70·856	43·373	2·8806	3·0492	0·9447	5·9299	0·16864
1·80	103·1324	1·2432	71·228	43·433	2·9422	3·1075	0·9468	6·0496	0·16530
1·82	104·2783	1·2495	71·593	43·497	3·0049	3·1669	0·9488	6·1719	0·16203
1·84	105·4242	1·2559	71·952	43·556	3·0689	3·2277	0·9508	6·2965	0·15882
1·86	106·5702	1·2619	72·303	43·615	3·1340	3·2897	0·9527	6·4237	0·15567
1·88	107·7161	1·2680	72·649	43·666	3·2005	3·3530	0·9545	6·5535	0·15259
1·90	108·8620	1·2739	72·987	43·720	3·2682	3·4177	0.9562	6·6859	0·14957
1·92	110·0079	1·2797	73·319	43·770	3·3372	3·4838	0·9579	6·8210	0·14661
1·94	111·1538	1·2854	73·645	43·816	3·4075	3·5512	0.9595	6·9588	0·14370
1·96	112·2997	1·2910	73·966	43·864	3·4792	3·6201	0·9611	7·0993	0·14086
1·98	113·4456	1·2964	74·274	43·910	3·5523	3·6904	0·9626	7·2427	0·13807
2·00	114·5916	1·3017	74·584	43·950	3·6269	3·7622	0·9640	7·3891	0·13534
2·02	115·7375	1·3070	74·886	43·993	3·7028	3·8355	0·9654	7·5383	0·13266
2·04	116·8834	1·3122	75·183	44·032	3·7803	3·9103	0·9667	7·6906	0·13003
2·06	118·0293	1·3173	75·472	44·070	3·8593	3·9867	0·9680	7·8460	0·12745
2·08	119·1752	1·3222	75·758	44·108	3·9398	4·0647	0·9693	8·0045	0·12493
2·10	120·3211	1·3271	76·037	44·145	4·0219	4·1443	0·9705	8·1662	0·12246
2·12	121·4671	1·3319	76·311	44·177	4·1055	4·2256	0·9716	8·3311	0·12003
2·14	122·6130	1·3365	76·578	44·208	4·1909	4·3085	0·9727	8·4994	0·11765
2·16	123·7589	1·3412	76·843	44·239	4·2779	4·3932	0·9737	8·6711	0·11533
2·18	124·9048	1·3457	77·102	44·270	4·3666	4·4797	0·9748	8·8463	0·11304
2·20	126·0507	1·3501	77·354	44·297	4·4571	4·5679	0·9757	9·0250	0·11080
2·22	127·1966	1·3544	77·603	44·327	4·5494	4·6580	0·9767	9·2073	0·10861
2·24	128·3425	1·3587	77·848	44·352	4·6434	4·7499	0·9776	9·3933	0·10646
2·26	129·4885	1·3628	78·084	44·378	4·7394	4·8437	0·9785	9·5831	0·10435
2·28	130·6344	1·3669	78·320	44·402	4·8372	4·9395	0·9793	9·7767	0·10228
2·30	131·7803	1·3710	78·549	44·425	4·9370	5·0372	0·9801	9·9742	0·10026
2·32	132·9262	1·3748	78·773	44·449	5·0387	5·1370	0·9809	10·1757	0·09827
2·34	134·0721	1·3787	78·996	44·469	5·1424	5·2388	0·9816	10·3812	0·09633
2·36	135·2180	1·3825	79·212	44·490	5·2483	5·3427	0·9823	10·5909	0·09442
2·38	136·3640	1·3862	79·425	44·511	5·3562	5·4487	0·9830	10·8049	0·09255
2·40	137·5099	1·3899	79·633	44·532	5·4662	5·5569	0·9837	11·0232	0·09072
2·42	138·6558	1·3934	79·836	44·549	5·5785	5·6674	0·9843	11·2459	0·08892
2·44	139·8017	1·3969	80·037	44·565	5·6929	5·7801	0·9849	11·4730	0·08716
2·46	140·9476	1·4003	80·233	44·582	5·8097	5·8951	0·9855	11·7048	0·08543
2·48	142·0935	1·4037	80·426	44·598	5·9288	6·0125	0·9861	11·9413	0·08374
2·50	143·2394	1·4070	80·615	44·616	6·0502	6·1323	0·9866	12·1825	0·08208
2·60	148·9690	1·4227	81·504	44·683	6·6947	6·7690	0·9890	13·4637	0·07427
2·70	154·6986	1·4366	82·310	44·741	7·4063	7·4735	0·9910	14·8797	0·06721
2·80	160·4282	1·4493	83·040	44·787	8·1919	8·2527	0·9926	16·4446	0·06081
2·90	166·1578	1·4609	83·701	44·828	9·0596	9·1146	0·9940	18·1741	0·05502

TABLE 46.—*Hyperbolic functions*—Continued.

u	u In degrees.	v	v In degrees.	θ	$\sinh u$ $= \tan v$	$\cosh u$ $= \sec v$	$\tanh u$ $= \sin v$	e^{u}	e^{-v}
	°		°	°					
3·00	171·8873	1·4713	84·301	44·861	10·0179	10·0677	0·9951	20·0855	0·04979
3·10	177·6169	1·4808	84·841	44·883	11·0765	11·1215	0·9959	22·1980	0·04505
3·20	183·3465	1·4894	85·331	44·906	12·2459	12·2866	0·9967	24·5325	0·04076
3·30	189·0761	1·4971	85·775	44·925	13·5379	13·5748	0·9973	27·1126	0·03688
3·40	194·8057	1·5041	86·177	44·936	14·9654	14·9987	0·9978	29·9641	0·03337
3·50	200·5352	1·5104	86·541	44·948	16·5426	16·5728	0·9982	33·1155	0·03020
3·60	206·2648	1·5162	86·870	44·961	18·2854	18·3128	0·9985	36·5982	0·02732
3·70	211·9944	1·5214	87·168	44·966	20·2113	20·2360	0·9988	40·4473	0·02472
3·80	217·7240	1·5261	87·445	44·971	22·3394	22·3618	0·9990	44·7012	0·02237
3·90	223·4535	1·5303	87·681	44·975	24·6911	24·7113	0·9992	49·4024	0·02024
4·00	229·1831	1·5342	87·901	44·980	27·2899	27·3082	0·9993	54·5981	0·01832
5·00	286·4789	1·5573	89·227	44·989	74·202	74·208	0·9999	148·41	0·006738
6·00	343·7747	1·5658	89·716	44·993	201·71	201·72	0·9999	403·43	0·002479
7·00	401·0705	1·5690	89·895	45·000	548·35	548·35	1·0000	1096·6	0·000912
8·00	458·3662	1·5701	89·960	45·000	1490·5	1490·5	1·0000	2981·0	0·000335
9·00	515·6620	1·5705	89·986	45·000	4051·6	4051·6	1·0000	8103·1	0·000123
10·00	572·9578	1·5706	89·995	45·000	11013·2	11013·2	1·0000	22026·5	0·000045
∞	∞	1·5708	90·000	45·000	∞	∞	1·0000	∞	0

$$\sinh u = \frac{e^{u} - e^{-u}}{2},\ \cosh u = \frac{e^{u} + e^{-u}}{2},\ \tanh u = \frac{\sinh u}{\cosh u} = \frac{e^{u} - e^{-u}}{e^{u} + e^{-u}}.$$

TABLE 47.—*Period of a wave.*

Depth of water (h).	Length of wave in feet (λ).								
	1	10	100	1 000	10 000	100 000	1 000 000	10 000 000	100 000 000
Feet.	*Seconds.*	*Seconds.*	*Seconds.*	*Seconds.*	*Seconds.*	*Seconds.*	*Seconds.*	*Seconds.*	*Seconds.*
1	0·442	1·873	17·641	176·29	1762·9	17629	176295	1762947	17629473
10	0·442	1·398	5·922	55·789	557·51	5575·1	55751	557508	5575085
100	0·442	1·398	4·419	18·726	176·41	1762·9	17629	176295	1762947
1 000	0·442	1·398	4·419	13·975	59·218	557·89	5575·1	55751	557508
10 000	0·442	1·398	4·419	13·975	44·192	187·26	1764·1	17629	176295
100 000	0·442	1·398	4·419	13·975	44·192	139·75	592·18	5579	55751

The period (τ) of a wave is determined by the equation

$$\tau^2 = \frac{2\pi\lambda}{g} \Big/ \tanh\frac{2\pi h}{\lambda},\ = \frac{0{\cdot}1953\ \lambda}{\tanh 6{\cdot}283185\frac{h}{\lambda}}$$

where g is taken equal to 32·1722 feet per second, as in this table; or

$$\tau^2 = \frac{0{\cdot}195373\ \lambda}{\tanh 6{\cdot}283185\frac{h}{\lambda}}$$

if g is taken equal to 32·16.

TABLE 48.—*Wave velocity.*

Depth of water (h).	Length of wave in feet (λ).									
	1	10	100	1 000	10 000	100 000	1 000 000	10 000 000	100 000 000	Infinite.
Feet.	*Ft./sec.*	*Ft./sec.*	*Ft./sec.*	*Ft./sec.*	*Ft./sec.*	*Ft./sec.*	*Ft./sec.*	*Ft./sec.*	*Ft./sec.*	*Ft./sec.*
1	2·262	5·340	5·668	5·672	5·672	5·672	5·672	5·672	5·672	5·672
10	2·262	7·156	16·89	17·92	17·94	17·94	17·94	17·94	17·94	17·94
100	2·262	7·156	22·63	53·40	56·68	56·72	56·72	56·72	56·72	56·72
1 000	2·262	7·156	22·63	71·56	168·9	179·2	179·4	179.4	179·4	179·4
10 000	2·262	7·156	22·63	71·56	226·3	534·0	566·8	567·2	567·2	567·2
100 000	2·262	7·156	22·63	71·56	226·3	715·6	1689	1793	1794	1794

The wave velocity, i. e., velocity of propagation, is

$$\lambda/\tau.$$

Tables 47 and 48 are adapted from Airy's Tides and Waves.

TABLE 49.—*Ratio of vertical to horizontal axes of elliptic orbits of water particles.*

$\frac{y}{\lambda}$	$2\pi\frac{y}{\lambda}$	Ratio of axes.	$\frac{y}{\lambda}$	$2\pi\frac{y}{\lambda}$	Ratio of axes.	$\frac{y}{\lambda}$	$2\pi\frac{y}{\lambda}$	Ratio of axes.
0·00	0·0000	0·0000	0·10	0·6283	0·5568	0·40	2·5133	0·9869
0·01	0·0628	0·0627	0·12	0·7540	0·6375	0·50	3·1416	0·9962
0·02	0·1257	0·1250	0·14	0·8796	0·7062	0·60	3·7699	0·9989
0·03	0·1885	0·1863	0·16	1·0053	0·7638	0·70	4·3982	0·9995
0·04	0·2513	0·2461	0·18	1·1310	0·8113	0 80	5·0265	0·9999
0·05	0·3142	0·3042	0·20	1·2566	0·8501	0 90	5·6549	0·9999
0·06	0·3770	0·3601				1·00	6·2832	0·9999
0·07	0·4398	0·4134	0·25	1·5708	0·9171			
0·08	0·5027	0·4642	0·30	1·8850	0·9549	10·00	62·8319	1·0000
0·09	0·5655	0·5120	0·35	2·1991	0·9757	∞	∞	1·0000

The maximum horizontal displacement at the bottom (where $y=0$) being A, and $2A$, the distance between the foci of the elliptic orbits, the maximum displacements for other depths are:

$$\mathbf{x}=A\cosh ly=A\cosh 2\pi\frac{y}{\lambda},$$

$$\mathbf{y}=A\sinh ly=A\sinh 2\pi\frac{y}{\lambda};$$

$$\therefore\ \frac{\mathbf{y}}{\mathbf{x}}=\tanh 2\pi\frac{y}{\lambda},$$

which is the ratio tabulated above.

TABLE 50.—*Propagation of a free tide wave along a uniform channel.*

Depths.		Velocity of propagation.			Wave length, statute miles.	Time required to travel			Difference in phase of tide wave		
Fathoms.	Feet.	Feet per second.	Knots, or nautical miles, per hour.	Statute miles per hour.		1 foot.	1 naut. mile.	1 stat. mile.	per statute mile.	per foot.	
						s.	*h.*	*h.*	°	°	*Radians.*
0	0	0·000	0·000	0·000	0·00					[0·00	[0·0000
	1	5·672	3·358	3·867	48·03	0·1763	0·2978	0·2586	7·4953	14196	2478
	2	8·022	4·750	5·469	67·93	0·1247	0·2105	0·1828	5·2996	10037	1752
	3	9·824	5·817	6·698	83·20	0·1018	0·1719	0·1493	4·3269	08195	1430
	4	11·344	6·717	7·735	96·07	0·08818	0·1489	0·1293	3·7472	07097	1239
	5	12·683	7·510	8·648	107·41	0·07886	0·1332	0·1156	3·3517	06348	1108
1	6	13·894	8·226	9·473	117·66	0·07199	0·1216	0·1056	3·0597	05795	1011
	7	15·007	8·886	10·232	127·09	0·06662	0·1125	0·09775	2·8327	05365	0936
	8	16·043	9·499	10·938	135·86	0·06234	0·1053	0·09141	2·6498	05019	0876
	9	17·016	10·075	11·602	144·10	0·05877	0·09921	0·08621	2·4983	04732	0826
	10	17·937	10·620	12·230	151·90	0·05574	0·09416	0·08177	2·3700	04489	0783
	11	18·812	11·139	12·826	159·31	0·05316	0·08985	0·07794	2·2597	04280	0747
2	12	19·649	11·634	13·397	166·40	0·05089	0·08598	0·07463	2·1635	04098	0715
	13	20·451	12·109	13·944	173·19	0·04890	0·08258	0·07174	2·0785	03937	0687
	14	21·223	12·566	14·470	179·73	0·04713	0·07955	0·06911	2·0030	03794	0662
	15	21·968	13·007	14·978	186·04	0·04552	0·07686	0·06676	1·9351	03665	0640
	16	22·688	13·434	15·469	192·14	0·04407	0·07446	0·06464	1·8737	03549	0619
	17	23·387	13·847	15·945	198·05	0·04275	0·07220	0·06270	1·8177	03443	0601
3	18	24·065	14·249	16·408	203·79	0·04156	0·07018	0·06094	1·7664	03345	0584
4	24	27·787	16·453	18·946	235·32	0·03598	0·06079	0·05277	1·5298	02897	0506
5	30	31·067	18·395	21·182	263·09	0·03219	0·05435	0·04721	1·3683	02591	0452
6	36	34·032	20·151	23·204	288·21	0·02939	0·04963	0·04310	1·2491	02366	0413
7	42	36·759	21·765	25·063	311·30	0·02720	0·04593	0·03990	1·1564	02190	0382
8	48	39·297	23·268	26·794	332·79	0·02545	0·04297	0·03733	1·0818	02049	0358
9	54	41·681	24·680	28·419	352·98	0·02399	0·04052	0·03519	1·0199	01932	0337
10	60	43·936	26·014	29·956	372·07	0·02276	0·03845	0·03338	0·9675	01832	0320
15	90	53·810	31·861	36·688	455·69	0·01858	0·03139	0·02726	0·7900	01496	0261
20	120	62·134	36·790	42·364	526·19	0·01610	0·02718	0·02361	0·6842	01296	0226
30	180	76·099	45·058	51·885	644·45	0·01314	0·02219	0·01927	0·5586	01058	0185
40	240	87·871	52·029	59·912	744·14	0·01138	0·01922	0·01669	0·4838	00916	0160
50	300	98·243	58·170	66·984	831·98	0·01018	0·01719	0·01493	0·4327	00820	0143
60	360	107·620	63·722	73·377	911·39	0·009294	0·01569	0 01363	0·3950	00748	0131
70	420	116·243	68·828	79·256	984·41	0·008606	0·01453	0·01262	0·3657	00693	0121
80	480	124·268	73·580	84·728	1052·38	0·008045	0.01359	0·01180	0·3421	00648	0113
90	540	131·807	78·043	89·868	1116·22	0·007587	0·01281	0·01113	0·3225	00611	0107
100	600	138·936	82·265	94·729	1176·60	0·007199	0·01216	0·01056	0·3060	00579	0101
150	900	170·162	100·754	116·019	1441·03	0·005877	0·00992	0·00862	0·2498	00473	0083
200	1200	196·486	116·340	133·968	1663·96	0·005089	0·008598	0·007463	0·2163	00410	0072
300	1800	240·645	142·487	164·076	2037·92	0·004156	0·007018	0·006094	0·1766	00334	0058
400	2400	277·873	164·530	189·459	2353·19	0·003598	0·006079	0·005277	0·1530	00290	0051
500	3000	310·671	183·950	211·821	2630·95	0·003219	0·005435	0·004721	0·1368	00259	0045
600	3600	340·323	201·507	232·039	2882·06	0.002939	0·004963	0·004310	0·1249	00237	0041
700	4200	367·591	217·653	250·630	3112·98	0·002720	0·004593	0·003990	0·1156	00219	0038
800	4800	392·971	232·680	267·935	3327·91	0·002545	0·004297	0·003733	0·1082	00205	0036
900	5400	416·809	246·795	284·188	3529·79	0·002399	0·004052	0·003519	0·1020	00193	0034
1000	6000	439·356	260·145	299·561	3720·72	0·002276	0·003845	0.003338	0·0967	00183	0032
1500	9000	538·098	318·611	366·885	4556·94	0·001858	0·003139	0·002726	0·0790	00150	0026
2000	12000	621·342	367·900	423·643	5261·90	0·001610	0·002718	0·002361	0·0684	00130	0023
3000	18000	760·986	450·584	518·854	6444·48	0·001314	0·002219	0·001927	0·0559	00106	0019
4000	24000	878·711	520·289	599·121	7441·44	0·001138	0·001922	0·001669	0·0484	00092	0016
5000	30000	982·428	581·701	669·838	8319·78	0·001018	0·001719	0·001493	0·0433	00082	0014
6000	36000	1076·20	637·222	733·770	9113·87	0·000929	0·001569	0·001363	0·0395	00075	0013
7000	42000	1162·43	688·278	792·563	9844·10	0·000861	0·001453	0·001262	0·0366	00069	0012
8000	48000	1242·68	735·800	847·283	10523·79	0·000804	0·001359	0·001180	0·0342	00065	0011
9000	54000	1318·07	780·434	898·682	11162·17	0·000759	0·001281	0·001113	0·0323	00061	0011
10000	60000	1389·36	822·650	947·294	11765·96	0·000720	0·001216	0·001056	0·0306	00058	0010

Velocity $=\sqrt{gh}$

where $h=$ the undisturbed depth in feet and $g=$ the acceleration of gravity, assumed to be 32·1722 feet per second in this computation.

If

$\tau=$ the periodic time ($=$ 12·4206012 solar hours or $=$ ½ lunar day for the tide wave), then

λ, or wave-length, $=\tau\sqrt{gh}$ feet $=\frac{\tau}{5280}\sqrt{gh}$ miles.

Difference in phase $=\frac{360°}{\lambda}$.

The nautical mile is taken as 6080 feet.

TREASURY DEPARTMENT

UNITED STATES COAST AND GEODETIC SURVEY

W. W. DUFFIELD
SUPERINTENDENT

PHYSICAL HYDROGRAPHY

MANUAL OF TIDES

PART III

By ROLLIN A. HARRIS

APPENDIX No. 7—REPORT FOR 1894

WASHINGTON
GOVERNMENT PRINTING OFFICE
1895

TREASURY DEPARTMENT

UNITED STATES COAST AND GEODETIC SURVEY

W. W. DUFFIELD
SUPERINTENDENT

PHYSICAL HYDROGRAPHY

MANUAL OF TIDES

PART III

By ROLLIN A. HARRIS

APPENDIX No. 7—REPORT FOR 1894

WASHINGTON
GOVERNMENT PRINTING OFFICE
1895

APPENDIX No. 7—1894.

MANUAL OF TIDES.

PART III.

SOME CONNECTIONS BETWEEN HARMONIC AND NONHARMONIC QUANTITIES, INCLUDING APPLICATIONS TO THE REDUCTION AND PREDICTION OF TIDES.

By ROLLIN A. HARRIS.

Submitted for publication December 22, 1894.

PREFACE TO PART III.

The following pages have been prepared for the purpose of supplying some of the immediate wants of the Tidal Division. They constitute Part III of a proposed manual of tides, the plan of which may be outlined thus:

Part I. History of tides, including old methods of treatment.

Part II. Tidal observation, equilibrium theory, and harmonic analysis.

Part III. Connections between harmonic and nonharmonic quantities.

Part IV. Tidal theory and astronomical quantities deduced from tidal observations.

Part V. Tidal currents.

Tables. Auxiliary tables and a collection of tidal constants, including cotidal charts.

As no considerable portion of the material constituting Part III is elsewhere available, it has seemed desirable to publish it in advance of Parts I and II, whose subject-matter has been largely treated in various publications.

The object of this paper is, in a general way, indicated by its title; but it may not be out of place to comment here upon certain of its aims and features.

Owing to a lack of reasonably precise definitions and adequate modes of reduction, the nonharmonic constants (i. e. those referring to high and low water) have too often been of little service where an accurate and satisfactory knowledge of the tide was desired. It is true that quite accurate predictions have been made, based upon the tables resulting from nonharmonic discussions; but it is to be observed (1) that the tables so used were designed chiefly for particular stations and so were of little or no use generally; (2) that long series of observations were necessary for the construction of such tables; (3) that most nonharmonic discussions have been confined to stations where the diurnal inequality is so small as to give no serious trouble. The present attempt to overcome these objections provides an approximate method of analyzing a single month's observations upon high and low water for obtaining both the harmonic and the nonharmonic constants, regardless of the amount of diurnal inequality, and without difficult computations. It also provides tables for the analysis and prediction of tides, designed for general use, regardless of the location of the station.

Little attention has heretofore been given to the classification of tides for the purpose of selecting the best possible port of reference for a given subordinate station, proximity in geographic position having been the sole criterion with few exceptions. Now, it is believed, are given for the first time such harmonic and nonharmonic criteria as will readily show how the types of tide at different stations compare. In this connection are given rules for referring one station to another.

The chapter entitled "Prediction of Tides" describes a proposed tide-predicting machine combining the merits of the Thomson and the Ferrel machines; a general graphic process enabling one (by means of a set of curves constructed once for all from the harmonic constants of the given station) to predict tides without a machine and without computation, the amount of work involved depending upon the accuracy desired; also an approximate arithmetical method for predicting high and low waters from certain tidal constants.

By correlating harmonic and nonharmonic quantities a better insight into tides is obtained than from the consideration of either alone. In this way observable phenomena may be inferred from harmonic constants, inconsistencies may be detected, and missing quantities supplied; also particular values of nonharmonic quantities may be reduced to their mean values, and conversely.

§§ 46, 47 are given mainly for the purpose of furnishing a key to the study of Ferrel's nonharmonic analyses. These discussions, covering years of observations, throw much light upon the nature of tidal inequalities.

The appended tables, already alluded to, are based principally upon those of Baird and Darwin. The considerable computation involved in their preparation has been performed by members of the Tidal Division. Their scope differs somewhat from that of the paper accompanying them. This remark applies especially to Tables 1 to 13, which are designed for the harmonic analysis of tides from hourly ordinates, a subject to be treated in Part II.

I have to acknowledge much assistance from Mr. L. P. Shidy, whose long experience in tidal work has rendered his advice particularly valuable, and whose cooperation has enabled the work to go on.

CONTENTS OF PART III.

CHAPTER I.

PROPERTIES OF COMPOUND WAVE HAVING A PREDOMINATING COMPONENT.

CHAPTER II.

COMPUTATION OF NONHARMONIC QUANTITIES FROM HARMONIC TIDAL CONSTANTS.

CHAPTER III.

REDUCTIONS OF OBSERVATIONS MADE UPON HIGH AND LOW WATERS.

CHAPTER IV.

TO REDUCE RESULTS TO THEIR MEAN VALUES.

CHAPTER V.

ON THE CLASSIFICATION OF TIDES.

CHAPTER VI.

PREDICTION OF TIDES.

TABLES.

CHAPTER I.

PROPERTIES OF A COMPOUND WAVE HAVING A PREDOMINATING COMPONENT.

1. The equation of a compound wave may be written

$$y = A \cos (at + \alpha) + B \cos (bt + \beta) + C \cos (ct + \gamma) + \quad . \quad . \quad . \tag{1}$$

where A, a, α refer to the predominating component (i. e. the one which determines the number and approximate times of the maximum and minimum values of y) and where B, b, β; C, c, γ; . . . refer to the subordinate components.

The values of t rendering y a maximum or a minimum are roots of the derived equation

$$Aa \sin (at + \alpha) + Bb \sin (bt + \beta) + Cc \sin (ct + \gamma) + \quad . \quad . \quad . \quad = 0. \tag{2}$$

Any root of (2) which causes y to become a maximum may be written

$$t = \frac{2\,n\,\pi}{a} - \frac{v}{a} - \frac{\alpha}{a}, \tag{3}$$

and a minimum

$$t = \frac{(2\,n + 1)\,\pi}{a} - \frac{w}{a} - \frac{\alpha}{a}, \tag{4}$$

where n is an integer or zero and v, w, quantities usually small in comparison with π. v/a (or w/a) is the amount by which the time of any maximum (or minimum) of the compound wave precedes the corresponding time of the predominating component.

2. *Approximate expressions for v, w, and y.*

When v is small and b not many times greater than a, we may assume

$$\cos \frac{b}{a}\, v = \cos v \tag{5}$$

$$\sin \frac{b}{a}\, v = \frac{b}{a} \sin v, \tag{6}$$

and so for $\cos \frac{c}{a} v$ and $\sin \frac{c}{a}\, v$. When b differs little from a these are approximations even if v be not so small. Of course the same approximations are true when w replaces v.

From equations (2), (3), and (4) we have

$$\tan v = a\, \frac{Aa \sin 2\,n\,\pi + Bb \sin \frac{b}{a}\left(2\,n\,\pi + \frac{a}{b}\beta - \alpha\right) + Cc \sin \frac{c}{a}\left(2\,n\,\pi + \frac{a}{c}\gamma - \alpha\right) + \quad . \quad . \quad .}{Aa^2 \cos 2\,n\,\pi + Bb^2 \cos \frac{b}{a}\left(2\,n\,\pi + \frac{a}{b}\beta - \alpha\right) + Cc^2 \cos \frac{c}{a}\left(2\,n\,\pi + \frac{a}{c}\gamma - \alpha\right) + \quad . \quad . \quad .}, \tag{7}$$

$$\tan w = a\, \frac{Aa \sin \overline{2\,n+1}\,\pi + Bb \sin \frac{b}{a}\left(\overline{2n+1}\,\pi + \frac{a}{b}\beta - \alpha\right) + Cc \sin \frac{c}{a}\left(\overline{2\,n+1}\,\pi + \frac{a}{c}\gamma - \alpha\right) + \ . \ . \ .}{Aa^2 \cos \overline{2\,n+1}\,\pi + Bb^2 \cos \frac{b}{a}\left(\overline{2\,n+1}\,\pi + \frac{a}{b}\beta - \alpha\right) + Cc^2 \cos \frac{c}{a}\left(\overline{2\,n+1}\,\pi + \frac{a}{c}\beta - \alpha\right) + \ . \ . \ .}, \tag{8}$$

and the corresponding maximum and minimum values of y are

$$y=\Big[A\cos 2n\pi+B\cos\frac{b}{a}\Big(2n\pi+\frac{a}{b}\beta-\alpha\Big)+C\cos\frac{c}{a}\Big(2n\pi+\frac{a}{c}\gamma-\alpha\Big)+\ .\ .\ .\Big]\cos v$$
$$+\Big[A\sin 2n\pi+B\frac{b}{a}\sin\frac{b}{a}\Big(2n\pi+\frac{a}{b}\beta-\alpha\Big)+C\frac{c}{a}\sin\frac{c}{a}\Big(2n\pi+\frac{a}{c}\gamma-\alpha\Big)+\ .\ .\ .\Big]\sin v, \quad (9)$$

$$y=\Big[A\cos\overline{2n+1}\,\pi+B\cos\frac{b}{a}\Big(\overline{2n+1}\,\pi+\frac{a}{b}\beta-\alpha\Big)+C\cos\frac{c}{a}\Big(\overline{2n+1}\,\pi+\frac{a}{c}\gamma-\alpha\Big)+\ .\ .\ .\Big]\cos w$$
$$+\Big[A\sin\overline{2n+1}\,\pi+B\frac{b}{a}\sin\frac{b}{a}\Big(\overline{2n+1}\,\pi+\frac{a}{b}\beta-\alpha\Big)+C\frac{c}{a}\cos\frac{c}{a}\Big(\overline{2n+1}\,\pi+\frac{a}{c}\gamma-\alpha\Big)+\ .\ .\ .\Big]\sin w. \quad (10)$$

3. *To find the approximate average value of the maxima of a compound wave when the speeds of the components are independent of one another.*

Since

$$\cos v=1-\frac{\tan^2 v}{2}+\frac{3}{2\cdot 4}\tan^4 v-\ .\ .\ .\ .\ , \quad (11)$$

equation (9) may be written

$$y=\Big[1-\frac{\tan^2 v}{2}+\ .\ .\ .\Big]\Big[A\cos 2n\pi+B\cos\frac{b}{a}\Big(2n\pi+\frac{a}{b}\beta-\alpha\Big)$$
$$+C\cos\frac{c}{a}\Big(2n\pi+\frac{a}{c}\gamma-\alpha\Big)+\ .\ .\ .\Big]+\tan v\Big[1-\frac{\tan^2 v}{2}+\ .\ .\ .\Big]$$
$$\Big[A\sin 2n\pi+B\frac{b}{a}\sin\frac{b}{a}\Big(2n\pi+\frac{a}{b}\beta-\alpha\Big)+C\frac{c}{a}\sin\frac{c}{a}\Big(2n\pi+\frac{a}{c}\gamma-\alpha\Big)+\ .\ .\ .\Big]. \quad (12)$$

It is required to find an expression for the average value of y; that is, the value of

$$\frac{1}{\infty}\sum_{n=o}^{\infty}y_n, \quad (13)$$

$a, b, c,\ .\ .\ .$ being independent of one another.

The part of (13) which is independent of v reduces to A. The part of (13) which contains $\tan v$ to first power is

$$\frac{1}{\infty}\sum_{n=o}^{\infty}\frac{\Big[Aa\sin 2n\pi+Bb\sin\frac{b}{a}\Big(2n\pi+\frac{a}{b}\beta-\alpha\Big)+Cc\sin\frac{c}{a}\Big(2n\pi+\frac{a}{c}\gamma-\alpha\Big)\ .\ .\ .\Big]^2}{Aa^2\cos 2n\pi+Bb^2\cos\frac{b}{a}\Big(2n\pi+\frac{a}{b}\beta-\alpha\Big)+Cc^2\cos\frac{c}{a}\Big(2n\pi+\frac{a}{c}\gamma-\alpha\Big)+\ .\ .\ .}. \quad (14)$$

The approximate value of the denominator of (14) is Aa^2. The numerator consists of products and squares of its terms; the average value of any such product is zero, while for the squares we have, since the average value of $(\text{sine})^2=\frac{1}{2}$,

$$\tfrac{1}{2}\,[B^2b^2+C^2c^2+\ .\ .\ .].$$

Thus (14) is approximately equal to

$$\frac{1}{2\,Aa^2}\,[B^2b^2+C^2c^2+\ .\ .\ .]. \quad (15)$$

In like manner the part of (13) containing $\tan v$ to the second power is approximately equal to

$$-\frac{1}{4\,Aa^2}\,[B^2b^2+C^2c^2+\ .\ .\ .]. \quad (16)$$

The sum of (15) and (16) being

$$\frac{1}{4\,Aa^2}\,[B^2\,b^2 + C^2\,c^2 + \quad . \quad . \quad .], \tag{17}$$

it follows that

The average value of the maxima of a compound wave is, approximately,

$$A + \frac{1}{4\,Aa^2}\,[B^2\,b^2 + C^2\,c^2 + \quad . \quad . \quad . \quad] \tag{18}$$

provided $a, b, c,$ . . . are each incapable of expression in terms of the others, and the quantity added to A is relatively small.

This also represents the average depression of the minima; or, it is the average amplitude or semirange of the compound wave.

When there are fixed relations between the speeds of the components, the speeds of no two being commensurable with each other, formula (18) may still give the semirange of the compound wave, while the average elevation of the maxima or depression of the minima may not be given with sufficient accuracy. For, any alteration in the average range on account of such a relation implies an alteration in the average of the maxima or minima or both, while the average of the maxima or minima may be altered without necessitating an alteration in the average range.

4. *A wave composed of two simple waves of equal periods.*

Let its equation be

$$y = A\cos at + B\cos(at + \beta). \tag{19}$$

$$\therefore -\tan at = \tan v = \tan w = \frac{B\sin\beta}{A + B\cos\beta}. \tag{20}$$

$$y = \pm\sqrt{A^2 + B^2 + 2\,AB\cos\beta}. \tag{21}$$

The average value of this radical, supposing β to take all values from zero to π*, is

$$\frac{1}{\pi}\int_0^\pi \sqrt{A^2 + B^2 + 2\,AB\cos\beta}\;d\,\beta. \tag{22}$$

This integral represents the average value of the third side of a triangle two of whose sides are the fixed lengths A, B, while the included angle (or its supplement β) varies uniformly.

The integral

$$\int_0^\pi \sqrt{A^2+B^2+2\,AB\cos\beta}\;d\,\beta \tag{23}$$

represents one-half the periphery of an ellipse whose semiaxis major is $A + B$, and whose semiaxis minor is $A - B$.

By writing $2\,\varphi$ for β and $\sin\theta$ for $2\sqrt{\frac{AB}{A+B}}$, this integral becomes $2\,(A+B)$ times the integral

$$E\,(\sin\theta,\,\varphi)$$

which is tabulated by Legendre in his Traité des Fonctions Elliptiques, Table IX. However the average value of y be computed, the result will serve as a test of the accuracy of formula (18) for this special case, even when B is not small in comparison with A. Compare the values in the last column of Table 16 with column 9, Table 21, where the special form of (18),

$$A + \frac{1}{4\,Aa^2}\,[B^2\,b^2] \text{ or } A + \frac{B^2}{4\,A}, \tag{24}$$

has been tabulated.

---

* So far as obtaining an average value is concerned, it is indifferent whether we regard the speeds as equal and vary β, or regard the speeds as differing by an infinitesimal and make β constant or zero. In the latter case the radical may be written

$$\sqrt{A^2 + B^2 + 2\,AB\cos\omega}$$

where

$$\omega = (a \sim b)\,t.$$

It will be seen that the agreement is very close when B is comparatively small; when $B=A$ the discrepancy amounts to a trifle over 2 per cent of A or B.

It follows from this that if a and b denote the semiaxes of an ellipse, the periphery of the same is, as is well known, approximately

$$\pi\left[(a+b)+\frac{(a-b)^2}{4(a+b)}\right].$$

To find

$$\frac{1}{\beta'}\int_0^{\beta'}\sqrt{A^2+B^2+2\,A\,B\cos\beta}\;d\,\beta$$

when β' is small.

Upon replacing $\cos\beta$ by $1-2\sin^2\frac{\beta}{2}$, the radical may be written

$$A+B-\frac{2\,A\,B\sin^2\frac{\beta}{2}}{A+B}\quad\cdot\;\cdot\;\cdot\;,$$

which becomes

$$A+B-\frac{\beta^2}{2}\,\frac{A\,B}{A+B}$$

when β is small. The average value of β^2 when β varies from 0 to β' is $\frac{\beta'^2}{3}$; this gives for the required result

$$A+B-\frac{\beta'^2}{6}\,\frac{A\,B}{A+B}. \tag{25}$$

In like manner the value of

$$\frac{1}{\beta'}\int^{\pi+\beta'}\sqrt{A^2+B^2+2\,A\,B\cos\beta}\;d\,\beta,$$

which is the average value of the radical between $\beta=\pi$ and $\beta=\pi+\beta'$, becomes, when β' is small

$$A-B+\frac{\beta'^2}{6}\,\frac{A\,B}{A-B}. \tag{26}$$

5. *A wave composed of two simple waves, the period of one being twice that of the other.*

Let its equation be

$$y=A\cos at+B\cos(bt+\beta). \tag{27}$$

$$\therefore\;4A\sin\tfrac{1}{2}v+B\tan\tfrac{1}{2}v\cos\beta-B\sin\beta=0 \tag{28}$$

where $v=-at$ and $b=\frac{1}{2}a$. The angle $\frac{1}{2}v$ can be found by the following graphic process:

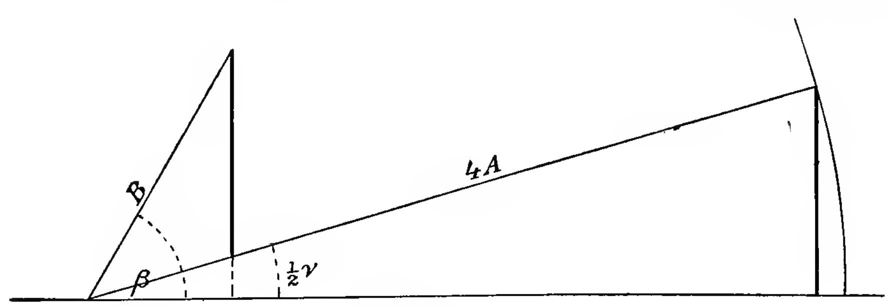


Fig. 1

Lay off β from the horizontal line, figure 1, and let B be laid off in the direction thus determined. Draw an arc to the radius $4A$; move this radius until the heavy lines are equal. The angle so determined is $\frac{1}{2}v$. It is convenient to work upon cross-section paper, using a thread for the radius $4A$.

The values of v are shown on Plate I (Table 17) for each value of β. The corresponding values of y are given on Plate II (Table 18). Formula (18) now becomes

$$A + \frac{1}{4\,A\,a^2}\,[B^2 b^2] \text{ or } A + \frac{B^2}{16\,A} \tag{29}$$

whose values are given in column 15, Table 21. The values in this column, compared with those obtainable from Table 18, afford a means of testing formula (18) for the special case here considered, even when B is greater than A.

6. If the formula

$$A + \frac{1}{4\,Aa^2}\,[B^2\,b^2] \tag{30}$$

expresses the average amplitude of a wave composed of the simple waves A and B, even when B is not small in comparison with A, then the formula

$$A + \frac{1}{4\,Aa^2}\,[B^2\,b^2 + C^2\,c^2] \tag{31}$$

expresses the average amplitude of a wave composed of the simple waves A, B, and C, provided c differs from b by an infinitesimal and $B^2 + C^2$ of (31) is not greater than B^2 of (30).

The amplitude of B and C combined is

$$\sqrt{B^2 + C^2 + 2\,B\,C\cos\,(b \sim c)\,t}. \tag{32}$$

Replacing B^2 of (30) by the square of this expression, the average value of B^2 becomes

$$B^2 + C^2.$$

In like manner, if formula (31) apply to three waves, the formula

$$A + \frac{1}{4\,Aa^2}\,[B^2\,b^2 + C^2\,c^2 + D^2\,d^2] \tag{33}$$

will apply to four waves, provided c and d differ from b, and so from each other, by an infinitesimal and $B^2 + C^2 + D^2$ of (33) is not greater than $B^2 + C^2$ of (31) or B^2 of (30). So one can proceed for any number of simple waves.

Tables 16, 18, and 21 show that (30) is nearly true when a differs from b or from $2\,b$ by an infinitesimal, even if B is not small in comparison with A. In the former case B may be equal to A and in the latter it may be equal to $2\,A$ without sensibly affecting the truth of (30).

7. *The effect of a subordinate wave having a variable amplitude of the form*

$$B'\cos\,(b't + \beta') + B'', \tag{34}$$

where $B'' \gtreqless B'$, *in increasing the average amplitude of a given wave.*

In deriving formula (18) all amplitudes were assumed to be essentially positive; therefore in applying it to the present case the variable amplitude must not become negative.

Since $B'' \gtreqless B'$, all values of $\cos\,(b't + \beta')$ from $+1$ to -1 are to be used, and the average value of B^2, or

$$[\mathrm{B}'\cos\,(b't + \beta') + B'']^2, \tag{35}$$

becomes

$$\tfrac{1}{2}\,B'^2 + B''^2. \tag{36}$$

When $B'' = 0$, this becomes

$$\tfrac{1}{2}\,B'^2; \tag{37}$$

i. e. the effect is one-half as much as it would have been had the amplitude of the subordinate wave been constantly equal to B'. (It may be noted here that the average numerical value of $B'\cos\,(b't + \beta')$ is $2\,B'/\pi$, the square of which is about $\frac{2}{5}\,B'^2$ instead of $\frac{1}{2}\,B'^2$.)

8. *To find the speed of a wave composed of two simple waves having approximately equal but not commensurable speeds.*

If n' be such a value of n that

$$\frac{b}{a}\left(2\,n'\,\pi + \frac{a}{b}\,\beta - \alpha\right) = 0,$$

then there is no displacement of the maximum of A due to B; i. e., A and B conspire at this common maximum. For the following minimum,

$$\frac{b}{a}\left(\overline{2\,n'+1}\,\pi + \frac{a}{b}\,\beta - \alpha\right) = \frac{b}{a}\,\pi.$$

At this maximum of A the phase of B is zero, at the minimum it is $\frac{b}{a}\,\pi$; the phase of B at a time midway between this maximum and minimum of A is equal to $\frac{b}{2\,a}\,\pi$. At the rth subsequent maximum, minimum, and midtime the phases of B are

$$\frac{b}{a}2\,r\,\pi,\ \frac{b}{a}\,\overline{2\,r+1}\,\pi,\ \frac{b}{2\,a}\,\overline{4\,r+1}\,\pi,$$

respectively. If

$$\frac{b}{2\,a}\,\overline{4\,r+1}\,\pi = z,$$

then

$$\frac{b}{a}\,2\,r\pi = z - \frac{b}{a}\frac{\pi}{2}$$

and

$$\frac{b}{a}\,\overline{2\,r+1}\,\pi = z + \frac{b}{a}\frac{\pi}{2}.$$

By means of § 2 we obtain for the length of a half period expressed in hours, say, corresponding to a phase of B equal to z,

$$\tfrac{1}{2}\,P_z = \frac{180^\circ}{a} + 57{\cdot}3\,\frac{Bb\,\sin\left(z + \frac{b}{a}90^\circ\right)}{Aa^2 - Bb^2\cos\left(z + \frac{b}{a}90^\circ\right)} + 57{\cdot}3\,\frac{Bb\,\sin\left(z - \frac{b}{a}90^\circ\right)}{Aa^2 + Bb^2\cos\left(z - \frac{b}{a}90^\circ\right)}. \qquad (39)$$

If t denote the number of hours between the time of conspiring of A and B, and the given time for which the length of a half period is required, we have

$$z = (b - a)\,t. \qquad (40)$$

Finally, the speed for this time is equal to

$$\frac{360^\circ}{P_z}. \qquad (41)$$

At the time of conspiring the speed is approximately equal to

$$a + a\,\frac{Bb\,(b-a)}{Aa^2 + Bb^2}, \qquad (42)$$

and, at the time of interfering,

$$a - a\,\frac{Bb\,(b-a)}{Aa^2 - Bb^2}. \qquad (43)$$

9. *Small components having speeds commensurable with the speed of the predominating one.* Suppose now that in equation (1) there are terms of the form

$$A_i\cos\,(i\,at + \alpha_i) \qquad (44)$$

where i is an integer greater than unity.

So long as all speeds are independent of one another the average value of v or w will be zero. When the harmonics of A are included, the average values of (7) and (8) become

$$\tan v = a \frac{\sum_i A_i\, ai \sin i\left(\frac{\alpha_i}{i} - \alpha\right)}{Aa^2 + \sum_i A_i\,(ia)^2 \cos i\left(\frac{\alpha_i}{i} - \alpha\right)}, \tag{45}$$

and

$$\tan w = a \frac{\sum_i A_i\, ai \sin i\left(\pi + \frac{\alpha_i}{i} - \alpha\right)}{Aa^2 + \sum_i A_i\,(ia)^2 \cos i\left(\pi + \frac{\alpha_i}{i} - \alpha\right)}, \tag{46}$$

where

$$i = 2, 3, \text{etc.}$$

The respective values of each maximum and minimum of a wave *composed of A and its harmonic only* are

$$y = A \cos v + \sum_i A_i \cos i\left(\frac{\alpha_i}{i} - \alpha - v\right) \tag{47}$$

and

$$y = -A \cos w + \sum_i A_i \cos i\left(\pi + \frac{\alpha_i}{i} - \alpha - w\right). \tag{48}$$

These individual values are equivalent to average values, because they recur at each period of A.

If there are also very small components of the form

$$A_j \cos (j\, at + \alpha_j) \tag{49}$$

where j is a vulgar fraction, the effects of one such term upon the average value of the maxima and minima are

$$\frac{A_j}{m} \sum_{n=0}^{n=m-1} \cos j\left(2\, n\, \pi + \frac{\alpha_j}{j} - \alpha\right), \tag{50}$$

and

$$\frac{A_j}{m} \sum_{n=0}^{n=m-1} \cos j\left(\overline{2\, n+1}\, \pi + \frac{\alpha_j}{j} - \alpha\right), \tag{51}$$

where m is the denominator of the fraction j. Special case: When $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ . . . . each of these expressions reduces to zero.

10. *Conditions for a predominating component.*

In order to have a simple wave determine the number of maxima and minima when combined with a series of other simple waves, it is necessary that its greatest slope be greater than the greatest possible slope of the series. For the two steepest parts of the predominating wave fall somewhere upon the serial wave; and the upward and downward slopes at these points cause the resultant curve to slope upward or downward at these points. But a maximum or a minimum implies the existence of both an upward and a downward slope.

Since the greatest slope of any simple wave is proportional to amplitude ÷ wave-length, the necessary condition for a predominating wave is

$$Aa > Bb + Cc + \;.\;.\;.\;.\;. \tag{52}$$

This condition is also sufficient provided the greatest curvature of the predominating component be not less than the greatest possible curvature of the serial wave. Now, the greatest curvature of any component is at a maximum or minimum point; consequently the greatest possible

curvature of a series of waves would be the curvature at a time when all the maxima or the minima fall together. Writing the serial wave, we have

$$y = B\cos(bt+\beta) + C\cos(ct+\gamma) + \quad . \quad . \quad . \quad . \quad , \tag{53}$$

$$\frac{-dy}{dt} = +Bb\sin(bt+\beta) + Cc\sin(ct+\gamma) + \quad . \quad . \quad . \quad . \quad ,$$

$$\frac{-d^2y}{dt^2} = +Bb^2\cos(bt+\beta) + Cc^2\cos(ct+\gamma) + \quad . \quad . \quad . \quad .$$

Let the origin of time be taken at this common maximum, say; then

$$\frac{-dy}{dt} = 0$$

$$-\left(\frac{d^2y}{dt^2}\right)_0 = Bb^2 + Cc^2 + \quad . \quad . \quad . \quad .$$

These values substituted in the expression for the radius of curvature,

$$\left[1 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}} \div \frac{d^2y}{dt^2},$$

gives at this point

$$\rho_0 = \frac{1}{Bb^2 + Cc^2 + \quad . \quad . \quad .};$$

or

$$\text{curvature} = Bb^2 + Cc^2 + \quad . \quad . \quad . \quad .$$

In like manner the greatest curvature of the predominating component is Aa^2 . $\therefore$ the remaining and, with (52), sufficient condition for a predominating component is

$$Aa^2 \gtreqless Bb^2 + Cc^2 + \quad . \quad . \quad . \quad . \tag{54}$$

CHAPTER II.

COMPUTATION OF NONHARMONIC QUANTITIES FROM HARMONIC TIDAL CONSTANTS.

11. In the preceding chapter on waves nothing has been said concerning the cause producing them. It will now be assumed that the maximum of a component follows its apparent cause, or fictitious moon, by a certain angle called the epoch of the component, and whose value is such that if it be divided by the speed of the component, the quotient is the time elapsing between the transit of the fictitious moon and the occurrence of the maximum value of the component. For this reason the terms which have been written in the form

$$A \cos (at + \alpha)$$

will now be written in the form

$$A \cos (at + \arg_0 A - A^\circ).$$

A may be used to denote either the component A itself or its amplitude;
a is the speed of the component expressed in degrees per mean solar hour;
α is the initial argument or phase of the component A; i. e. its phase when $t = 0$;
$\arg_0 A$ denotes the initial argument or hour angle of the fictitious moon producing A; for brevity it is usually called the initial argument or initial equilibrium argument of the component A. To avoid misunderstanding $at + \alpha$ will be spoken of as the *phase* rather than as the *argument* of the component.
A° denotes the epoch or lag of the component A behind its fictitious moon;

$$\therefore \alpha = \arg_0 A - A^\circ.$$

A° / a is the epoch expressed in time and may then be called the interval of the component.

For harmonics of A we always have

$$a_i = i a_1,$$

and usually

$$\arg_0 A_i = i \arg_0 A_1. \tag{55}$$

The mean range, the spring range, etc., and the high or low water inequality in height.

For some purposes it is convenient to give names to various ranges of tide.

Mean range is the average value of the semidaily range of tide.

Spring range is the greatest periodic semidaily range, occurring usually one or two days after new and full moon.

Neap range is the smallest periodic semidaily range, occurring usually one or two days after the moon is in quadrature, that is, after the first and third quarters.

Perigean range is the greatest periodic semidaily range of tide, occurring usually from one to three days after the moon is in perigee.

Apogean range is the smallest periodic semidaily range, occurring usually from one to three days after the moon is in apogee.

Great diurnal range is the difference between the mean of all the higher high waters and the mean of all the lower low waters of each day during one or more half tropical or declinational months.

Small diurnal range is the difference between the mean of all the lower high waters and the mean of all the higher low waters of each day during one or more half tropical or declinational months.

Great tropic range is the greatest periodic daily range of tide, usually occurring soon after the moon is farthest north or south from the equator and therefore near one of the tropics.

Small tropic range is the smallest periodic daily range of tide, usually occurring soon after the moon is farthest north or south from the equator and therefore near one of the tropics.

Tides determining these ranges, or of simultaneous occurrence, may be referred to as *spring, neap, perigean, tropic*, etc.; a like remark is applicable to lunitidal intervals, and occasionally to other quantities.

Diurnal inequality as here used generally denotes the greatest periodic difference in height between two consecutive high waters or low waters, usually occurring soon after the moon is farthest north or south from the equator; this inequality is therefore determined by the tropic tides.

12. *To compute the mean lunitidal interval or corrected establishment.*

When the tide contains no components whose speeds are commensurable with that of M_2, one can assume that the high water establishment is, in hours,

$$\frac{M_2^\circ}{28^\circ{\cdot}9841}, \text{ or } \frac{M_2^\circ}{m_2}, \tag{56}$$

and the low-water establishment is the same increased, say, by one-fourth of a mean lunar day, that is

$$\frac{M_2^\circ}{m_2} + 6^{h}{\cdot}2103 \; (= 6^{h}\, 12^{m}{\cdot}62). \tag{57}$$

When M_2 has harmonics, which are denoted by M_4, M_6, etc., the high water establishment is

$$\frac{M_2^\circ - v}{m_2} \tag{58}$$

and the low water

$$\frac{M_2^\circ - w}{m_2} + 6^{h}{\cdot}2103 \tag{59}$$

where from § 9, since $\alpha_i = i \arg_0 A_1 - A_i^\circ$,

$$\tan v = \frac{2\, M_4 \sin(2\, M_2^\circ - M_4^\circ) + 3\, M_6 \sin(3\, M_2^\circ - M_6^\circ) + \quad . \quad . \quad .}{1^2\, M_2 + 2^2\, M_4 \cos(2\, M_2^\circ - M_4^\circ) + 3^2\, M_6 \cos(3\, M_2^\circ - M_6^\circ) + \quad . \quad . \quad .}, \tag{60}$$

$$\tan w = \frac{2\, M_4 \sin(2\, M_2^\circ - M_4^\circ) - 3\, M_6 \sin(3\, M_2^\circ - M_6^\circ) + \quad . \quad . \quad .}{-\, 1^2\, M_2 + 2^2\, M_4 \cos(2\, M_2^\circ - M_4^\circ) - 3^2\, M_6 \cos(3\, M_2^\circ - M_6^\circ) + \quad . \quad . \quad .}. \tag{61}$$

Less accurate formulæ for v and w are

$$\tan v = \frac{2\, M_4 \sin(2\, M_2^\circ - M_4^\circ) + 3\, M_6 \sin(3\, M_2^\circ - M_6^\circ),}{M_2} \tag{62}$$

$$\tan w = \frac{2\, M_4 \sin(2\, M_2^\circ - M_4^\circ) - 3\, M_6 \sin(3\, M_2^\circ - M_6^\circ).}{-\, M_2} \tag{63}$$

13. *To compute the mean range of tide from given harmonic constants.*

In formula (18) substitute M_2 for A, and other components whose speeds are incommensurable with m_2 for B, C, etc. Multiply the result by 2. This should be increased by the excess of (47) — (48) over 2 M_2, which is

$$\begin{aligned} &M_2 (\cos v + \cos w) \\ &+ M_4 [\cos(2\, M_2^\circ - M_4^\circ - 2\, v) - \cos(2\, M_2^\circ - M_4^\circ - 2\, w)] \\ &+ M_6 [\cos(3\, M_2^\circ - M_6^\circ - 3\, v) + \cos(3\, M_2^\circ - M_6^\circ - 3\, w)] \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &- 2\, M_2 \end{aligned} \tag{64}$$

where v and w have the values given in § 12.

Since v and w are assumed to be comparatively small, and as v is often approximately equal to $-w$, the above may be written

$$M_2 (\cos v + \cos w) + 0{\cdot}035\ M_4 (v - w) \sin (2\ M_2{}^\circ - M_4{}^\circ) + 2\ M_6 \cos (3\ M_2{}^\circ - M_6{}^\circ) - 2\ M_2$$

where v and w are expressed in degrees.

The complete formula now becomes

$$2\ M_2 + \frac{1}{2\ M_2\ m_2^2} [S_2^2\ s_2^2 + N_2^2\ n_2^2 + \quad . \quad . \quad . \quad + K_1^2\ k_1^2 + O_1^2\ o_1^2 + \ . \ . \ .],$$

$$+ M_2 (\cos v + \cos w) + 0{\cdot}035\ M_4 (v - w) \sin (2\ M_2{}^\circ - M_4{}^\circ) + 2\ M_6 \cos (3\ M_2{}^\circ - M_6{}^\circ) - 2\ M_2 \quad (65)$$

where the second part is due to the harmonics of M_2 and is usually small except at river stations. Table 21 has been prepared for the purpose of avoiding the labor of each time computing the residual effects of S_2, N_2, etc., upon the range of tide, as indicated in (65). See also Table 22.

The above formula for computing the range of tide is not vitiated because of certain fixed relations between the speeds of some of the components.

Let the parallax wave, where $n_2 + l_2 = 2\ m_2$, $m_2 + 2\ n = 2\ n_2$, be first considered. Its amplitude at any particular time is approximately of the form

$$N_2 - L_2 \cos 2x + 2\ N \cos x, \quad (66)$$

the average value of which is N_2. But in reality the residual effect of L_2 and $2\ N$ upon N_2 is

$$\frac{L_2^2\ l_2^2}{4\ N_2\ n_2^2} + \frac{(2\ N)^2 (2\ n)^2}{4\ N_2\ n_2^2}$$

instead of zero. Therefore in finding the residual effect of the parallax wave upon M_2, the amplitude to be used would be approximately

$$N_2 + \frac{L_2^2\ l_2^2}{4\ N_2 n_2^2} + \frac{(2\ N)^2\ (2\ n)^2}{4\ N_2\ n_2^2} - L_2 \cos 2\ x + 2\ N \cos (-x)$$

The average value of the square of this expression (see § 7) multiplied by n_2^2 is, very nearly,

$$N_2^2\ n_2^2 + L_2^2\ l_2^2 + (2\ N)^2\ (2\ n)^2.$$

The theoretical values of N_2, L_2, $2\ N$, give, for each value of x, an amplitude with which to enter Table 16. From Table 15 a correction to the phase $(-x)$ is obtained. The mean of the corresponding tabular values shows that the fixed speed relations do not sensibly increase the range of tide.

For the declinational wave the fixed relations are $k_1 + o_1 = m_2$, $k_1 + p_1 = s_2$; the amplitude at any given time is approximately of the form

$$K_1 + O_1 \cos y + P_1 \cos z. \quad (67)$$

Now increase K_1 by the residual effects of O_1 and P_1 upon it, and (§ 16, remark based upon Table 18) multiply the average value of the square of the amplitude so altered by the square of the speed. When the amplitude of the declinational wave is great, the speed is a trifle greater than m_1; when the amplitude is small, the speed exceeds k_1.* For the present purpose the speed should be between m_1 and k_1, but nearer to m_1. The result indicated above becomes nearly equal to

$$K_1^2\ k_1^2 + O_1^2\ o_1^2 + P_1^2\ p_1^2.$$

Having regard to the derivation of (15) and (16), it is not difficult to convince one's self that small components like K_2 and S_1, S_4, S_6, require no special treatment because of their speed relations to K_1 and S_2, respectively.

---

*By means of Table 27, the speed of the $K_1\ O_1$ wave may be ascertained for any given time; it is approximately equal to

$$14{\cdot}6 + \frac{(2.8)^\delta}{400}$$

where δ is the number of days before or after the nearest extreme declination of the moon, or rather maximum declinational effect. See also § 8.

14. *The mean range of tide as determined from the tidal components is invariably less than the mean range of tide determined from observed high and low waters. The difference between the two determinations is constant from year to year at a given station.*

In summing for any component, the curve is read at *fixed times* (which are determined in advance) depending in no way upon the height of the sea. In summing for high and low waters the tidal curve is actually read at times which could be only approximately determined in advance, and which times depend to a considerable extent upon the height of the sea. In the former case the elevations and depressions other than those due to the component (with its harmonics) sought will eventually become eliminated. In the latter case we have the liberty of selecting readings at *about* the fixed times of maxima of the wave got by combining all the tidal components. The mean range of the observed tide will always be greater than that of the theoretical tide. For, in order to get this observed range of tide to agree with that of the theoretical tide, we would have to observe the height of the sea *at the theoretical* or *predictable times of high and low water.*

The magnitude of the discrepancy just pointed out will depend largely upon who observes the tides, and whether the high and low waters are observed upon a staff or are obtained from the record of an automatic tide gauge.

Let it be assumed that what is meant by high and low water heights are heights obtained from a continuous curve by a tabulator who reads, as nearly as his judgment will permit, a curve which will eliminate such irregularities as are obviously not predictable. With this definition of high and low waters, and in want of definite information, the discrepancy may be assumed to be perhaps 1 per cent of the computed range. For high and low waters as ordinarily tabulated this discrepancy probably amounts to about 2 per cent.

15. *To compute the hourly heights of the* M *tide.*

The height of the M tide h hours after high water is

$$\begin{aligned} &\mathrm{M}_2 \cos\{28^\circ\, 59'\, h - v\} + \mathrm{M}_4 \cos\{57^\circ\, 58'\, h + 2\,\mathrm{M}_2{}^\circ - \mathrm{M}_4{}^\circ - 2\,v\} \\ &\qquad + \mathrm{M}_6 \cos\{86^\circ\, 57'\, h + 3\,\mathrm{M}_2{}^\circ - \mathrm{M}_6{}^\circ - 3\,v\}; \end{aligned} \tag{68}$$

and h hours after low water

$$\begin{aligned} &-\mathrm{M}_2 \cos\{28^\circ\, 59'\, h - w\} + \mathrm{M}_4 \cos\{57^\circ\, 58'\, h + 2\,\mathrm{M}_2{}^\circ - \mathrm{M}_4{}^\circ - 2\,w\} \\ &\qquad - \mathrm{M}_6 \cos\{86^\circ\, 57'\, h + 3\,\mathrm{M}_2{}^\circ - \mathrm{M}_6{}^\circ - 3\,w\}. \end{aligned} \tag{69}$$

16. *To compute the speed and range of the diurnal wave when the moon is far from the equator; also the mean range of tide at these times.*

At the conspiring of K_1 and O_1 the speed of the combined wave is, by § 8,

$$\mathrm{k}_1 - \frac{\mathrm{k}_1\, \mathrm{O}_1\, \mathrm{o}_1\, (\mathrm{k}_1 - \mathrm{o}_1)}{\mathrm{K}_1\, \mathrm{k}_1{}^2 + \mathrm{O}_1\, \mathrm{o}_1{}^2}; = 14^\circ{\cdot}6 \tag{70}$$

when the value of $\mathrm{K}_1/\mathrm{O}_1$ is assumed to be 1·4066. In other words, it is very nearly 0°·1 greater than m_1.

Calling this speed d_1, it follows that the mean semirange of the diurnal wave when the moon is far from the equator is

$$\mathrm{K}_1 + \mathrm{O}_1 + \frac{1}{4\,(\mathrm{K}_1 + \mathrm{O}_1)\,\mathrm{d}_1{}^2}\,[\mathrm{P}_1{}^2\,\mathrm{p}_1{}^2 + \mathrm{Q}_1{}^2\,\mathrm{q}_1{}^2 + \;\cdot\;\cdot\;\cdot\;]; \tag{71}$$

$$= (1{\cdot}0236)\,(\mathrm{K}_1 + \mathrm{O}_1) \tag{72}$$

when P_1 and Q_1 are given their theoretical values. Since K_1 and O_1 separate 13° 32′ in one-half of a D_1 day (i. e. in $12^{\mathrm{h}}{\cdot}33$), $\mathrm{K}_1 + \mathrm{O}_1$ should be multiplied by 0·9977, formula (25), thus making the required range

$$2\,(1{\cdot}021)\,(\mathrm{K}_1 + \mathrm{O}_1). \tag{73}$$

In computing the mean range of tides at these times it should be noted: First, that the K_2 moon is approximately 180° from the M_2 moon; second, the large wave whose amplitude is $K_1 + O_1$ has a fixed phase with respect to M_2; third, from Table 18 it is seen that the mean range of a wave composed of a semidiurnal and a diurnal component is not sensibly affected by the phase of the latter with respect to the former.

The mean range of tide at these times, which it is convenient to denote by Mc, differs from Mn, formula (65), by the substitution of $(K_1 + O_1)^2\, d_1^2$ in place of $K_1^2\, k_1^2 + O_1^2\, o_1^2$ and by the subtraction of $2\, K_2$ and the residual effect of K_2. That is

$$\text{Mc} = \text{M n} + \frac{(K_1 + O_1)^2\, d_1^2 - K_1^2\, k_1^2 - O_1^2\, o_1^2 - K_2^2\, k_2^2}{2\, M_2\, m_2^2} - 2\, K_2, \tag{74}$$

$$= \text{Mn} + \frac{0{\cdot}127\, (K_1 + O_1)^2 - 0{\cdot}135\, K_1^2 - 0.116\, O_1^2 - 0.540\, K_2^2}{M_2} - 2\, K_2.$$

Mc will exceed Mn whenever

$$(K_1 + O_1)^2\, d_1^2 - K_1^2\, k_1^2 - O_1^2\, o_1^2 > 4\, M_2\, K_2\, m_2^2 - K_2^2\, k_2^2; \tag{75}$$

or, roughly, whenever

$$K_1\, O_1 > 8\, M_2\, K_2; \tag{76}$$

which is about equivalent to saying whenever

$$D_1 > 2\, \triangle_2, \tag{77}$$

D_1 being the tropic amplitude of the diurnal wave and $\triangle_2$ the mean amplitude of the semidiurnal wave.

17. *Given the epochs of two components whose speeds are nearly equal, to find how much the time of their conspiring lags behind the time of conjunction of their fictitious moons.*

When the moons are in conjunction the angle between the crests of the component tides is

$$B^\circ - A^\circ.$$

The time required for the crests to come together will be

$$\tau = \frac{B^\circ - A^\circ}{b - a} \tag{78}$$

where a mean solar hour is taken for the time unit. Whenever A° and B° appear to differ by more than 180°, ± 360° should be applied to one of them.

τ also represents the amount of lagging of the interference behind the time of opposition of the fictitious moons.

For the components M_2 and S_2

$$\tau = \frac{S_2{}^\circ - M_2{}^\circ}{1{\cdot}0159} = 0{\cdot}984\, (S_2{}^\circ - M_2{}^\circ). \tag{79}$$

This may be written $\tau\, (S_2;\, M_2)$, meaning the *age*, or *retard*, *of* S_2 *relative to* M_2, expressed in hours. Assuming that $\genfrac{}{}{0pt}{}{\text{spring}}{\text{neap}}$ tides occur when M_2 and S_2 have $\genfrac{}{}{0pt}{}{\text{like}}{\text{opposite}}$ phases, this is the interval between $\genfrac{}{}{0pt}{}{\text{new or full moon}}{\text{moon in quadrature}}$ and $\genfrac{}{}{0pt}{}{\text{spring}}{\text{neap}}$ tides.

In like manner

$$\tau\, (N_2;\, M_2) = \frac{N_2{}^\circ - M_2{}^\circ}{-0{\cdot}5444} = \frac{M_2{}^\circ - N_2{}^\circ}{0{\cdot}5444} = 1.837\, (M_2{}^\circ - N_2{}^\circ). \tag{80}$$

Assuming that the parallax wave $\genfrac{}{}{0pt}{}{\text{conspires}}{\text{interferes}}$ with M_2 when M_2 and N_2 have $\genfrac{}{}{0pt}{}{\text{like}}{\text{opposite}}$ phases, this is the interval between $\genfrac{}{}{0pt}{}{\text{moon in perigee}}{\text{moon in apogee}}$ and the greatest $\genfrac{}{}{0pt}{}{\text{increase}}{\text{decrease}}$ of range of tide due to parallax.

Similarly

$$\tau\, (O_1;\, K_1) = \frac{O_1{}^\circ - K_1{}^\circ}{-1{\cdot}0980} = \frac{K_1{}^\circ - O_1{}^\circ}{1{\cdot}0980} = 0{\cdot}911\, (K_1{}^\circ - O_1{}^\circ) \tag{81}$$

is the number of hours between the time $\genfrac{}{}{0pt}{}{\text{of the moon's extreme north or south declination}}{\text{when the moon crosses the equator}}$ and the time of the $\genfrac{}{}{0pt}{}{\text{greatest}}{\text{least}}$ range of the diurnal wave. $\tau(S_2;\ M_2)$, $\tau(N_2;\ M_2)$, and $\tau(O_1;\ K_1)$ may be referred to as the ages of the phase, parallax, and diurnal inequalities, respectively.

18. *To compute spring and neap ranges.*

The phase wave consists of S_2 and μ_2. S_2 conspires with M_2 at spring tide and interferes at neap tide. The argument of the μ_2 moon exceeds that of the S_2 moon by four times the mean longitude of the sun less that of the moon. Therefore, whenever the moon is new, full, or in quadrature, the S_2 moon and that of μ_2 are in conjunction. For this reason, and assuming that the epochs of S_2 and μ_2 are equal, S_2 and μ_2 conspire at the times of spring or neap tides.

The mean range of tide with the residual effect of S_2 excluded is

$$\mathrm{Mn} - \tfrac{1}{2}\,\frac{S_2^2\, s_2^2}{M_2\, m_2^2}. \tag{82}$$

Disregarding the time perturbations other than those due to phase, also the slight separation of M_2 and S_2 during a quarter lunar day, the $\genfrac{}{}{0pt}{}{\text{spring}}{\text{neap}}$ range ought to be expression (82) $\genfrac{}{}{0pt}{}{\text{increased}}{\text{decreased}}$ by $2\,(S_2 + \mu_2)$.

The perturbations just referred to diminish the direct effect of S_2 and μ_2 by about

$$0{\cdot}02 + 0{\cdot}04\left(\frac{K_1 + O_1}{M_2}\right)^2$$

part of their value; see formula (18). The diminution in S_2 due to the separation in a quarter lunar day is by formulæ (25) and (26) very small. Approximate values of the required ranges are

$$\mathrm{Sg} = \mathrm{Mn} + 1{\cdot}96\,(S_2 + \mu_2) - 0{\cdot}08\left(\frac{K_1 + O_1}{M_2}\right)^2 (S_2 + \mu_2) - \frac{S_2^2}{2\,M_2}, \tag{83}$$

$$\mathrm{Np} = \mathrm{Mn} - 1{\cdot}96\,(S_2 + \mu_2) + 0{\cdot}08\left(\frac{K_1 + O_1}{M_2}\right)^2 (S_2 + \mu_2) - \frac{S_2^2}{2\,M_2}; \tag{84}$$

$$\therefore\ \mathrm{Sp} + \mathrm{Np} = 2\,\mathrm{Mn} - \frac{S_2^2}{M_2}. \tag{85}$$

19. *To compute perigean and apogean ranges.*

These terms are here used to denote ranges analogous to the spring and neap, but referring to perigee and apogee instead of syzygy and quadrature.

The parallax wave consists chiefly of N_2, L_2, and $2\,N$. The argument of M_2 (i. e. of its fictitious moon) exceeds that of $\genfrac{}{}{0pt}{}{N_2\text{ by the}}{2\,N\text{ by twice the}}$ mean longitude of the moon from its perigee; in other words, $\genfrac{}{}{0pt}{}{\text{by the}}{\text{by twice the}}$ moon's mean anomaly.

The argument of L_2 (simple) is greater than that of M_2 by the mean longitude of the moon from its perigee $+\,180°$. From this it follows that when the moon is in mean perigee the fictitious N_2 moon is in conjunction with the fictitious M_2 moon while that of L_2 is in opposition. If conspiring and interference took place immediately, the amplitude of the parallax wave when the moon is in mean perigee or apogee would be, approximately,

$$N_2 - L_2 \pm 2\,N.$$

Proceeding as in the case of spring and neap ranges, but remembering that true and mean perigee differ considerably, we obtain for the required ranges the approximate values

$$\mathrm{Pn} = \mathrm{Mn} + \left[2{\cdot}1 - \tfrac{1}{2}\left(\frac{S_2}{M_2}\right)^2\right](N_2 - L_2 + 2\,N) - 0{\cdot}08\,\frac{(K_1 + O_1)^2}{M_2^2}(N_2 - L_2 + 2\,N) - \frac{N_2^2}{2\,M_2}, \tag{86}$$

$$\mathrm{An} = \mathrm{Mn} - \left[2{\cdot}1 - \tfrac{1}{2}\left(\frac{S_2}{M_2}\right)^2\right](N_2 - L_2 - 2\,N) + 0{\cdot}08\,\frac{(K_1 + O_1)^2}{M_2^2}(N_2 - L_2 - 2\,N) - \frac{N_2^2}{2\,M_2}; \tag{87}$$

$$\therefore\ \mathrm{Pn} + \mathrm{An} = 2\,\mathrm{Mn} - \frac{N_2^2}{M_2} + 4\,[2\,N]. \tag{88}$$

20. *To find the heights and lunitidal intervals of the tropic tides which have a fixed order of occurrence.*

Compute the semirange of the tide by formula (65), omitting the diurnal components, and diminish the result by K_2. The theoretical value of this is

$$1{\cdot}064\ M_2 - K_2,$$

or, more accurately,

$$1{\cdot}006\ M_2 + 0{\cdot}27\ \frac{S_2^2}{M_2} - K_2. \tag{89}$$

Where K_2 is not known it may be taken equal to $0{\cdot}272\ S_2$.

The semirange of the diurnal wave when K_1 and O_1 conspire is, formula (73),

$$1{\cdot}021\ (K_1 + O_1). \tag{90}$$

(90) divided by (89) gives the ratio with which to enter Tables 17 and 18.

By § 2 we see that for finding the acceleration in the times of maxima or minima, or the resultant heights, certain angles are involved which are of the forms

$$\frac{b}{a}\left(2\,n\,\pi + \frac{a}{b}\,\beta - \alpha\right), \frac{b}{a}\left(\overline{2\,n+1}\,\pi + \frac{a}{b}\,\beta - \alpha\right). \tag{91}$$

It will be convenient to refer to these as the *high water* and *low water phases*, respectively; and when no value of n is specified it is assumed to be zero.

The argument of the O_1 moon added to the argument of the K_1 moon is equal to that of the M_2 moon. When K_1 and O_1 conspire, the argument of either component or the half sum of these arguments is the argument of the diurnal wave. Putting $b = \frac{1}{2}\,a$, § 16, the high and low water phases become

$$\tfrac{1}{2}\,(2\,n\,\pi + M_2^\circ - K_1^\circ - O_1^\circ),\ \tfrac{1}{2}\,(\overline{2\,n+1}\,\pi + M_2^\circ - K_1^\circ - O_1^\circ). \tag{92}$$

Making $n = 0$ and 1, or any other even and odd integers, these phases become, after rejecting multiples of 360°,

High water phase.	*Low water phase.*	
$\frac{1}{2}\,(M_2^\circ - K_1^\circ - O_1^\circ)$,	$\frac{1}{2}\,(M_2^\circ - K_1^\circ - O_1^\circ) + 90^\circ$,	(93)
$\frac{1}{2}\,(M_2^\circ - K_1^\circ - O_1^\circ) + 180^\circ$,	$\frac{1}{2}\,(M_2^\circ - K_1^\circ - O_1^\circ) + 270^\circ$.	

These are the phases with which to enter Tables 17 and 18.

The heights of Table 18 are given in terms of the amplitude of the semidiurnal wave (89). The tropic lunitidal intervals are obtained from the mean values of the high and low water intervals, § 12, by subtracting from them the tabular values, Table 17, divided by 28·984.

When the harmonics of M_2 are large, M_2° should be replaced by $M_2^\circ - v$ in the above expressions which refer to the maxima of M_2, and by $M_2^\circ - w$ in those referring to its minima.

Having now determined the amplitude and phase arguments, the required quantities are readily obtained from the values in Tables 17 and 18.*

21. *To find the height inequalities.*

When the order of the tides determining a height inequality is fixed its value becomes known by the preceding paragraph. Here it is proposed to find an expression for a height inequality when the sequence of the tides upon which it depends is subject to alteration. Suppose that we are here concerned with the high water inequality, HWQ, and let the value from the table be denoted by $\overline{\mathrm{HWQ}}$.

For that portion of the time during which the change of order is not possible because the effect of the P_1 wave in producing inequality in the heights of the high waters is less than $\overline{\mathrm{HWQ}}$, we have

$$\mathrm{HWQ} = \overline{\mathrm{HWQ}}. \tag{94}$$

*Having thus obtained the heights of the tropic tides, each should be diminished by Mf cos Mf°. This is of importance when tropic LLW is required for the plane of reference. See also § 14.

Let x denote the distance in degrees of the maximum or minimum of P_1 from the maximum of M_2; then $2\,P_1 \cos x$ is the inequality in heights of the high waters which P_1 tends to introduce. Suppose that for the other portion of the time $2\,P_1 \cos x \overline{\gtrless} \overline{HWQ}$. During this time one half of the inequalities will be in their normal condition and the other half will have their condition altered. The values of the former are represented by

$$2\,P_1 \cos x + \overline{HWQ}, \tag{95}$$

and of the latter by

$$2\,P_1 \cos x - \overline{HWQ}, \tag{96}$$

where

$$1 \overline{\gtrless} \cos x \overline{\gtrless} \frac{\overline{HWQ}}{2\,P_1} = l, \text{ say.} \tag{97}$$

The average value of $\cos x$ between the limits $\cos x = 1$ and $\cos x = l$, or x between 0 and $\cos^{-1} l$, is

$$\frac{\sqrt{1 - l^2}}{\cos^{-1} l};$$

this substituted for $\cos x$ in (95) and (96) will give the required average values. The respective weights to be given to (94), (95), and (96) are

$$\frac{\pi}{2} - \cos^{-1} l,\ \frac{1}{2} \cos^{-1} l, \text{ and } \frac{1}{2} \cos^{-1} l.$$

The value of the required inequality is

$$\begin{aligned} \text{HWQ} &= \left(1 - \frac{2 \cos^{-1} l}{\pi}\right) \overline{HWQ} + \frac{4\,P_1}{\pi} \sqrt{1 - l^2} \\ &= \left[1 - \frac{(\cos^{-1} l)^\circ}{90^\circ}\right] \overline{HWQ} + \frac{4\,P_1}{3 \cdot 1416} \sqrt{1 - l^2}. \end{aligned} \tag{98}$$

The fraction representing the percentage of (tropic) high water inequalities which have the order of their tides changed is

$$\begin{aligned} &\frac{\cos^{-1} l}{\pi} \\ &= \frac{1}{\pi} \cos^{-1} \frac{(\overline{HWQ})}{2\,P_1} = \text{X, say.} \end{aligned} \tag{99}$$

(If quite limited periods be considered, for example, particular years, P_1 will remain constant from year to year, but $\overline{HWQ}$ will vary; see § 48. Consequently values must be given to $\overline{HWQ}$ suited to the particular years in order to ascertain the number of high water inequalities such that the high waters upon which they depend will have their order reversed.)

22. *To compute the great and small tropic ranges, and the heights of the tides between which they occur.*

These are the greatest and least ranges of the tide upon a day when K_1 and O_1 conspire. Denoting them by Gc and Sc, respectively, we have

$$\text{Gc} + \text{Sc} = 2\,\text{Mc}, \tag{100}$$
$$\text{Gc} - \text{Sc} = \text{HWQ} + \text{LWQ}; \tag{101}$$
$$\text{Gc} = \text{Mc} + \tfrac{1}{2}\,(\text{HWQ} + \text{LWQ}), \tag{102}$$
$$\text{Sc} = \text{Mc} - \tfrac{1}{2}\,(\text{HWQ} + \text{LWQ}). \tag{103}$$

Suppose the inequality in low water to be the more pronounced; Table 18 gives the values of tropic low waters, § 20. The tropic high waters are then obtained by the equations

$$\text{Tropic HHW} = \text{tropic LLW} + \text{Gc}, \tag{104}$$

$$\text{Tropic LHW} = \text{tropic HLW} + \text{Sc}. \tag{105}$$

23. *To find the average range of the diurnal wave; also, roughly, the great and small diurnal ranges of tide.*

The average semirange of the diurnal wave is, formula (18),

$$K_1 + \frac{1}{4\,K_1\,k_1^2}\,[O_1^2\,o_1^2 + P_1^2\,p_1^2 + \;\ldots\;]. \tag{106}$$

The theoretical value of this is 1·14 K_1, or 1·61 O_1. If this wave had a fixed position with respect to M_2, the difference between the great and small diurnal ranges of tide would be almost directly proportional to the range of the diurnal wave. This can be seen by referring to Table 18. When the moon is far from the equator—i. e. when the diurnal wave is large—the position is nearly fixed. Assuming the fixed position to hold true for all declinations of the moon, the above statement may be written

$$\frac{Gt - Sl}{Gc - Sc} = \frac{Gt - Mn}{Gc - Mc} = \frac{\text{range of diurnal wave}}{\text{tropic range of diurnal wave}} \tag{107}$$

since

$$Gt + Sl = 2\,Mn, \tag{108}$$

$$Gc + Sc = 2\,Mc. \tag{109}$$

(107) is theoretically equal to

$$\frac{1{\cdot}14\,K_1}{1{\cdot}02\,(K_1 + O_1)} = 0{\cdot}65. \tag{110}$$

Since the above assumption is not correct when the declination of the moon is small, and more especially since one is supposed to choose the greater range regardless of the sequence of the tides (see § 21), the above ratio ought to be increased by the positive quantity ζ representing these effects. For stations where there are usually two tides daily, ζ may be taken equal to 0·10, and for the present purpose Mc may be taken equal to Mn. Instead of (110) one may write

$$\frac{Gt - Mn}{Gc - Mn} = \frac{3}{4}; \tag{111}$$

$$\therefore Gt = \frac{3}{4}\,Gc + \frac{1}{4}\,Mn. \tag{112}$$

If the tide had no semidiurnal components Mn would be zero, and the sequence of tides would not enter into the question. In such a case

$$\frac{Gt}{Gc} = 0{\cdot}65. \tag{113}$$

For stations where there is usually but one tide a day, ζ must be very small. Mn can not be obtained from observation; but it may be replaced by a rough value 2 (1·2) M_2, while the ratio

$$\frac{Gt - Mn}{Gc - Mn}$$

may be put equal to $\frac{2}{3}$; consequently

$$Gt = \frac{2}{3}\,Gc + \frac{4}{5}\,M_2. \tag{114}$$

At stations having tides of this character, the diurnal range, Gt, is often called the mean range of tide.

24. *To compute roughly the average values of the height inequalities; application to planes of reference.*

If the sequence of tides remained fixed for all declinations of the moon, the average values of the height inequalities would be to their maximum values (HWQ, LWQ) as the amplitude of the diurnal wave is to its tropic amplitude, or as 0·65 is to 1·00, §§ 21, 23.

Considering now the more pronounced inequality, low water, say, the average value ought to be

0·65 LWQ + a fraction of (Gc − Mn), say.

This assumed, the fraction must be such that it vary directly as Gc − Mn and inversely as LWQ. The expression now becomes

$$0{\cdot}65\ \mathrm{LWQ} + \varepsilon\,\frac{\mathrm{Gc} - \mathrm{Mn}}{\mathrm{LWQ}}\,(\mathrm{Gc} - \mathrm{Mn}) \tag{115}$$

where ε is supposed to be constant. The special case LWQ = HWQ gives, making use of (111), $\varepsilon = 0{\cdot}10$.

Assuming that the mean of the two tropic low waters is the same as mean low water, it follows that the depression of the lower low waters below mean low water is

$$\frac{\mathrm{LWQ}}{3} + \frac{1}{\mathrm{LWQ}}\left[\frac{\mathrm{Gc} - \mathrm{Mn}}{5}\right]^2. \tag{116}$$

The average value of the other and less pronounced inequality is obtained by subtracting the one already obtained from the quantity Gt − Mn.

The following equations are obtained from Table 18 and show how the sum of the two tropic high or low waters differs from Mc, the mean tropic range of tide:

$$\text{Tropic HHW} + \text{tropic LHW} = \mathrm{Mc} - 0{\cdot}126\,\frac{(\mathrm{K}_1 + \mathrm{O}_1)^2}{\mathrm{M}_2}\cos(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{O}_1{}^\circ); \tag{117}$$

$$\text{Tropic LLW} + \text{tropic HLW} = -\,\mathrm{Mc} - 0{\cdot}126\,\frac{(\mathrm{K}_1 + \mathrm{O}_1)^2}{\mathrm{M}_2}\cos(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{O}_1{}^\circ). \tag{118}$$

Less accurately we have

$$\text{Mean HHW} + \text{mean LHW} = \mathrm{Mn} - 0{\cdot}08\,\frac{(\mathrm{K}_1 + \mathrm{O}_1)^2}{\mathrm{M}_2}\cos(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{O}_1{}^\circ); \tag{119}$$

$$\text{Mean LLW} + \text{mean HLW} = \mathrm{Mn} - 0{\cdot}08\,\frac{(\mathrm{K}_1 + \mathrm{O}_1)^2}{\mathrm{M}_2}\cos(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{O}_1{}^\circ). \tag{120}$$

The mean of both high and low water heights is the height of mean sea level, or rather half tide level (HTL), above the line about which the components are supposed to oscillate; that is, above mean sea level as determined from hourly ordinates (MSL). From § 13 the mean of high and low water gives the elevation of mean sea level due to the harmonics of M_2.

$$\therefore\ \mathrm{HTL} = \mathrm{MSL} + \mathrm{M}_4\cos(2\,\mathrm{M}_2{}^\circ - \mathrm{M}_4{}^\circ) - 0{\cdot}04\,\frac{(\mathrm{K}_1 + \mathrm{O}_1)^2}{\mathrm{M}_2}\cos(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{MO}_1{}^\circ). \tag{121}$$

25. *Stations where the tide is usually diurnal.*

The tropic lunitidal intervals may be found by means of formulæ (58)–(61) after replacing $6^{\mathrm{h}}{\cdot}2103$ by $12^{\mathrm{h}}{\cdot}33$, M_2, $\mathrm{M}_2{}^\circ$, m_2 by $\mathrm{K}_1 + \mathrm{O}_1$, $\frac{1}{2}(\mathrm{K}_1{}^\circ + \mathrm{O}_1{}^\circ)$, d_1; and M_4, $\mathrm{M}_4{}^\circ$, m_4 by M_2, $\mathrm{M}_2{}^\circ$ m_2, respectively, omitting the higher harmonics.

The (great) tropic range of tide is by formulæ (65) and (73)

$$2\,(1{\cdot}021)\,(\mathrm{K}_1{+}\mathrm{O}_1){+}(\mathrm{K}_1{+}\mathrm{O}_1)\,(\cos v{+}\cos w){-}0{\cdot}035\,\mathrm{M}_2\,(v{-}w)\sin(\mathrm{M}_2{}^\circ{-}\mathrm{K}_1{}^\circ{-}\mathrm{O}_1{}^\circ){-}2\,(\mathrm{K}_1{+}\mathrm{O}_1), \tag{122}$$

the corresponding high and low water heights being

$$(\cos v + 0.021)\,(\mathrm{K}_1 + \mathrm{O}_1) + \mathrm{M}_2\cos(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{O}_1{}^\circ) - 0.035\,\mathrm{M}_2\,v\sin(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{O}_1{}^\circ),$$

and

$$-(\cos w + 0.021)\,(\mathrm{K}_1 + \mathrm{O}_1) + \mathrm{M}_2\cos(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{O}_1{}^\circ) - 0.035\,\mathrm{M}_2\,w\sin(\mathrm{M}_2{}^\circ - \mathrm{K}_1{}^\circ - \mathrm{O}_1{}^\circ),$$

where v and w are expressed in degrees.

When M_2 is small, this range is nearly equal to

$$2\,(1{\cdot}021)\,(\mathrm{K}_1{+}\mathrm{O}_1). \tag{123}$$

Mean daily low water is then about

$$\frac{2}{3}\,(\mathrm{K}_1 + \mathrm{O}_1)$$

below MSL.

CHAPTER III.

REDUCTIONS OF OBSERVATIONS MADE UPON HIGH AND LOW WATERS.

26. The main object of this chapter is to determine, from a series of observed high and low waters, certain harmonic and nonharmonic quantities which are of special importance in determining the best port of reference for a given station. In so far as the amplitudes and epochs of various components are determined, the reductions constitute a rude harmonic analysis.* The nonharmonic quantities obtained are of interest because of their obvious connection with observed tidal phenomena. The astronomical items used (transits, phases, etc.) are taken directly from the Greenwich ephemeris; the only change consists in using civil time.

The tables given at the close of this paper would, by some alterations and extensions, enable one to use mean motions throughout, thus doing away with some of the inaccuracies which attend the process of reduction as carried out below. It is believed, however, that such errors are too small to be of much consequence in an analysis of the kind proposed. There are some advantages, especially when the series is short, in using the real motions of the bodies.

For instance, the epoch of S_2 obtained from observing a few tides and referring them (without correction) to the mean sun will be much more in error than would have been the case had the true sun been used. But it will be seen from Table 31 that the correction to be applied to the epoch of S_2 because of T_2 and the solar part of K_2 is practically the equation of time.

FIRST REDUCTION.†

27. In this reduction the first process consists in tabulating the times and heights of high and low water, together with the times of transit of the moon across some known meridian. The next step is the subtraction of the transits from the times of the tides, usually selecting such transits as will give the smallest possible interval without introducing too many negative values. Before taking means of the intervals and heights, the length of series to be used must be decided upon.

Length of series suitable for finding the semidiurnal range of tide and the mean lunitidal interval.—Since the variation in range, from day to day depends chiefly upon the manner in which the semidiurnal components fall upon M_2, the required length of series should be, as nearly as possible, a multiple of the synodic periods of S_2, N_2, L_2, K_2, . . . with M_2; i. e. of the form

$$\frac{a\ S_2\ 14{\cdot}765 + b\ (N_2 + L_2)\ 27{\cdot}555 + c\ K_2\ 13{\cdot}661 + \ \ .\ \ .\ \ .}{S_2 + N_2 + L_2 + K_2 + \ \ .\ \ .\ \ .} \tag{124}$$

where a, b, c, . . . are integers. The weights S_2, $N_2 + L_2$, K_2, . . . are given to the synodic periods because the *direct* effect of any component in altering the range of the tide is nearly proportional to its amplitude.

For the purpose of eliminating the diurnal inequality as far as possible, the $\frac{\text{high}}{\text{low}}$ water which ends the series should be of the opposite kind from the $\frac{\text{high}}{\text{low}}$ water beginning it. As a length of

* For a more elaborate mode of analysis, see a paper by Prof. G. H. Darwin entitled "On the harmonic analysis of tidal observations of high and low water," Roy. Soc. Proc., Vol. 48 (1890), pages 278–340, see also a correction to this paper, ibid., Vol. 52 (1892), pages 388, 389.

† For an example see § 30.

series suited to the high waters is generally also suited to the low waters, one should solve the above equation to the nearest lunar half day. This having been done, the results expressed in mean solar days are as follows:

29, 58, 87, 105, 134, 163, 192, 221, 250, 279, 297, 326, 355, 384.

It is often convenient to divide the series, whatever its length, into periods of 29 days each.

Since the transits of the true (not the mean) moon are used, the lunar semidiurnal components N_2, L_2, lunar K_2, . . . have little effect upon the interval. Consequently the synodic month of $29\frac{1}{2}$ days might seem to be a better length of series for the determination of the interval. But on account of the diurnal inequality in interval it is probable that even for this purpose 29 days would usually be preferable to $29\frac{1}{2}$.

Having fixed upon the length of series to be used, the sums and means of the intervals and the heights on staff are then taken. The resulting intervals are reduced to a value which would have been obtained had local transits been used by adding

$$S - L + 0{\cdot}035\ (E - L) \tag{125}$$

where E, S, and L are the west longitude in time of the meridian of the transits, i. e. of the ephemeris used, of the (standard) time meridian, and of the meridian of the station, respectively.

Using now the corrected intervals or establishments (HWI, LWI) expressed in hours, the epoch of M_2 is (unless M_6 be large)

$$14{\cdot}4921\ (\text{HWI} + \text{LWI} \mp 6{\cdot}2103) = M_2°. \tag{126}$$

The $\substack{\text{upper}\\ \text{lower}}$ sign is used when the low water interval is taken $\substack{\text{greater}\\ \text{less}}$ than the high water.

The resulting LW height subtracted from the HW height gives the range of tide for the period of the observations.

The intervals and ranges just considered refer to tides occurring twice daily. For this reason when evanescent tides occur such days' observations should be omitted in the summations. If only a few occur they may be interpolated, or if a continuous record be available the points of maximum curvature may be selected for the missing high and low waters.*

For some purposes the average height of the higher high and lower low water may be useful, particularly for determining the great diurnal range of tide and the plane of reference defined by lower low water. These heights are always obtainable, as one high water and one low water occur each day, even when there are evanescent tides. In the first reduction, mark such heights and take the mean of the quantities so marked. The length of series used for this purpose should be an integral number of tropical months, or, less accurately, of half tropical months.

The following numbers represent such periods expressed in mean solar days:

(14), 27, (41), 55, (68), 82, (96), 109, (123), 137, (150), 164, (178), 191, (205), 219, (232), 246, (260), 273, (287), 301, (314), 328, (342), 355, (369), 382.

28. *Determination of mean sea level and long-period tides.*

Where hourly heights of the tide are given, it is obvious that a suitable period of time for determining mean sea level would be one in which each of the principal tidal components makes an integral number of oscillations.

By taking multiples of the hourly speed of each component, it is readily seen that the period of 29 mean solar days and zero hours is the best value obtainable unless the given heights of the tide are taken at intervals of less than one hour. The same period may be adopted for the determination of mean sea level (half-tide level, § 24) from observations of high and low waters. This mean sea level is the half sum of the heights of mean high and low water for the 29-day period or group.

For long series of observations, it is desirable to have the periods conform to certain dates made out in advance. These dates should be the same for all stations in order that the fluctuations

* This implies, as it should, that the small tropic range becomes negative.

of mean sea level at various places may be more readily compared. The following dates are supposed to refer to any year; the dates marking the middle of each group divide the average year into twelve equal parts; the length of group is 29 days, as is shown by the terminal dates:

Groups for mean sea level.		
Beginning (midnight preceding).	Middle (noon).	End (midnight following).
Jan. 2	Jan. 16	Jan. 30
Feb. 1	Feb. 15	Feb. 29
Mar. 3	Mar. 17	Mar. 31
Apr. 3	Apr. 17	May 1
May 3	May 17	May 31
June 3	June 17	July 1
July 3	July 17	July 31
Aug. 3	Aug. 17	Aug. 31
Sept. 2	Sept. 16	Sept. 30
Oct. 2	Oct. 16	Oct. 30
Nov. 2	Nov. 16	Nov. 30
Dec. 2	Dec. 16	Dec. 30

For all spring and summer months the groups begin on the 3d of each month, while for all fall and winter months (except February) they begin on the 2d.

In harmonically analyzing the twelve ordinates thus obtained, it should be borne in mind that the first ordinate is 15·218 days after 0^h (midnight preceding) January 1; this increases the $V_0 + u$ of Sa, Table 3, by 15°. If $V_0 + u$ be found for date of beginning of series used in analyzing short-period tides, then it must be increased by the number of days from this date to the middle of the first 29-day group used, multiplied by 0·9856. Supposing the groups to be so taken as to conform to the above scheme, then it is clear that the values of Table 3 are simply increased by the amount

$$30^\circ \times m - 15^\circ$$

where m is the number of the calendar month approximately covered by the group.

Besides the yearly change in sea level, a small change having a period of about 428 days may be considered here. This depends upon the 428-day component of the variation of latitude. The annual and 428-day tidal components may be nearly separated by combining 7 years' observations. If the annual inequality in sea level has been determined from analyzing the monthly heights just considered for 7 years, it may be taken away from the sea-level curve and the residual curve read for the 428-day tide, using the true lengths of period for the particular years.

The fictitious moon has the longitude of the instantaneous minimum north latitude of the place so far as the latitude depends upon the 428-day component; this moon goes around the earth once in 428 days from west to east. Suppose the time of minimum north latitude to be given for Greenwich, i. e. the time when the north pole of the earth's figure passes the instantaneous meridian of Greenwich; the time for a place L hours west is

$$\text{Time of Greenwich min. lat.} - \frac{15\,L}{\text{daily speed}} \text{ days.}$$

The average value of the daily speed is 0°·840. Formulæ for more accurate values, together with a table of times of minimum north latitude or of high water of 428-day tide, are given below:

No. of periods from 1865. E	Time of high water at Greenwich of 428-day component tide. T_1	Amplitude of 428-day component of the latitude variation.	Amplitude of 428-day tide for latitude 45°.
	Civil date.	$''$	*Feet.*
−40	1818, June 16	0·184	0·062
−35	1824, Apr. 13	·170	·057
−30	1830, Jan. 28	·146	·049
−25	1835, Nov. 11	·118	·040
−20	1841, Aug. 29	·096	·032
−15	1847, July 1	·085	·029
−10	1853, May 18	·090	·030
− 5	1859, Apr. 20	·108	·037
0	1865, Mar. 31	·135	·046
+ 5	1871, Mar. 13	·162	·055
+10	1877, Feb. 12	·180	·061
+15	1882, Dec. 31	·185	·063
+20	1888, Nov. 1	·174	·059
+25	1894, Aug. 20	·152	·051
+30	1900, June 3	0·124	0·042

$$\begin{aligned} \text{Amplitude} &= 0''\!\cdot\!135 + 0''\!\cdot\!05 \sin \Psi, \\ \text{Period} &= 428^{d}\!\cdot\!6 + 5^{d}\!\cdot\!26 \cos \Psi, \\ T_1 &= 1865{\cdot}25 + 428{\cdot}6\, E + 55 \sin \Psi, \end{aligned}$$

where

$$\Psi = (\,t - 1865{\cdot}25)\; 5^{\circ}\!\cdot\!48,$$

t denoting the year.

[These values are taken from Dr. S. C. Chandler's paper on latitude variation in No. 322 (July 10, 1894) of the Astronomical Journal.]

At the times of local upper transit of the fictitious moon,

$$\text{Height of } 428^{d} \text{ tide} = \frac{\sin 2\,\lambda}{3} \times \text{change in latitude*} \tag{127}$$

where the height is reckoned in feet from mean sea level, the (north) latitude, λ, is expressed in degrees, and the change in latitude, column 3, is expressed in seconds.†

This fluctuation or tide is of little or no practical importance. It is noticed here because, like the annual tide, it may be determined from a sufficient number of observations upon high and low water.

In the special reductions which follow, the length of the series will usually be about one month. Certain modifications will be found necessary if, for any reason, other lengths be used.

---

* The rounded value $\frac{1}{3}$ is written for 0·338.

† A ready means of comparing theory and observation is to tabulate the times of local transits of the fictitious moon, as the times of high and low waters of the 428-day tide, and to find the reading of mean sea level, at such times using a 29-day group on each occasion. The sea level is then corrected for the annual component, and finally 6 high water or 6 low water groups (approximately 7 years) are combined and the mean taken.

From hourly readings (1870–1887) at Pulpit Harbor, Maine, (lat. 44° 09′ N., long. 68° 53′ W.), the mean for the component high water was 10·495 feet, and for low water 10·357, showing, upon application of the group factor 1·007, an amplitude of 0·069 feet. The theoretical value of this, formula (127), is about 0.059 feet.

From hourly readings (1869–1887) at San Francisco, Cal., (lat. 37° 50′ N., long. 122° 24′ W.), the corresponding observed quantities were 8·285, 8·206, giving an amplitude of 0·040 feet, the theoretical value being 0·057.

REDUCTION OF SPRING AND NEAP TIDES.

29. In one column of the sheet headed "Ephemeris" will be found the times of the moon's phases expressed in the kind of time used in making the observations. Select from the first reduction, groups of observations, each group beginning about a day and a half before the times of new moon, full moon, or quadrature; then fill out the forms for spring and neap tides. The correction for parallax is taken from Table 25, using for the amplitude of N_2 the value $\frac{1}{11}$ Mn, and for its relative age the value at a neighboring station taken as $1{\cdot}837\ (M_2{}^{\circ} - N_2{}^{\circ})$. For a series about 220 days in length the parallax correction becomes zero. Since transits of the true (not the mean) moon are used, the lunitidal intervals need not be corrected for parallax, especially when the springs or neaps are taken in pairs.

The age of the solar tide.—The lunitidal intervals found in the column headed "Average" are plotted one-half lunar day apart, at times found in the average time column. The plotted values of each group are then connected, as nearly as possible, with a straight line. The values most distant from the time of spring or neap tide, as the case may be, will, theoretically, depart most widely from the straight line when the latter is properly drawn. The time of spring or neap tide is found by noting where the mean value of the interval (as determined from the first reduction before (125) has been applied) falls upon the straight line. Each such time is then diminished by the time of the appropriate phase of the moon. The mean of these differences is the age of the solar tide relative to the lunar.

Spring and neap ranges.—One range is obtained from the reduction of spring tides, and one from that of neap tides. The age applied to the times of one or more phases of the moon will show what group factors must be used in reducing the amplitude of the solar wave as brought out from the spring and neap ranges. The group used in determining the ranges need not be as extensive as the group used in determining the age.

According to § 18, the amplitude of the solar wave should be further increased by the factor

$$1{\cdot}02 + 0{\cdot}04\ \frac{\mathrm{HWQ}^2 + \mathrm{LWQ}^2}{\mathrm{Mn}^2}\,. \tag{128}$$

HWQ and LWQ may be taken from the first reduction with sufficient accuracy. This will be referred to as the *inequality factor.*

30. *Example.*

The following tabulations, relating to a month's observations at Sitka, Alaska, show how these reductions are carried out:

Sitka, Alaska.

First reduction.

Date.	Moon's transits.		Time of— HW		Time of— LW		Lunitidal interval. HW		Lunitidal interval. LW		Height of— HW	Height of— LW
1893.	*h.*	*m.*	*h.*	*m.*	*h.*	*m.*	*h.*	*m.*	*h.*	*m.*	*Feet.*	*Feet.*
	(13	08)										
July 1	1	34	1	08	8	09	(12	00)	6	35	14·6	3·4
	(13	59)	14	51	19	54	13	17	(5	55)	12·8	8·4
2	2	24	2	01	8	42	(12	02)	6	18	14·4	3·9
	(14	48)	15	25	20	50	13	01	(6	02)	13·0	8·3
3	3	11	2	30	9	25	(11	42)	6	14	14·0	4·0
	(15	34)	15	58	21	14	12	47	(5	40)	12·8	7·8
4	3	37	3	12	9	54	(11	38)	5	57	13·2	4·3
	(16	19)	16	37	22	10	12	40	(5	51)	12·9	7·6
5	4	41	4	05	10	27	(11	46)	5	46	12·5	5·0
	(17	04)	17	23	23	23	12	42	(6	19)	13·2	7·2
6	5	26	5	09	11	14	(12	05)	5	48	12·0	5·9
	(17	48)	18	03	------		12	37	------		13·6	---
7	6	11	6	22	0	32	(12	34)	(6	44)	11·4	6·6
	(18	36)	18	46	12	18	12	35	6	07	14·0	6·7
8	7	00	7	49	1	49	(13	13)	(7	13)	11·2	5·8
	(19	26)	19	44	13	03	12	44	6	03	14·6	7·6
9	7	53	9	02	2	52	(13	36)	(7	26)	11·5	5·0
	(20	22)	20	37	14	15	12	44	6	22	15·0	8·3
10	8	51	10	28	3	50	(14	06)	(7	28)	11·9	3·9
	(21	23)	21	30	15	25	12	39	6	34	15·5	8·6
11	9	55	11	31	4	50	(14	08)	(7	27)	12·5	2·9
[$11^d\ 23^h.5$]	(22	28)	22	29	16	36	12	34	6	41	15·7	8·4
P 12	11	01	12	17	5	40	(13	49)	(7	12)	12·9	2·1
	(23	34)	23	26	17	29	12	25	6	28	16·1	8·0
13	------		------		6	28	------		(6	54)	----	1·9
	12	07	13	09	18	21	(13	35)	6	14	13·5	7·8
14	(0	38)	0	22	7	18	12	15	(6	40)	16·3	1·8
	13	09	13	55	19	19	(13	17)	6	10	13·9	7·6
15	(1	38)	1	11	8	01	12	02	(6	23)	16·2	2·4
	14	06	14	50	20	13	(13	12)	6	07	14·6	7·8
16	(2	32)	1	56	8	46	11	50	(6	14)	16·3	3·6
	14	57	15	19	21	04	(12	47)	6	07	15·1	7·9
17	(3	20)	2	50	9	31	11	53	(6	11)	15.7	4·3
	15	43	15	59	21	47	(12	39)	6	04	14·7	7·3
18	(4	05)	3	47	9	56	12	04	(5	51)	14·0	5·0
	16	27	16	38	22	50	(12	33)	6	23	14·2	7·0
19	(4	48)	4	40	10	35	12	13	(5	47)	12·7	5·9
	17	09	17	22	------		(12	34)	------		13·8	---
20	(5	30)	5	40	0	00	12	31	6	51	11·6	6·8
	17	51	18	08	11	21	(12	38)	(5	51)	13·7	7·1
21	(6	12)	6	52	0	48	13	01	6	57	11·2	6·9
	18	33	18	46	12	10	(12	34)	(5	58)	13·8	8·3
22	(6	55)	8	12	2	00	13	39	7	27	11·0	6·8
	19	17	19	34	13	08	(12	39)	(6	13)	13·4	9·0
23	(7	40)	9	30	3	18	14	13	8	01	11·0	6·0
[$24^d\ 2^h{\cdot}2$]	20	04	20	28	14	41	(12	48)	(7	01)	13·4	9·2
A 24	(8	28)	11	11	4	07	15	07	8	03	11·3	5·3
	20	53	21	34	15	21	(13	06)	(6	53)	13·5	9·3
25	(9	19)	11	40	4	47	14	47	7	54	11·8	4·8
	21	45	22	20	16	34	(13	01)	(7	15)	14·1	9·2
26	(10	11)	12	08	5	27	14	23	7	42	12·7	4·6
	22	37	23	02	17	04	(12	51)	(6	53)	15·3	9·6
27	(11	03)	12	45	6	11	14	08	7	34	13·2	4·7
	23	29	23	43	17	54	(12	40)	(6	51)	15·1	9·1
28	(11	54)	------		6	40	------		7	11	----	4·0
	------		13	14	18	30	13	45	(6	36)	13·2	8·5
29	0	20	0	25	7	16	(12	31)	6	56	15·1	3·8
	(12	44)	13	44	19	00	13	24	(6	16)	13·5	8·2
30	1	08	0	57	7	46	(12	13)	6	38	15·4	3·8
	(13	32)	14	16	19	47	13	08	(6	15)	14·0	7·9
31	1	55	1	38	8	22	(12	06)	6	27	15·3	4·1
	(14	18)	14	38	20	28	12	43	(6	10)	15·0	7·3

Remarks.

Lat. $= 57^\circ\ 03'$ N.
Long. $= 125\ \ 18 = 9^h\ 01^m$ W.

Observations in mean local civil time.

Greenwich transits.

To convert Greenwich transits to local transits, observation time, add $L - S + 0{\cdot}035\,(L - E) = 18^m{\cdot}9$.

Greenwich civil time of perigee, *P*, and apogee, *A*.

P and *A* hour of transit $= 9\frac{1}{8}$.

	HWQ	LWQ
	Feet.	*Feet.*
July 11, 12	3·2	6·3
" 26	2·6	5·0
Mean	2·9	5·6

Sequence HHW to LLW.

July 1–29, 1893.

	HWI	LWI	HW	LW
	56	56	56	56
	h. m.	*h. m.*	*Feet*	*Feet.*
Sum	722 04	369 38	760·2	351·2
Mean	12 53·6	6 36·0	13·58	6·27

Interval $= 9^h\ 44^m{\cdot}8$. Mn $= 7{\cdot}31$ feet.
To correct for transits subtract $18^m{\cdot}9$.
HWI $= 12^h\ 34^m{\cdot}7$. LWI $= 6^h\ 17^m{\cdot}1$.
HWI $= 12^h\ 34^m{\cdot}7 - 12^h\ 25^m{\cdot}2 = 0^h\ 9^m{\cdot}5$.
HWI $-$ LWI $= 6^h\ 26^m{\cdot}6 = 6^h{\cdot}443$.
$6^h{\cdot}443 - 6^h{\cdot}210 = 0^h{\cdot}233$.
$14^\circ{\cdot}492 \times 0{\cdot}233 = 3^\circ{\cdot}4 = M_2^\circ$.
Duration of fall $= 6^h\ 07^m{\cdot}6$.

July 1–27, 1893.

HHW	LLW	Gt
26	25	
Feet.	*Feet.*	*Feet.*
376·5	110·2	
14·48	4·41	10·07

Sitka, Alaska, 1893.

Ephemeris.

		h.	m.	
Longitude of ephemeris	(E)	0	00	W.
Longitude of time meridian	(S)	9	01	"
$E-S=$		-9	01	

Moon.	Greenwich civil time.	Observation time. Gr. civ. time $+E-S$.	Observation time increased by assumed age.	Observation time increased by uncorrected age.
	d. h.	d. h.	d. h.	d. h.
○ Full moon	June 29 6·4	28 21·4		29 23·9
☾ Last quarter	July 6 22·1	6 13·1		7 15·6
● New moon	" 13 12·8	13 3·8		14 6·3
☽ First quarter	" 20 17·0	20 8·0		21 10·5
○ Full moon	" 28 20·2	28 11·2		29 13·7
Apogee	June 26 13·7	26 4·7	28 3·7	28 1·7
(Mid-time)	July 4 6·6	3 21·6	5 20·6	5 18·6
Perigee	" 11 23·5	11 14·5	13 13·5	13 11·5
(Mid-time)	" 18 0·8	17 15·8	19 14·8	19 12·8
Apogee	" 24 2·2	23 17·2	25 16·2	25 14·2
(Mid-time)	Aug. 1 0·0	31 15·0	2 14·0	2 12·0
Perigee	" 8 21·7	8 12·7	10 11·7	10 9·7
On equator	July 6 2·9	5 17·9		6 11·4
Farthest N.	" 12 8·9	11 23·9		12 17·4
On equator	" 18 18·3	18 9·3		19 2·8
Farthest S.	" 26 5·2	25 20·2		26 13·7

Sitka, Alask , 1893.

Spring tides.

		h.	m.	
Longitude of ephemeris	(E)	0	00	W.
Longitude of time meridian	(S)	9	01	"
$E-S=$		-9	01	

From first reduction, July 1–29, 1893: Mean (uncorrected) interval $=9^h\ 44^m{\cdot}8$; Mn $=7{\cdot}31$ ft.
Assumed: $N_2=\frac{1}{11}$ Mn $=0{\cdot}665$ ft.; $\tau\ (N_2;\ M_2)=47^h$, from analysis at St. Paul, Kadiak Id.

Moon new and full $+E-S$.	Time. HW	Time. LW	Time. Average.	Lunitidal interval. HW	Lunitidal interval. LW	Lunitidal interval. Average.	Height. HW	Height. LW
d. h.	d. h. m.	d. h. m.	d. h. m.	h. m.	h. m.	h. m.	Feet.	Feet.
	11 22 29	11 16 36		12 34	6 41	10 04·0	15·7	8·4
	12 12 17	12 5 40		13 49	7 12	9 58·5	12·9	2·1
July 13 3·8	12 23 26	12 17 29	13 3 08	12 25	6 28	9 50·5	16·1	8·0
	13 13 09	13 6 28		13 35	6 54	9 44·5	13·5	1·9
	14 0 22	13 18 21		12 15	6 14	9 36·5	16·3	7·8
	14 13 55	14 7 18		13 17	6 40	9 32·2	13·9	1·8
	15 1 11	14 19 19		12 02	6 10	9 26·8	16·2	7·6
	15 14 50	15 8 01		13 12	6 23		14·6	2·4
							8	8
							119·2	40·0
							14·90	5·00
		Group $0^d\ 2^h$ after perigean tides.		Correction in terms of N_2			— ·90	+ ·90
				" " feet			— ·60	+ ·60
							14·30	5·60
	26 23 02	27 6 11		12 51	7 34	10 21·0	15·3	4·7
	27 12 45	27 17 54		14 08	6 51	10 12·5	13·2	9·1
July 28 11·2	27 23 43	28 6 40	28 9 32	12 40	7 11	10 03·0	15·1	4·0
	28 13 14	28 18 30		13 45	6 36	9 57·0	13·2	8·5
	29 0 25	29 7 16		12 31	6 56	9 46·8	15·1	3·8
	29 13 44	29 19 00		13 24	6 16	9 37·8	13·5	8 2
	30 0 57	30 7 46		12 13	6 38	9 33·5	15·4	3·8
	30 14 16	30 19 47		13 08	6 15		14·0	7·9
							8	8
							114·8	50·0
							14·35	6·25
		Group $3^d\ 6^h$ after apogean tides.		Correction in terms of N_2			+ ·60	·60
				" " feet			+ ·40	·40
							14·75	5·85

Age $=\frac{1}{2}\ (12+29)=20{\cdot}5$
Spring range $=8{\cdot}80$

Sitka, Alaska, 1893.

Neap tides.

		h.	m.	
Longitude of ephemeris	(*E*)	0	00	W.
Longitude of time meridian	(*S*)	9	01	"
	E − *S* =	−9	01	

Moon in quadrature +*E* − *S*.	Time.			Lunitidal interval.			Height.	
	H W	L W	Average.	H W	L W	Average.	H W	L W
d. *h.*	*d.* *h.* *m.*	*d.* *h.* *m.*	*d.* *h.* *m.*	*h.* *m.*	*h.* *m.*	*h.* *m.*	*Feet.*	*Feet.*
	5 4 05	5 10 27		11 46	5 46	9 08·2	12·5	5·0
	5 17 23	5 23 23		12 42	6 19	9 13·5	13·2	7·2
July 6 13·1	6 5 09	6 11 14	6 14 44	12 05	5 48	9 18·5	12·0	5·9
	6 18 03	7 0 32		12 37	6 44	9 30·5	13·6	6·6
	7 6 22	7 12 18		12 34	6 07	9 37·2	11·4	6·7
	7 18 46	8 1 49		12 35	7 13	9 46·0	14·0	5·8
	8 7 49	8 13 03		13 13	6 03	9 51·5	11·2	7·6
	8 19 44	9 2 52		12 44	7 26		14·6	5·0
							8	8
							102·5	49·8
							12·81	6·22
Group 1[d] 7[h] after mid. tides (*A* to *P*). Correction in terms of N_2							— ·20	+ ·20
" " feet							— ·13	+ ·13
							12·68	6·35
	19 4 40	18 22 50		12 13	6 23	9 14·2	12·7	7·0
	19 17 22	19 10 35		12 34	5 47	9 25·8	13·8	5·9
July 20 8·0	20 5 40	20 0 00	20 8 47	12 31	6 51	9 27·8	11·6	6·8
	20 18 08	20 11 21		12 38	5 51	9 36·8	13·7	7·1
	21 6 52	21 0 48		13 01	6 57	9 37·5	11·2	6·9
	21 18 46	21 12 10		12 34	5 58	9 54·5	13·8	8·3
	22 8 12	22 2 00		13 39	7 27	9 59·5	11·0	6·8
	22 19 34	22 13 08		12 39	6 13		13·4	9·0
							8	8
							101·2	57·8
							12·65	7·22
Group 1[d] 6[h] after mid. tides (*P* to *A*). Correction in terms of N_2							+ ·28	— ·28
" " feet							+ ·19	— ·19
							12·84	7·03

Age = ½ (36 + 29) = 32·5

Age from springs and neaps = ½ (20·5 + 32·5) = 26·5

Average date of phases + age = July 18

No. of tides before springs or neaps $= 6 + \frac{26·5}{6·2} = 10·3$

" " " after " " " = 16 − 10·3 = 5·7

Spring range = 8·80

Neap " = 6·07

31. *Treatment of phase reduction results.*

STATION.

Spring and neap tides.

Amplitude relations.

(1) Spring range from reduction.
(2) Neap range from reduction.
(3) $\frac{1}{4}[(1)-(2)]=$ approx. S_2.
(4) (3) × group factor, Table 35.
(5) (4) × inequality factor.
(6) (5) × F_2, Table 33.
(7) $(6)-0{\cdot}01\ \mathrm{Mn}=S_2$.

$$(1)+2\left\{(6)-(5)+\frac{\mathrm{Mn}-2(6)}{2\ \mathrm{Mn}}[(4)-(3)]\right\}=\text{spring range, Sg.}$$

$$(2)-2\left\{(6)-(5)+\frac{\mathrm{Mn}+2(6)}{2\ \mathrm{Mn}}[(4)-(3)]\right\}=\text{neap range, Np.}$$

Sg and Np are reduced to their mean values by Chapter IV, as soon as K_1+O_1 becomes approximately known by § 39 or 44.

Epoch relations.

(8) Age from reduction (hours).
(9) $M_2°+(8)\times1{\cdot}016=S_2°$.

SITKA, ALASKA, 1893.

Spring and neap tides, July 18, 1893.

Amplitude relations.

(1) 8·80.
(2) 6·07.
(3) $\frac{1}{4}[8{\cdot}80-6{\cdot}07]=0{\cdot}682$.
(4) $0{\cdot}682\times1{\cdot}17=0{\cdot}798$.
(5) $0{\cdot}798\times1{\cdot}070=0{\cdot}854$.
(6) $0{\cdot}854\times1{\cdot}318=1{\cdot}126$.
(7) $1{\cdot}126-0{\cdot}073=1{\cdot}053=S_2$.

$$8{\cdot}80+2\{0{\cdot}272+0{\cdot}347[0{\cdot}115]\}=9{\cdot}43=\text{Sg.}$$

$$6{\cdot}07-2\{0{\cdot}272+0{\cdot}653[0{\cdot}115]\}=5{\cdot}36=\text{Np.}$$

Epoch relations.

(8) $26{\cdot}5=\tau(S_2;\ M_2)$.
(9) $3{\cdot}4+26{\cdot}5\times1{\cdot}016=3{\cdot}4+26{\cdot}9=30{\cdot}3=S_2°$.

REDUCTION OF PERIGEAN, APOGEAN, AND MIDTIME TIDES.

32. *Midtime tides.*

A number of tides occurring between perigee and apogee, also between apogee and perigee, are tabulated in the form headed "Midtime tides" for the purpose of ascertaining the relative age of the parallax wave. It is generally sufficient to begin the tabulation about a day and a half before the midtimes and to use on each occasion, say, 24 tides, thus covering nearly 6 lunar days. It is necessary to take pairs of midtimes because the age as determined from the one following perigee may differ a day or two from the age as determined from the one following apogee. The phase correction employed in this reduction is obtained from Table 24, column headed "0 tides," using the amplitude of S_2 uncorrected for K_2 and T_2. The phase correction becomes zero for a series about 220 days in length; i. e. when 8 midtime groups of the same kind are combined into one. The last column shows the corrected range of tide for the times given in the column headed "Average."

The next step is to plot these ranges at the proper times—which times are one-half lunar day apart, very nearly—and to draw a curve fitting the plotted values. Since these values usually appear as a double row of points, a third row should be obtained lying midway between the other two. It may be seen by aid of Table 16 that a parallax inequality whose age is zero and which is due to N_2, L_2, and $2N$ should cause the range of tide at midtime to be about equal to its mean value. Consequently one should mark upon each curve the time when the range becomes equal to its mean value taken from the first reduction. The times of occurrence of these ranges, diminished by the appropriate midtimes, give, in the mean, the age of the parallax inequality.

33. *Perigean and apogean tides.*

The age of the parallax inequality, when added to the observation times of perigee and apogee, gives the times of greatest and least parallax effect upon the range of tide. Before and after each of these times copy down, say, 6 tides, and determine the perigean and apogean ranges. Table 24, column headed "6 tides," is employed in making the correction for phase. The high and low water diurnal inequalities are taken at the times of greatest and least parallax effect.

The inequality factor for a short series is approximately equal to

$$0{\cdot}95+\left(\frac{S_2}{Mn}\right)^2+0{\cdot}08\,\frac{\dfrac{\Sigma_n\,(\text{HW inequality})^2+\Sigma_n\,(\text{LW inequality})^2}{n}}{Mn^2} \tag{129}$$

where n is the number of perigees and apogees taken. For a long series the factor becomes

$$0{\cdot}95+\left(\frac{S_2}{Mn}\right)^2+0{\cdot}04\,\frac{HWQ^2+LWQ^2}{Mn^2}. \tag{130}$$

34. *Example.*

A month's observations at Sitka, Alaska, will illustrate what has been said concerning the reduction of parallax tides.

Sitka, Alaska, 1893.

Midtime tides.

		h.	m.	
Longitude of ephemeris	(E)	0	00	W.
Longitude of time meridian	(S)	9	01	"
	$E-S=$	−9	01	

From first reduction, July 1–29, 1893: Mn = 7·31 feet.
From phase reduction: $S_2 = (5) - 0{\cdot}01$ Mn $= 0{\cdot}781$ foot; $\tau\,(S_2;\ M_2) = 26^{h}{\cdot}5$.

Midtime $+E-S$.	Time. HW	Time. LW	Time. Average.	Range. Observed.	Range. Average.	Range. ½ Correction.	Range. Correction.	Range. Corrected.
d. h.	*d. h. m.*	*d. h. m.*	*d. h. m.*	*Feet.*	*Feet.*		*Feet.*	*Feet.*
	2 15 25	2 20 50		4·7				
	3 2 30	3 9 45		10·0	7·35	−0·47 S_2	−0·73	6·62
July 3 21·6	3 15 58	3 21 14	4 00 34	5·0	7·50	−0·29	−0·45	7·05
	4 3 12	4 9 54		8·9	6·95	−0·08	−0·12	6·83
A to *P*	4 16 37	4 22 10		5·3	7·10	+0·13	+0·22	7·32
	5 4 05	5 10 27		7·5	6·40	+0·37	+0·58	6·98
	5 17 23	5 23 23		6·0	6·75	+0·60	+0·94	7·69
	6 5 09	6 11 14		6·1	6·05	+0·83	+1·30	7·35
	6 18 03	7 0 32		7·0	6·55	+1·00	+1·56	8·11
	7 6 22	7 12 18	7 3 19	4·7	5·85	+1·13	+1·77	7·62
	7 18 46	8 1 49		8·2	6·45	+1·18	+1·84	8·29
	8 7 49	8 13 03		3·6	5·90	+1·13	+1·77	7·67
	16 1 56	16 8 46		12·7				
	16 15 19	16 21 04		7·2	9·95	−0·61	−0·95	9·00
July 17 15·8	17 2 50	17 9 31	17 12 32	11·4	9·30	−0·45	−0·70	8·60
	17 15 59	17 21 47		7·4	9·40	−0·29	−0·42	8·98
P to *A*	18 3 47	18 9 56		9·0	8·20	0·00	0·00	8·20
	18 16 38	18 22 50		7·2	8·10	+0·26	+0·41	8·51
	19 4 40	19 10 35		6·8	7·00	+0·48	+0·75	7·75
	19 17 22	20 0 00		7·0	6·90	+0·70	+1·09	7·99
	20 5 40	20 11 21		4·5	5·75	+0·90	+1·41	7·16
	20 18 08	21 0 48	20 14 59	6·8	5·65	+1·05	+1·64	7·29
	21 6 52	21 12 10		2·9	4·85	+1·15	+1·80	6·65
	21 18 46	22 2 00		7·0	4·95	+1·16	+1·81	6·76

Age = ½ (20 + 70) = 45

Sitka, Alaska, 1893.

Perigean and apogean tides.

		h.	m.	
Longitude of ephemeris	(E)	0	00	W.
Longitude of time meridian	(S)	9	01	"
	$E-S=$	-9	0	

Perigee, apogee, $+E-S+45^h$.	Time. HW	Time. LW	Time. Average.	Height. HW	Height. LW
d. h.	*d. h. m.*	*d. h. m.*	*d. h. m.*	*Feet.*	*Feet.*
	11 22 29	12 5 40		15·7	2·1
	12 12 17	12 17 29		12·9	8·0
July 13 11·5	12 23 26	13 6 28	13 9 20	16·1	1·9
	13 13 09	13 18 21		13·5	7·8
	14 0 22	14 7 18		16·3	1·8
	14 13 55	14 19 19		13·9	7·6
				6 / 88·4	6 / 29·2
				14·73	4·87
	Group 0d 21h before springs.	Correction in terms of S_2		$-$ 0·87	$+$ 0·87
		" " feet		$-$ 0·68	$+$ 0·68
				14·05	5·55
				Perigean range	$=$ 8·50
				HW ineq. $=$ 2·6	LW ineq. $=$ 5·9
	24 11 11	24 4 07		11·3	5·3
	24 21 34	24 15 21		13·5	9·3
July 25 14·2	25 11 40	25 4 47	25 13 46	11·8	4·8
	25 22 20	25 16 34		14·1	9·2
	26 12 08	26 5 27		12·7	4·6
	26 23 02	26 17 04		15·3	9·6
				6 / 78·7	6 / 42·8
				13·12	7·13
	Group 4d 00h before springs.	Correction in terms of S_2		$-$ 0·17	$+$ 0·17
		" " feet		$-$ 0·13	$+$ 0·13
				12·99	7·26
				Apogean range	$=$ 5·73
				HW ineq. $=$ 2·3	LW ineq. $=$ 4·4

35. *Treatment of parallax reduction results.*

STATION.

Parallax tides.

Amplitude relations.

(1) Perigean range from reduction.
(2) Apogean range from reduction.
(3) $\frac{1}{4}[(1)-(2)]=$ approx. (N_2-L_2).
(4) (3)$\times$ group factor, Table 35.
(5) (4)$\times$ inequality factor.
(6) $f(N_2)$, $f(L_2)$, Table 10; c, Table 34.
(7) $\dfrac{(5)}{cf(N_2)-\frac{1}{7}f(L_2)}=N_2$.

$(1)+2\{N_2(1-c)+0{\cdot}4[(4)-(3)]\}=$ perigean range, Pn.

$(2)-2\{N_2(1-c)+0{\cdot}6[(4)-(3)]\}=$ apogean range, An.

Pn and An are reduced to their mean values by Chapter IV.

Epoch relations.

(8) Age from reduction (hours).
(9) $M_2^\circ-(8)\times 0{\cdot}544=N_2^\circ$.

SITKA, ALASKA, 1893.

Parallax tides July 19, 1893.

Amplitude relations.

(1) 8·50.
(2) 5·73.
(3) $\frac{1}{4}[8{\cdot}50-5{\cdot}73]=0{\cdot}692$.
(4) $0{\cdot}692\times 1{\cdot}02=0{\cdot}706$.
(5) $0{\cdot}706\times 1{\cdot}041=0{\cdot}735$.
(6) $f(N_2)=0{\cdot}967$, $f(L_2)=0{\cdot}64$, $c=1{\cdot}05$.
(7) $\dfrac{0{\cdot}735}{1{\cdot}05\times 0{\cdot}967-\frac{1}{7}0{\cdot}68}=\dfrac{0{\cdot}735}{0{\cdot}918}=0{\cdot}801=N_2$.

$8{\cdot}50+2\{0{\cdot}801\times(-0{\cdot}05)+0{\cdot}4[0{\cdot}014]\}=8{\cdot}53=$ Pn.

$5{\cdot}73-2\{0{\cdot}801\times(-0{\cdot}05)+0{\cdot}6[0{\cdot}014]\}=5{\cdot}65=$ An.

Epoch relations.

(8) $5{\cdot}45=\tau(N_2;\ M_2)$.
(9) $M_2^\circ-24{\cdot}5=-21{\cdot}1=338{\cdot}9=N_2^\circ$.

REDUCTION OF DECLINATIONAL TIDES.

36. *Moon near the equator.*

Whenever the amplitude of the diurnal wave is small in comparison with that of the semidiurnal, we have

$$\text{Range of diurnal wave} = \sqrt{(\text{HW inequality})^2 + (\text{LW inequality})^2} \qquad (131)$$

The minimum value of this range and the time of its occurrence are found by means of the reduction headed "Minimum diurnal tides." The time of this event diminished by the observation time of moon on equator is the uncorrected age of the diurnal inequality, or of O_1 relative to K_1.

In determining the minimum diurnal tide, several days' observations should be used, beginning shortly before the moon crosses the equator. The HW inequality or LW inequality should be made to change sign whenever either passes through zero. The following process may be employed:

Plot the numerically greater inequality, and thus determine when it becomes zero. The equation of this straight line may be written

$$y = mt \qquad (132)$$

The lesser inequality when plotted will approximately follow a straight line whose equation is

$$\frac{t}{a} + \frac{y}{b} = 1 \qquad (133)$$

The sum of the squares of the two y's becomes a minimum when

$$t = \frac{b^2 a}{a^2 m^2 + b^2} \qquad (134)$$

If the speed of the diurnal wave is less than m_1, prefix the minus sign to the minimum range.

The minimum ranges should be taken in pairs in order to eliminate Q_1 from O_1.

37. *Moon far north or south.*

The uncorrected age of the diurnal inequality having been found, increase the times of extreme declination thereby, thus obtaining the times when the amplitude of the diurnal wave becomes a maximum. At these times tropic tides occur; in the reduction, one or two or more high waters and as many low waters are taken on either side of each time. It may be noted that for any one determination of the quantities directly connected with the tropic tides, the lunitidal intervals having approximately equal values should be written in the same column. From a reduction of the high waters and of the low waters, according to the accompanying form, the following uncorrected quantities are obtained:

Intervals, heights, Gc, Sc, Mc, $\overline{\text{HWQ}}$*, LWQ.

Both heights and intervals of the great tropic tides can always be obtained from the reductions; but the small tropic tides can not be so obtained where evanescent tides occur. At such places Sc may be roughly inferred from the equation

$$\text{Gc} + \text{Sc} = 2\ \text{Mn}.$$

The tropic range of the diurnal wave for a given time becomes approximately known by the formula

$$2\ \text{D}_1 = \sqrt{\text{HWQ}^2 + \text{LWQ}^2}. \qquad (137)$$

The amplitude D_2 of the semidiurnal wave when the moon is far from the equator is approximately equal to

$$\tfrac{1}{2}\ \text{Mc} - \tfrac{1}{8}\frac{\text{D}_1^2}{\text{Mc}} \qquad (138)$$

* It is here supposed that LWQ > HWQ. $\overline{\text{HWQ}}$ indicates merely that the order of taking the tropic higher high and lower high water, has remained the same throughout the series.

At stations where the order of the high waters, say, is sometimes reversed, even when the moon is far from the equator, a period of six months, or some multiple thereof, should be used in finding HWQ and X, § 21. These quantities are reduced to their mean values by Chapter IV.

When a short series is used, X and HWQ should be computed from $P_1 = 0{\cdot}331\ K_1$, §§ 39, 40, and the corrected $\overline{\text{HWQ}}$, by means of the formulæ

$$\text{Cos X } 180° = \frac{\overline{\text{HWQ}}}{2\ \text{P}_1}, \qquad (135)$$

$$\text{HWQ} = 2\ \text{P}_1\ [(1 - 2\ \text{X}) \cos \text{X } 180° + \frac{2}{\pi} \sin \text{X } 180°] \qquad (136)$$

where Mc is the mean of Gc and Sc. If Mc is poorly determined, the formula

$$D_2 = 0{\cdot}44\ \text{Mn} - 0{\cdot}06\ \frac{D_1^2}{\text{Mn}} \tag{139}$$

should be used.

Having found D_2, two ratios are obtained by dividing $\overline{\text{HWQ}}$ and LWQ by 2 D_2. The intersection of two curves upon Plate III which correspond to these ratios gives an amplitude (in terms of D_2) and a HW phase.

Stations where the tide is usually diurnal.—The quantities to be obtained from the reduction are the intervals (properly distinguished) and the heights of the tropic tides; also, approximately, mean sea level and the mean semidaily range of tide. For most stations, the two latter quantities can be found whenever the moon is near the equator, and especially about the times of the equinoxes. From these results are determined the ratio Gc ÷ Mn, the duration of the great tropic range (always taken to be less than a half lunar day), and whether the tropic high or tropic low water, departs farther from mean sea level. By means of Table 20 an amplitude and HW phase become known.

Rule for distinguishing between the qurdrants:

Sequence HHW to LLW	Interval (HW or LW) marked (*a*)	HW phase falls in 1st quadrant.
	" " " (*b*)	" " " 3d "
Sequence LLW to HHW	HW interval marked (*a*)	" " " 4th "
	LW (*b*)	" " " " "
	HW interval marked (*b*)	" " " 2d "
	LW (*a*)	" " " " "

It is assumed that the HW interval is taken approximately equal to $M_2°/29$ and the LW interval, to $M_2°/29 + 6^h$; also that the mean intervals have the marks (*a*, *b*) belonging to the intervals of the great tropic tides, § 53.

38. *Example.*

Declinational reductions for a month's observations at Sitka, Alaska, are given below:

Sitka, Alaska, 1893.

Minimum diurnal tide.

		h.	m.
Longitude of ephemeris	(*E*)	0	00 W.
Longitude of time meridian	(*S*)	9	01 "
	E — *S* =	— 9	01

Moon on equator + *E* − *S*.		Time.													Height.		Inequality.		Time and value of minimum range.
		H W			Average.			L W			Average.				H W	L W	H W	L W	
d.	*h.*	*d.*	*h.*	*m.*	*d.*	*h.*	*m.*	*d.*	*h.*	*m.*	*d.*	*h.*	*m.*		*Feet.*	*Feet.*			
		3	15	58				3	9	25					12·8	4·0	+0·4	—3·8	
		4	3	12				3	21	14					13·2	7·8	+0·3	—3·5	
		4	16	37				4	9	54					12·9	4·3	—0·4	—3·3	
		5	4	05				4	22	10					12·5	7·6	—0·7	—2·6	
July 5	17·9	5	17	23	5	23	16	5	10	27	5	16	55		13·2	5·0	—1·2	—2·2	
		6	5	09				5	23	23					12·0	7·2	—1·6	—1·3	
		6	18	03				6	11	14					13·6	5·9	—2·2	—0·7	July 6, 8h
		7	6	22				7	0	32					11·4	6·6	—2·6	+0·1	" 5, 18
		7	18	46				7	12	18					14·0	6·7	—2·8	+0·9	14 = age.
		8	7	49				8	1	49					11·2	5·8			Range = 2·20
		16	1	56				15	20	13					16·3	7·8	+1·2	—4·2	
		16	15	19				16	8	46					15·1	3·6	+0·6	—4·3	
		17	2	50				16	21	04					15·7	7·9	+1·0	—3·6	
		17	15	59				17	9	31					14·7	4·3	—0·7	—3·0	
July 18	9·3	18	3	47	18	10	12	17	21	47	18	3	52		14·0	7·3	—0·2	—2·3	
		18	16	38				18	9	56					14·2	5·0	—1·5	—2·0	July 19, 6h
		19	4	40				18	22	50					12·7	7·0	—1·1	—1·1	" 18, 9
		19	17	22				19	10	35					13·8	5·9	—2·2	—0·9	21 = age.
		20	5	40				20	0	00					11·6	6·8	—2·1	+0·3	Range = 1·60
		20	18	08				20	11	21					13·7	7·1			

Uncorrected age = ½ (14 + 21) = 17·5
Average date of moon on equator + age = July 12
Minimum range = 1·90

Sitka, Alaska, 1893.

Tropic high waters.

		h.	m.
Longitude of ephemeris	(E)	0	00 W.
Longitude of time meridian (S)		9	01 "
	$E - S =$	9	01

From first reduction, July 1–29, 1893: Mean interval $= 9^h\ 45^m$.
From phase reduction: $S_2 = (5) - 0·01$ Mn $= 0·781$ foot; $\tau\ (S_2;\ M_2) = 26^h·5$.

Moon.				Time of high water.	Lunitidal interval.		Height.			
Extreme declination. $+ E - S + 17^h·5$.		Transit.			u n, l s, or	l n, u s.	For H H W.	Corr'n.	For L H W.	Corr'n.
		Time.	Kind.							
d.	*h.*	*d. h. m.*		*d. h. m.*	*h. m.*	*h. m.*	*Feet.*		*Feet.*	
July 12 N.	17·4	11 9 55	u n	11 22 29	12 34		15·7	— ·60 S_2		
		11 22 28	l n	12 12 17		13 49			12·9	— ·75 S_2
		12 11 01	u n	12 23 26	12 25		16·1	— ·83		
		12 23 34	l n	13 13 09		13 35			13·5	— ·90
July 26 S.	13·7	25 9 19	l s	25 22 20	13 01		14·1	— ·15		
		25 21 45	u s	26 12 08		14 23			12·7	— ·36
		26 10 11	l s	26 23 02	12 51		15·3	— ·52		
		26 22 37	u s	27 12 45		14 08			13·2	— ·68
					4	4	4	4	4	4
					49 111	54 115	61·2	—2·10	52·3	—2·69
				Mean	12 43	13 59	15·30	— ·52	13·08	— ·67
				Corrected for phase	12 15	13 31	14·89		12·56	

Sitka, Alaska, 1893.

Tropic low waters.

	h.	m.	
Longitude of ephemeris (*E*)	0	00	W
Longitude of time meridian (*S*)	9	01	"
$E - S =$	-9	01	

Moon. Extreme declination $+E-S+17{\cdot}5^h$.		Moon. Transit. Time.	Moon. Transit. Kind.	Time of low water.	Lunitidal interval. u n, l s, or	Lunitidal interval. l n, u s.	Height. For L L W.	Height. Corr'n.	Height. For H L W.	Height. Corr'n.
d.	*h.*	*d. h. m.*		*d. h. m.*	*h. m.*	*h. m.*	*Feet.*		*Feet.*	
		11 9 55	u n	11 16 36		6 41			8·4	+ ·53 S_2
July 12 N.	17·4	11 22 28	l n	12 5 40	7 12		2·1	+ ·69 S_2		
		12 11 01	u n	12 17 29		6 28			8·0	+ ·79
		12 23 26	l n	13 6 28	6 54		1·9	+ ·87		
		25 9 19	l s	25 16 34		7 15			9·2	+ ·05
July 26 S.	13·7	25 21 45	u s	26 5 27	7 42		4·6	+ ·27		
		26 10 11	l s	26 17 04		6 53			9·6	+ ·44
		26 22 37	u s	27 6 11	7 34		4·7	+ ·60		
					4 27 142	4 25 137	4 13·3	4 +2·43	4 35·2	4 +1·81
				Mean	7 20	6 49	3·32	+ ·61	8·80	+ ·45
				Corrected for phase	6 52	6 21	3·80		9·15	
				Mean of tropic ints. =	10 13					
				" int. first red'n =	9 45					
				Diff. =	28					

Average date of extreme declination + age	=	July 20
Great tropic range	=	11·09
Small tropic "	=	3·41
High water inequality	=	2·33
Low " "	=	5·35

39. *Treatment of declinational reduction results.*

STATION.

Declinational tides.

Amplitude relations.

(1) Minimum semirange of diurnal wave from reduction.
(2) $F(K_1)$, Table 10; c_1, Table 31. For av. time of min. diurnal tides.
(3) $\overline{HWQ}$ from reduction.
(4) LWQ " "
(5) (3) × group factor.
(6) (4) × " "
(6′) Gc from reduction + ½ [(5) + (6) — (3) — (4)].
(6″) Sc " " — ½ [(5) + (6) — (3) — (4)].
(6‴) Tropic LLW from reduction — ½ [(6) — (4)].
(7) $\sqrt{(5)^2 + (6)^2} =$ approx. $2\ D_1$.
(8) $0{\cdot}88\ Mn - 0{\cdot}03 \frac{(7)^2}{Mn} = 2\ D_2$.
(9) $\frac{(5)}{(8)} = \frac{\overline{HWQ}}{2\ D_2}$.
(10) $\frac{(6)}{(8)} = \frac{LWQ}{2\ D_2}$.
(11) Amplitude and HW phase from Table 19.
(12) $F(K_1)$, $F(O_1)$, Table 10; c_{11}, Table 31. For av. time of tropic tides.
(13) $\frac{(1) + (11)}{c_1 + c_{11}} = K_1'$
(14) $\frac{c_1 (11) - c_{11} (1)}{c_1 + c_{11}} = O_1'$
(15) $K_1' \times F(K_1) = K_1$.
(16) $O_1' \times F(O_1) = O_1$.
(17) I and F (Mn) for middle of series, Tables 6, 14.
(18) Mn from first reduction.
(19) (18) × F(Mn) = Mn
(20) $\frac{S_2}{\frac{1}{2} Mn}, \frac{K_1 + O_1}{\frac{1}{2} Mn}$.
(21) Tabular value, Table 23.
(22) ½ (19) × (21) = $(1 + \varepsilon)\ M_2$.

Nonharmonic quantities (see Chapter IV).

$$\frac{1{\cdot}02\ (K_1 + O_1)}{c_{11}\ K_1^1 + O_1^1} = 1{\cdot}02\ F_1$$

(5) × $1{\cdot}02\ F_1 = \overline{HWQ}$. (6) × $1{\cdot}02\ F_1$ = LWQ

[tropic HHWI from reduction — HWI*] × $1{\cdot}02\ F_1$ + HWI† = tropic HHWI.
[tropic LHWI from reduction — HWI] " " = tropic LHWI.
[tropic LLWI from reduction — LWI] " LWI = tropic LLWI.
[tropic HLWI from reduction — LWI] " " = tropic HLWI.
[(6′) — (18)] × $1{\cdot}02\ F_1$ + (19) = Gc.
[(6″) — (18)] " " = Sc.
[Gt from first reduction — (18)] × $1{\cdot}02\ F_1$ + (19) = Gt.
2 (19) — Gt = Sl.
Mean LW from first reduction — ½ [(19) — (18)] = LW.
[(6‴) — mean LW from first reduction] × $1{\cdot}02\ F_1$ + LW = tropic LLW.
[LLW from first reduction — mean LW from first reduction] × $1{\cdot}02\ F_1$ + L W = LLW.

Epoch relations.

(23) Age from reduction (hours).
(24) (23) × 1·098 = approx. ($K_1° - O_1°$).
(25) (24) + acc. in K_1 due to P_1, Table 31, = $K_1° - O_1°$.
0·911 ($K_1° - O_1° = \tau\ (O_1;\ K_1)$.
(26) Mean lunitidal interval exceeds mean tropic interval (minutes).
(27) (26) × 0·483.
(28) 2 HW phase from amplitude relations.
(29) (27) + (28) — acc. in K_1 due to P_1, Table 31, = $M_2° - K_1° - O_1°$.
(30) $\frac{M_2° + (25) - (29)}{2} = K_1°$.
(31) $\frac{M_2° - (25) - (29)}{2} = O_1°$

* Not corrected by (125). † Corrected by (125).

SITKA, ALASKA, 1893.

Declinational tides, July, 1893.

Amplitude relations.

(1) 0·95.
(2) $F(K_1) = 0{\cdot}905$, $c_1 = 1.243$; July 12.
(3) 2·33.
(4) 5·35.
(5) $2{\cdot}33 \times 1{\cdot}01 = 2{\cdot}353$.
(6) $5{\cdot}35 \times 1{\cdot}01 = 5{\cdot}404$.
(6′) $11{\cdot}09 + \frac{1}{2}[0{\cdot}077] = 11{\cdot}13$.
(6″) $3{\cdot}41 - \frac{1}{2}[0{\cdot}077] = 3{\cdot}37$.
(6‴) $3{\cdot}81 - \frac{1}{2}[0{\cdot}054] = 3{\cdot}783$.
(7) $\sqrt{2{\cdot}353^2 + 5{\cdot}404^2} = \sqrt{34{\cdot}740} = 5{\cdot}894$.
(8) $0{\cdot}88 \times 7{\cdot}31 - 0{\cdot}03\,\frac{34{\cdot}740}{7{\cdot}31} = 6{\cdot}433 - 0{\cdot}143 = 6{\cdot}290 = 2\,D_2$.
(9) $\frac{2{\cdot}353}{6{\cdot}290} = 0{\cdot}374$.
(10) $\frac{5{\cdot}404}{6{\cdot}290} = 0{\cdot}859$.
(11) $D_1 = 0{\cdot}945\,D_2 = 2{\cdot}972$, HW phase $= 246{\cdot}1$.
(12) $F(K_1) = 0{\cdot}905$, $F(O_1) = 0{\cdot}854$, $c_{11} = 1{\cdot}196$; July 20.
(13) $\frac{3{\cdot}922}{2{\cdot}439} = 1{\cdot}608 = K_1'$
(14) $\frac{2{\cdot}558}{2{\cdot}439} = 1{\cdot}049 = O_1'$
(15) $1{\cdot}608 \times 0{\cdot}905 = 1{\cdot}455 = K_1$.
(16) $1{\cdot}049 \times 0{\cdot}854 = 0{\cdot}896 = O_1$.
(17) $I = 28{\cdot}23$, $F(Mn) = 1{\cdot}022$; July 15.
(18) 7·31.
(19) $7{\cdot}31 \times 1{\cdot}022 = 7{\cdot}471 = Mn$.
(20) $\frac{1{\cdot}053}{3{\cdot}736} = 0{\cdot}28$, $\frac{2{\cdot}351}{3{\cdot}736} = 0{\cdot}63$.
(21) 0·949.
(22) $3{\cdot}736 \times 0{\cdot}949 = 3{\cdot}545 = (1+\varepsilon)\,M_2 = 1{\cdot}02\,M_2$; $\therefore 3{\cdot}476 = M_2$.

Nonharmonic quantities.

$$\frac{1{\cdot}02\,(1{\cdot}455 + 0{\cdot}896)}{1{\cdot}196 \times 1{\cdot}608 + 1{\cdot}049} = 1{\cdot}02\left(\frac{2{\cdot}351}{2{\cdot}972}\right) = 1{\cdot}02 \times 0{\cdot}791 = 0{\cdot}807.$$

$2{\cdot}353 \times 0{\cdot}807 = 1{\cdot}899 =$ HWQ. $5{\cdot}404 \times 0{\cdot}807 = 4{\cdot}361 =$ LWQ.

[12 15 — 12 54] × 0·807 = — 31. — 31 + 12 35 = 12 04 = tropic HHWI.
[13 31 — 12 54] × 0·807 = + 30. + 30 + 12 35 = 13 05 = tropic LHWI.
[6 52 — 6 36] × 0·807 = + 13. + 13 + 6 17 = 6 30 = tropic LLWI.
[6 21 — 6 36] × 0·807 = — 12. — 12 + 6 17 = 6 05 = tropic HLWI.
[11·13 — 7·31] × 0·807 = + 3·083. + 3·083 + 7·471 = 10·55 = Gc.
[3·37 — 7·31] × 0·807 = — 3·180. — 3·180 + 7·471 = 4·29 = Sc.
[10·07 — 7·31] × 0·807 = + 2·227. + 2·227 + 7·471 = 9·70 = Gt.
14·94 — 9·70 = 5·24 = Sl.
6·27 — ½ [7·47 — 7·31] = 6·19 = LW.
[3·783 — 6·27] × 0·807 = — 2·007. — 2·01 + 6·19 = 4·18 = tropic LLW.
[4·41 — 6·27] × 0·807 = — 1·501. — 1·50 + 6·19 = 4·69 = LLW.

Epoch relations.

(23) 17·5.
(24) $17{\cdot}5 \times 1{\cdot}098 = 19{\cdot}2$.
(25) $19{\cdot}2 + (-8{\cdot}6) = 10{\cdot}6$, July 12.
$0{\cdot}911 \times 10{\cdot}6 = 9{\cdot}7 = r\,(O_1;\,K_1)$.
(26) — 28.
(27) $-28 \times 0{\cdot}483 = -13{\cdot}5$.
(28) $2 \times 246{\cdot}0 = 492{\cdot}0$.
(29) $-13{\cdot}5 + 492{\cdot}0 - (-11{\cdot}6) = 490{\cdot}1$, July 20.
(30) $\frac{3{\cdot}4 + 10{\cdot}6 - 490{\cdot}1}{2} = -238{\cdot}0 = 122{\cdot}0 = K_1^\circ$.
(31) $\frac{3{\cdot}4 - 10{\cdot}6 - 490{\cdot}1}{2} = -248{\cdot}6 = 111{\cdot}4 = O_1^\circ$.

40. *Inferred amplitudes and epochs of components.*

Component.	Amplitude.	Epoch.
K_2	$0{\cdot}272\ S_2$	$M_2^\circ + 1{\cdot}098\,\tau\,(S_2;\ M_2)$
L_2	$0{\cdot}145\ N_2$	$M_2^\circ + 0{\cdot}544\,\tau\,(N_2;\ M_2)$
ν_2	$0{\cdot}194\ N_2$	$M_2^\circ - 0{\cdot}472\,\tau\,(N_2;\ M_2)$
2N	$0{\cdot}133\ N_2$	$M_2^\circ - 1{\cdot}089\,\tau\,(N_2;\ M_2)$
P_1	$0{\cdot}331\ K_1$	$K_1^\circ - 0{\cdot}082\,\tau\,(O_1;\ K_1)$
Q_1	$0{\cdot}194\ O_1$	$K_1^\circ - 1{\cdot}642\,\tau\,(O_1;\ K_1)$
T_2	$0{\cdot}059\ S_2$	$M_2^\circ + \ \ 975\,\tau\,(S_2;\ M_2)$
M_4	$\frac{1}{5} M_2$ (duration fall $\sim 6^h{\cdot}21$)	$2\,M_2^\circ \mp 90^\circ$ *
$(MS)_4$	$M_4 \times \frac{2\,S_2}{M_2}$	$M_4^\circ + S_2^\circ - M_2^\circ$

$$K_2^\circ = M_2^\circ + 1{\cdot}081\ (S_2^\circ - M_2^\circ)$$
$$L_2^\circ = 2\,M_2^\circ - N_2^\circ$$
$$\nu_2^\circ = M_2^\circ - 0{\cdot}868\,(M_2^\circ - N_2^\circ)$$
$$(2N)^\circ = 2\,N_2^\circ - M_2^\circ$$
$$P_1^\circ = K_1^\circ - 0{\cdot}075\,(K_1^\circ - O_1^\circ)$$
$$Q_1^\circ = K_1^\circ - 1{\cdot}495\,(K_1^\circ - O_1^\circ)$$
$$T_2^\circ = M_2^\circ + 0{\cdot}960\ (S_2^\circ - M_2^\circ)$$
$$M_4^\circ = 2\,M_2^\circ \mp 90^\circ\,*$$
$$(MS)_4^\circ = M_4^\circ + S_2^\circ - M_2^\circ$$

41. *Collection of results, Sitka, Alaska.*

	From 29 days' high and low water, July 1–29, 1893.		From harmonic analysis of hourly ordinates 1 year, 1893–94.	
	Amplitude.	Epoch.	Amplitude.	Epoch.
	Feet.	°	*Feet.*	°
K_1	1·46	122	1·51	125
K_2	0·29	32	0·32	20
L_2	0·12	28	0·31	35
M_2	3·48	3·4	3·58	2·6
N_2	0·80	339	0·69	338
2N	0·11	314		
O_1	0·90	111	0·91	106
P_1	0·48	121	0·46	123
Q_1	0·17	106	0·14	106
S_2	1·05	30	1·14	34
ν_2	0·16	342	0·06	295

From 29 days' high and low water, July 1–29, 1893.

	h.	*m.*		*Feet.*
HWI	0	10	Mn	7·47
LWI	6	17	Gc	10·55
Tropic HHWI	—0	21 *b*	Sc	4·29
Tropic LHWI	0	40 *a*	Gt	9·70
Tropic LLWI	6	30 *b*	Sl	5·24
Tropic HLWI	6	05 *a*	Sg	9·59
Age of phase inequality	26		Np	5·52
Age of parallax inequality	45		Pn	8·67
Age of diurnal inequality	10		An	5·83
			HWQ	1·90
			LWQ	4·36
			LW (on staff)	6·19
			Tropic LLW (on staff)	4·18
			Mean LLW (on staff)	4·69

* Use $\substack{\text{upper}\\ \text{lower}}$ sign when duration of fall $\gtrless 6^h{\cdot}21$. The determination of M_4 in this manner is necessarily rough.

DATUM PLANES.

42. *To determine a plane of reference, having a given definition with respect to the tide, when only a few observations are available.*

It is here supposed that observations, or more likely predictions, simultaneous with the observations in question are given for a neighboring principal station having a similar type of tide; it is also supposed that the observations or predictions at the principal station have been reduced to a plane of reference having the required definition.

Determine from the observed heights at the subordinate station a plane approaching the one which definition would require. Call this the approximate plane and let h denote its height above the required plane. Find a range of tide by taking the mean of all observed low waters from the mean of all observed high waters.

Let H denote the height of the tides at the principal station which correspond to the tides used in determining the approximate plane at the subordinate station. Let

$$r = \frac{\text{observed range at subordinate station}}{\text{corresponding range at principal station}}; \tag{140}$$

then

$$h = r\,H. \tag{141}$$

The more nearly r remains constant, the better the selection of the neighboring principal station.

As a general rule field parties are supplied with tide tables giving predictions for the time in question. Now, unless the predicted heights are referred to a plane of reference having the same tidal definition as the one to be determined, they must be reduced to such a plane by adding a constant which we must suppose to be given in the tide tables. To dispense with this labor, predictions and soundings should be referred to planes having uniform definitions over a considerable area. To avoid confusion, which is sure to result near the limits of each such area, one definition should be used the world over.

43.* *To determine mean sea level from one month's observations.*—Use the period of 29 solar days. Take the mean of average high and average low water, or the mean of the hourly ordinates if a continuous record be available.

To determine mean low water from one month's observations.—Use the period of 29 solar days. Find the mean range of tide and mean low water for this period. Reduce this range by the factor F (Mn), Table 14. Depress the observed mean low water by

$$\tfrac{1}{2}\,(\text{mean range} - \text{observed mean range}). \tag{143}$$

The value of $(K_1 + O_1)/M_2$ which is needed in Table 14 may be inferred from some neighboring station with sufficient accuracy for the present purpose, or it may be found by means of § 44.

To determine mean lower low water from one month's observations.—Use the period of 27 solar days, i. e. a declinational month. Mark the lower low water of each day, and take the mean of the values so marked. Subtract this value from mean low water as determined from the 29-day period, and multiply the result by the factor 1·02 F_1, Table 32. Depress the corrected mean low water by the quantity just obtained, and the result will be mean lower low water.

This plane of reference is unsatisfactory in accurate work, because no one reducing factor, like F_1, applies well to all stations. This is especially true where the low water inequality (LWQ) is small in comparison with the high water inequality (HWQ).

44. *To determine the harmonic or Indian tide plane.*

This plane is, by definition,

$$M_2 + S_2 + K_1 + O_1 \tag{144}$$

---

*To correct the planes of reference mentioned in this paragraph for the annual and semiannual components, depress the observation result by

$$\text{Sa}\cos(h - \text{Sa}^\circ) + \text{Ssa}\cos(2h - \text{Ssa}^\circ) \tag{142}$$

where h is the mean longitude of the sun, Table 29. The amplitudes and epochs of these components may be taken from a neighboring station.

feet below mean sea level as determined from hourly ordinates. Usually it nearly coincides with what might be called the tropic lower low water springs. At stations where harmonic analyses are available the depression of this plane below mean sea level becomes known at once.

Where harmonic analyses are not available a month's observations upon high and low water may be used, as indicated below, for determining the required plane:

Select a time near each extreme declination of the moon when the sum of the high and low water inequalities is a maximum. Denoting these inequalities by HWQ and LWQ, the tropic range of the diurnal wave for the series is, approximately,

$$2\,\mathrm{D}_1 = \sqrt{\mathrm{HWQ}^2 + \mathrm{LWQ}^2}\ ; \tag{145}$$

$$\therefore\ \mathrm{K}_1 + \mathrm{O}_1 = \mathrm{D}_1 \times F_1\ , \tag{146}$$

Table 32.

From the observations determine spring and neap ranges, denoting them by Sg and Np; then

$$\mathrm{S}_2 = \tfrac{1}{4}\,(\mathrm{Sg} - \mathrm{Np})\left(1{\cdot}02 + 0{\cdot}04\,\frac{\mathrm{HWQ}^2 + \mathrm{LWQ}^2}{\mathrm{Mn}^2}\right) F_2 - 0{\cdot}01\ \mathrm{Mn}\ . \tag{147}$$

M_2 is obtained directly from Mn; i. e. the mean range from first reduction $\times\ F$ (Mn), Table 23. The quantities M_2, S_2, and $\mathrm{K}_1 + \mathrm{O}_1$, taken in connection with mean sea level as determined by the preceding paragraph, determine the required plane. This plane is in quite extensive use under the name of low water springs or Indian low water springs. It has the convenience of being so low that comparatively few low water heights are negative.

45. *Stations where the tides are usually diurnal.*

When possible, mean sea level should be determined from hourly ordinates. If determined from high and low waters, they should be taken in pairs, so that each high water taken shall be accompanied by an adjacent low water having approximately the same displacement from mean sea level. Mean low water (semidiurnal) is here of no consequence as a plane of reference. The correction for annual and semiannual components is made as in § 43.

The semidaily amplitude of tide when the moon is near the equator is, approximately,

$$1{\cdot}1\ \mathrm{M}_2 + \text{correction for phase, Table 24.} \qquad (148$$

S_2 may be assumed to be

$$\mathrm{M}_2\,\frac{\mathrm{S}_2\ \text{at neighboring station}}{\mathrm{M}_2\ \text{at neighboring station}}\,. \tag{149}$$

From the observed tropic range we have

$$\mathrm{K}_1 + \mathrm{O}_1 = \left[\frac{\mathrm{Gc}}{2} - 2\,\frac{(\mathrm{M}_2{}^2 + \mathrm{S}_2{}^2)}{\mathrm{Gc}}\right] \times F_1 \tag{150}$$

[obtained by aid of formulæ (65), (73), and (122)].

Having in this manner found M_2, S_2, and $\mathrm{K}_1 + \mathrm{O}_1$ from, say, one month's observations, the Indian tide plane becomes approximately known.

FERREL'S EXPRESSIONS FOR INEQUALITIES IN THE TIDE.

46. If the lunitidal intervals and heights be classified according to an argument x whose period is that of some tidal inequality, the resulting interval and amplitude may, according to Fourier's theorem, be written

$$B_0 + M_i{}' \sin x + N_i{}' \cos x + M_{ii}{}' \sin 2\,x + N_{ii}{}' \cos 2\,x + \ \ldots \qquad (7)\ (151)$$

$$\tfrac{1}{2}\ \mathrm{Mn} + M_i \cos x + N_i \sin x + M_{ii} \cos 2\,x + N_{ii} \sin 2\,x + \ \ldots \qquad (4)\ (152)$$

where B_0 denotes the mean lunitidal interval, Mn the mean range of tide, and i the characteristic of the inequality. These expressions may be written in the form

$$B_0 + B_i \sin\,(x - \varepsilon_i) + B_{ii} \sin\,(2\,x - \varepsilon_{ii}) + \ \ldots \qquad (7)\ (153)$$

$$\tfrac{1}{2}\ \mathrm{Mn}\,[1 + R_i \cos\,(x - \alpha_i) + R_{ii} \cos\,(2\,x - \alpha_{ii}) + \ \ldots] \qquad (1)\ (154)$$

where

$$B_{i} = \pm \sqrt{M_i'^2 + N_i'^2} = \frac{M_i'}{\cos \varepsilon_i}, \; B_{ii} = \pm \sqrt{M_{ii}'^2 + N_{ii}'^2} = \frac{M_{ii}'}{\cos \varepsilon_{ii}}, \quad . \; . \; . \; . \; ,$$
$$\tan \varepsilon_i = -\frac{N_i'}{M_i'}, \qquad \tan \varepsilon_{ii} = -\frac{N_{ii}'}{M_{ii}'}, \qquad . \; . \; . \; ; \qquad (8)\ (155)$$

$$\tfrac{1}{2} \text{ Mn } R_i = \pm \sqrt{M_i^2 + N_i^2} = \frac{M_i}{\cos \alpha_i}, \; \tfrac{1}{2} \text{ Mn } R_{ii} = \pm \sqrt{M_{ii}^2 + N_{ii}^2} = \frac{M_{ii}}{\cos \alpha_i}, \quad . \; . \; . \; . \; ,$$
$$\tan \alpha_i = \frac{N_i}{M_i}, \qquad \tan \alpha_{ii} = \frac{N_{ii}}{M_{ii}} \qquad . \; . \; . \; . \qquad (5)\ (156)$$

In reducing tides, the observations are taken in groups. For this reason the coefficients B_i, R_i, and B_{ii}, R_{ii} as determined above should be multiplied by the factors (a little greater than unity,)

$$\frac{x_q - x_p}{2 \sin \frac{1}{2} (x_q - x_p)} \text{ and } \frac{x_q - x_p}{\sin (x_q - x_p)}$$

where x_p and x_q are the values of x at the two limits of the group of observations.

ε_i and α_i denote the epoch or lag of the principal term in the expression for any inequality. These divided by the "speed" of the inequality (i. e. the change in x per mean solar hour, say) give the "age of the tide" from times and heights, respectively. In all cases we shall assume that x corresponds to the hour of (local) transit of the moon, or to the moon's anomaly, longitude, etc., as the case may be, and not to the time of high water, low water, or a mean of these times; otherwise the value of x would correspond to a time a constant amount in advance or in retard of the moon's transit or other astronomical argument. Ferrel adopted this assumption in his "Discussion of tides in New York Harbor."* In earlier papers, notably in his "Discussion of tides in Boston Harbor"† and in his "Tidal researches,"‡ he has not corrected the ε's and α's for the lunitidal interval. To make this correction, add the value of the lunitidal interval multiplied by the speed of the inequality when the ε's and α's are affected with one subscript; use twice this speed when there are two subscripts, and so on.

47. *To express Ferrel's constants in the harmonic notation.*

For most inequalities this is not an easy matter, especially if much accuracy is required. The chief difficulties are, first, the inequality may be due to several components; second, the argument x may not vary uniformly with the time. The work given below will illustrate some simple cases.

Inequality due to a single component B.—From § 2 we have

$$\tan v = \frac{a}{b} \frac{\frac{B\, b^2}{A\, a^2} \sin (x - \theta)}{1 + \frac{B\, b^2}{A\, a^2} \cos (x - \theta)} \qquad (157)$$

where

$$\theta = B^\circ - A^\circ,$$
$$x = (b - a)\, t,$$
$$t = \frac{2\, n\, \pi}{a} - \frac{\alpha}{a} = \text{the time of a high water of } A \text{ reckoned}$$

from the conjunction of the fictitious moons of A and B, expressed in hours.

If
$$\tan v' = \frac{e \sin (x - \theta)}{1 + e \cos (x - \theta)}$$

where e is a constant less than unity, then

$$v' = e \sin (x - \theta) - \tfrac{1}{2} e^2 \sin 2 (x - \theta) + \tfrac{2}{3} e^3 \sin 3 (x - \theta) - \; . \; . \; .$$
$$= e \cos \theta \sin x - e \sin \theta \cos x - \frac{e^2}{2} \cos 2\, \theta \sin 2\, x + \frac{e^2}{2} \sin 2\, \theta \cos 2\, x + \; . \; . \; . \; . \qquad (158)$$

* United States Coast Survey Report for 1875. This discussion, because of its comparative clearness and consistency, will be referred to in preference to his other works. The left-hand numbers in the above expressions refer to corresponding expressions in the "Discussion of tides in New York Harbor."

† United States Coast Survey Report, 1868.

‡ Ibid., 1874, Appendix.

when v is small, and especially when a is nearly equal to b,

$$\frac{v}{a} = \frac{1}{b}\left[e \cos\theta \sin x - e \sin\theta \cos x - \frac{e^2}{2}\cos 2\theta \sin 2x + \frac{e^2}{2}\sin 2\theta \cos 2x + \ \ . \ \ . \ \ . \right] \quad (159)$$

$$= -\left[M_i' \sin x + N_i' \cos x + M_{ii}' \sin 2x + N_{ii}' \cos 2x + \ \ . \ \ . \ \ . \right] \quad (160)$$

where

$$M_i' = -\frac{e}{b}\cos\theta = -\frac{B\,b}{A\,a^2}\cos(B^\circ - A^\circ),$$

$$N_i' = \frac{e}{b}\sin\theta = \frac{B\,b}{A\,a^2}\sin(B^\circ - A^\circ),$$

$$M_{ii}' = \frac{e^2}{2\,b}\cos 2\theta = \frac{B^2\,b^3}{2\,A^2\,a^4}\cos 2\,(B^\circ - A^\circ),$$

$$N_{ii}' = -\frac{e^2}{2\,b}\sin 2\theta = -\frac{B^2\,b^3}{2\,A^2\,a^4}\sin 2\,(B^\circ - A^\circ),$$

. . . .

Assuming the decrease of interval to be

$$\frac{v}{a} = -\left[B_i \sin(x - \varepsilon_i) + B_{ii}\sin(2x - \varepsilon_{ii}) + \ \ . \ \ . \ \ . \right], \quad (161)$$

then

$$B_i = \pm\sqrt{M_i'^2 + N_i'^2} = \frac{M_i'}{\cos\varepsilon_i} = -57{\cdot}3\,\frac{B\,b}{A\,a^2}\text{ hours,}$$

$$B_{ii} = \pm\sqrt{M_{ii}'^2 + N_{ii}'^2} = \frac{M_{ii}'}{\cos\varepsilon_{ii}} = 57{\cdot}3\,\frac{B^2\,b^3}{2\,A^2\,a^4}\text{ hours,} \quad (162)$$

. . . .

$$\tan\varepsilon_i = -\frac{N_i'}{M_i'};\ \ \varepsilon_i = \theta = B^\circ - A^\circ,\text{ a small angle,}$$

$$\tan\varepsilon_{ii} = -\frac{N_{ii}'}{M_{ii}'};\ \ \varepsilon_{ii} = 2\,\theta = 2\,(B^\circ - A^\circ),\text{ a small angle,} \quad (163)$$

. . . .

For a case as simple as this, the auxiliary quantities M_i', N_i', . . . are introduced merely for the purpose of illustration.

From § 3 we have

$$y = A + B\cos(x - \theta) + B\,\frac{b}{a}\tan v \sin(x - \theta) - \tfrac{1}{2}A\tan^2 v + \ \ . \ \ . \ \ .$$

$$+\text{ residual effects of components not involved in the inequality.} \quad (164)$$

This may be written, without introducing M_i, N_i, . . . ,

$$y = \tfrac{1}{2}\,\text{Mn}\left[1 + R_i\cos(x - \alpha_i) + R_{ii}\cos(2x - \alpha_{ii}) + \ \ . \ \ . \ \ . \right] \quad (165)$$

where

$$R_i = \frac{2\,B}{\text{Mn}},\ \ R_{ii} = -\frac{B^2\,b^2}{2\,A\,a^2\,\text{Mn}},\ \ . \ . \ . ; \quad (166)$$

$$\alpha_i = \theta = B^\circ - A^\circ,\ \ \alpha_{ii} = 2\,\theta = 2\,(B^\circ - A^\circ),\ \ . \ \ . \ \ . \quad (167)$$

When an inequality is due to more than one component, the harmonic expressions for B_i, ε_i, R_i, α_i, B_{ii}, . . . become more complicated than those just given. Forms (151) and (152) are, however, sufficiently general to include such cases.

The phase inequality. If A, B denote the components M_2, S_2, and if there be no tide whose speed is a linear function of m_2 and s_2, we have

$$B_1 = -57{\cdot}3 \frac{S_2\, s_2}{M_2\, m_2^2} \text{hours}, \quad B_{11} = 57{\cdot}3 \frac{S_2^2\, s_2^3}{2\, M_2^2\, m_2^4} \text{ hours}, \quad \ldots ,$$

$$\epsilon_1 = S_2{}^\circ - M_2{}^\circ \quad , \qquad \epsilon_{11} = 2\,(S_2{}^\circ - M_2{}^\circ), \quad \ldots ; \tag{168}$$

$$R_1 = \frac{2\, S_2}{Mn} \quad , \qquad R_{11} = -\frac{S_2^2\, s_2^2}{2\, M_2\, m_2^2\, Mn}, \quad \ldots ,$$

$$\alpha_1 = S_2{}^\circ - M_2{}^\circ \qquad \alpha_{11} = 2\,(S_2{}^\circ - M_2{}^\circ), \quad \ldots \tag{169}$$

When μ_2 is taken into account, R_1 becomes

$$\frac{2\,(S_2 + \mu_2)}{Mn}$$

This should be divided by the inequality factor belonging to the phase reduction, § 29.

The parallax inequality. Whatever parallax inequality in time there may be is due chiefly to the transit selected in taking the intervals. The time coefficients B_2, B_{22}, . . . of this inequality ought to be small, and so will not be considered here. In regard to the constants R_2, α_2 it may be noted that an approximate value for y is, § 3,

$$y = A + B \cos(x - \theta) + C \cos(x - \varkappa) + \quad \ldots , \tag{170}$$

and so

$$M_2{}^* = B \cos\theta + C \cos\varkappa$$

$$N_2{}^* = B \sin\theta + C \sin\varkappa$$

. . . .

$$(\tfrac{1}{2}\, Mn)^2 R_2^2 = M_2^{2*} + N_2^{2*} = B^2 + C^2 + 2\, B\, C \cos(\theta \sim \varkappa)$$

$$\tan\alpha_2 = \frac{B \sin\theta + C \sin\varkappa}{B \cos\theta + C \cos\varkappa}. \tag{171}$$

When $\theta = \varkappa$ or $\theta = \varkappa \pm 180^\circ$, $R_2 = 2\,(B + C)\,/\,Mn$ or $R_2 = 2\,(B - C)/\,Mn$, respectively. Here N_2, L_2 replace B, C, and θ differs from $\varkappa$ by about 180°.

$$R_2 = \frac{2\,(N_2 - L_2)}{Mn}, \tag{172}$$

$$\theta = M_2{}^\circ - N_2{}^\circ,$$

$$\varkappa = L_2{}^\circ - M_2{}^\circ \pm 180^\circ.$$

α_2 is found from the above expression for $\tan\alpha_2$. The value of R_2 should be divided by an inequality factor belonging to a parallax reduction where the mean, not the true, perigee is used.

The declinational inequality. The time inequality whose period is a half tropical month, is due to the transit selected and to the sun's declination; it is usually small. The height constants R_3, α_3 have for their approximate values (see § 16)

$$R_3 = \frac{2\, K_2}{Mn} - \frac{K_1\, O_1}{2\, Mn^2}, \tag{173}$$

$$\alpha_3 = K_2{}^\circ - M_2{}^\circ.$$

Diurnal tides.—In what has preceded no distinction has been made between the two high waters of a day, or between the two low waters. Let it now be supposed that the differences

* This is an auxiliary quantity, not a harmonic component.

between the high water heights and also between the low water heights have been found for each day. The amplitude of the diurnal wave for any particular day is approximately equal to

$$\tfrac{1}{2}\left[\,(\text{HW inequality})^2 + (\text{LW inequality})^2\,\right]^{\frac{1}{2}} \qquad (174)$$

Having found this amplitude for each value of the moon's longitude (λ), the constants involved in its expression may be determined. Let us assume the amplitude to be (see Ferrel's New York tides, § 22, l. c.)

$$K_1 + O_1 \sin\,(2\,\lambda - \alpha_1') + O_{11} \sin\,(4\,\lambda - \alpha_{11}'). \qquad (175)$$

The number of hours by which the maximum amplitude of the diurnal wave follows an extreme declination of the moon is

$$\alpha_1' - 90^\circ, = K_1{}^\circ - O_1{}^\circ,$$

divided by $k_1 - o_1$ or 1·098. This is the age of the diurnal inequality.

CHAPTER IV.

TO REDUCE RESULTS TO THEIR MEAN VALUES.

48. All tidal components which depend upon the moon are assumed to have slightly variable amplitudes in order to better adapt them to each year of the luni-solar cycle. The reason for this is that if they had fixed amplitudes many additional components would be necessary to take into account this irregularity of the tide; moreover, these additional components would have periods differing so little from the periods of the components with which they are now combined that a long series of observations would be necessary for their independent determination. Besides accounting for the small components due to the regression of the moon's node, the variation in the amplitudes may be made to take into account still other components whose speeds are nearly equal to the speeds of those components with which they are combined.

The factors F and f, for reducing the amplitudes of the components obtained from a particular series to their mean values and vice versa, are given in Tables 10 and 13, which are based upon, or copied from, the tables given in Baird's Manual for Tidal Observations. In Table 14 are given F for the mean range of tide, for $K_1 + O_1$, and X, based upon the factors of Baird's manual and the mean values of the coefficients as given in Darwin's report (1883). Table 14, for reducing the mean range of tide, may be tested by means of the observed ranges at Boston for the years 1848 to 1865, United States Coast Survey Report, 1868, page 81, Table VIII, last column; also at New York for the years 1856 to 1874, United States Coast Survey Report, 1875, page 198, Table V, eighth column. The ratio $(K_1 + O_1)/M_2$ is about 0·2 at Boston and 0·3 at New York.

49. The following approximate relations will suffice for reducing reduction results to their mean values. In fact, a consideration of the nature of these relations, together with the nature of the F's of the components, of Mn, $K_1 + O_1$, and X, shows that the left-hand part of these relations lies very near to its mean value when the quantities composing the right-hand part have their respective mean values. To reduce a quantity represented by the left-hand part to its mean value, observe the corresponding increase, say, in the right-hand part and apply it to the given quantity, adding or multiplying as the case may be.

$$\mathrm{Sg} = \mathrm{Mn} + 2\,\mathrm{S}_2. \tag{176}$$

$$\mathrm{Np} = \mathrm{Mn} - 2\,\mathrm{S}_2. \tag{177}$$

$$\mathrm{Pn} = \mathrm{Mn} + 2\,(\mathrm{N}_2 - \mathrm{L}_2). \tag{178}$$

$$\mathrm{An} = \mathrm{Mn} - 2\,(\mathrm{N}_2 - \mathrm{L}_2). \tag{179}$$

$$\overline{\mathrm{HWQ}},\ \overline{\mathrm{LWQ}},\ \text{proportional to } \mathrm{K}_1 + \mathrm{O}_1. \tag{180}$$

$$\mathrm{HWQ}^* = 2\mathrm{P}_1 \left[(1 - 2\,\mathrm{X}) \cos \mathrm{X}\,180^\circ + \frac{2}{\pi} \sin \mathrm{X}\,180^\circ\right] \tag{181}$$

where

$$\cos \mathrm{X}\,180^\circ = \frac{\overline{\mathrm{HWQ}}}{2\,\mathrm{P}_1}.$$

Mean duration of rise or fall $\sim 6^h.21$, proportional to M_4. (182)

Lunitidal interval for the great or small tropic tides $\sim$ mean lunitidal interval, proportional to $K_1 + O_1$. (183)

Duration of rise or fall for the great or small tropic tides $\sim$ mean duration of rise or fall, proportional to $K_1 + O_1$. (184)

$$\mathrm{Gc} = \mathrm{Mn} + \tfrac{1}{2}\,\mathrm{HWQ} + \tfrac{1}{2}\,\mathrm{LWQ}. \tag{185}$$

$$\mathrm{Sc} = 2\,\mathrm{Mn} - \tfrac{1}{2}\,\mathrm{HWQ} - \tfrac{1}{2}\,\mathrm{LWQ}. \tag{186}$$

$$\mathrm{Gt} = \tfrac{3}{4}\,\mathrm{Gc} + \tfrac{1}{4}\,\mathrm{Mn}. \tag{187}$$

$$\mathrm{Sl} = \tfrac{7}{4}\,\mathrm{Mn} - \tfrac{3}{4}\,\mathrm{Gc}. \tag{188}$$

Depression of tropic LLW or mean LLW below mean LW, proportional to LWQ. (189)

---

* This supposes HWQ $<$ LWQ; when LWQ $<$ HWQ, the low water values should replace the high water.

If K_1 and O_1 are each known, the variation in $K_1 + O_1$ should be found by means of Table 10 instead of Table 14. Quantities proportional to $K_1 + O_1$ are obviously reduced to their mean values by the factor

$$\frac{K_1 + O_1}{K_1' + O_1'} \tag{190}$$

where the primes indicate values at a particular time as distinguished from mean values. For a short series P_1 combines with K_1, and the factor becomes

$$\frac{1{\cdot}02\,(K_1 + O_1)}{c_{11}\,K_1' + O_1'} \tag{191}$$

to be computed by aid of Tables 10 and 31, when K_1 and O_1 are known; otherwise it may be taken equal to $1{\cdot}02\,F_1$, Table 32.

CHAPTER V.

ON THE CLASSIFICATION OF TIDES.

50. For places where continuous records of the tides have been obtained for any considerable length of time, it is usually possible to make predictions based upon the analysis of such observations. But any tide table would soon become too bulky and expensive if predictions were to be given in detail for every place where forecasts are desired. It therefore becomes important to find a classification of tides, with the object of referring all stations having similar tides to one principal station where full predictions are given.

The following assumptions implied in the proposed classification hold true, as regards the principal components, for nearly every station where an analysis has been made:

First. The intervals of the $\genfrac{}{}{0pt}{}{\text{semidiurnal}}{\text{diurnal}}$ components are approximately equal to one another at any particular station.

Second. The amplitudes of the $\genfrac{}{}{0pt}{}{\text{lunar or solar semidiurnal}}{\text{diurnal}}$ components bear approximately the same ratios to one another at all stations.

Granting the truth of these assumptions, the form of the tidal wave depends almost wholly upon

(a) The form of the wave composed of the tropic diurnal wave and the principal lunar semidiurnal component; that is, upon $(K_1 + O_1)\ /\ M_2$ and $M_2^\circ - K_1^\circ - O_1^\circ$.

(b) The ratio of the amplitudes of the principal solar and lunar semidiurnal component; that is, upon S_2/M_2.

(c) The form of the lunar semidiurnal wave; that is, upon M_4/M_2, $2\ M_2^\circ - M_4^\circ$, M_6/M_2, $3\ M_2^\circ - M_6^\circ$, where, as is nearly always the case, M_4 and M_6 are the principal harmonics of M_2.

These quantities being equal at two stations, the tides at the one may be inferred from those at the other. Not only are the times and heights of high and low water comparable, but also the height of the sea or tide at any intermediate hour. When only high and low waters are to be compared, the quantities in (c) need not be alike at both stations unless $(K + O_1)/M_2$ is a comparatively large quantity; in which case the form of the lunar semidiurnal wave should be approximately the same at both stations.

51. If nonharmonic constants are used the quantities upon which the form of the tide wave depends are

(a′) The ratios of the high and low water inequalities in height to the mean range of tide, together with the sequence of the four tides of a day; that is, upon HWQ/Mn, LWQ/Mn, and whether the order of the tides is from higher high to lower low, or from lower low to higher high, the moon being far from the equator.

(b′) The ratio of the semimensual height inequality to the mean range of tide; that is, (Sg — Np)/2 Mn.

(c′) The form of the lunar semidiurnal wave. The quantities here selected for determining this are the duration of fall and the heights of the sea referred to high and low waters expressed in terms of the mean range; that is, LWI — HWI, and heights of the sea or tide at various hours before and after high and low water expressed in terms of Mn, when the range has its mean value.

It will be observed that (a′), (b′), and (c′) are nearly equivalent to (a), (b), and (c), respectively, in fixing the form of the tide wave. Whenever the quantities in (a′), (b′), and (c′) are equal at two stations the tides are comparable throughout. If only high and low waters are to be com-

pared, the quantities in (c′) need not to be alike at both stations unless HWQ/Mn or LWQ/Mn is a comparatively large quantity; in which case the form of the lunar semidiurnal wave should be approximately the same at both stations.

At stations where the tide is usually diurnal the height inequalities are not easily determined. The form of tide may be determined from the duration of the (great) tropic range, together with the rise of tropic HHW and the fall of tropic LLW reckoned from mean sea level. The amount of rise compared with the amount of fall shows whether HWQ $\gtrless$ LWQ.

52. The *sequence of the tides* is determined by the following rule:

If the HW phase, i. e. $\frac{1}{2}(\mathrm{M}_2° - \mathrm{K}_1° - \mathrm{O}_1°)$, lie in the first or third quadrant, higher high precedes lower low; if in the second or fourth quadrant, higher high follows lower low. If twice the HW phase is used, it will lie in the first or second semicircle according as higher high precedes or follows lower low.

Having found that the tides at two stations are comparable, the next step is to give rules for referring the one to the other.

The times.—The quantities which must be applied to the times of the tides at the first station to obtain the times of the tides at the second station may be expressed thus:

$$\begin{aligned}\text{Difference for HW} &= (\mathrm{HW})_{,,} - (\mathrm{HW})_{,} = (\mathrm{HWI})_{,,} - (\mathrm{HWI})_{,} + S_{,} - S_{,,} \\ &\quad + 1\tfrac{1}{30}(L_{,,} - L_{,}) + n\,(12^{\mathrm{h}}\ 25^{\mathrm{m}}), \\ \text{Difference for LW} &= (\mathrm{LW})_{,,} - (\mathrm{LW})_{,} = (\mathrm{LWI})_{,,} - (\mathrm{LWI})_{,} + S_{,} - S_{,,} \\ &\quad + 1\tfrac{1}{30}(L_{,,} - L_{,}) + n\,(12^{\mathrm{h}}\ 25^{\mathrm{m}});\end{aligned} \tag{192}$$

where $L_{,}$, $L_{,,}$ are the longitudes of the stations and $S_{,}$, $S_{,,}$ the longitudes of the time meridians, all expressed in time and reckoned westward from Greenwich. $n\,(12^{\mathrm{h}}\ 25^{\mathrm{m}})$ is such a multiple of a half lunar day as will cause the diurnal inequality at the first station to properly reappear at the second. As the form of the tide wave changes slowly from one day to another, n can be so taken as to allow for the change in date which occurs in crossing the Pacific Ocean, at the same time keeping the tidal differences between the limits $-12^{\mathrm{h}}\ 25^{\mathrm{m}}$ and $+12^{\mathrm{h}}\ 25^{\mathrm{m}}$.

When the tidal waves at the two stations are of the same form, the time "difference for HW" is equal to the "difference for LW;" in fact, all like phases of the tide at the two stations are directly comparable by means of this time difference.

The transit.—Rules for computing the high and low water establishments and for determining the sequence of the tides from harmonic constants alone have already been given. The question which now arises is, To which transit must a highwater interval (whose value is nearer $\mathrm{M}_2°/28{\cdot}98$ than to $\mathrm{M}_2°/28{\cdot}98 \pm 12^{\mathrm{h}}\ 25^{\mathrm{m}}$) be applied in order to obtain a higher high water when the moon's declination is $\begin{smallmatrix}\text{north?}\\ \text{south?}\end{smallmatrix}$ The answer is, $\begin{smallmatrix}\text{an upper}\\ \text{a lower}\end{smallmatrix}$ transit must be used if the HW phase lies in the first or fourth quadrant, and $\begin{smallmatrix}\text{a lower}\\ \text{an upper}\end{smallmatrix}$ if in the second or third quadrant. In order to obtain a lower low water when the moon's declination is $\begin{smallmatrix}\text{north, an upper}\\ \text{south, a lower}\end{smallmatrix}$ transit must be used if the HW phase lies in the first or second quadrant, and $\begin{smallmatrix}\text{a lower}\\ \text{an upper}\end{smallmatrix}$ if in the third or fourth quadrant (supposing the low water interval to be nearer $\mathrm{M}_2°/28{\cdot}98 + 6^{\mathrm{h}}\ 13^{\mathrm{m}}$ than to $\mathrm{M}_2°/28{\cdot}98 + 6^{\mathrm{h}}\ 13^{\mathrm{m}} \pm 12^{\mathrm{h}}\ 25^{\mathrm{m}}$).

These statements enable one to decide whether n in the above equations is odd or even.

If the intervals at the two stations refer to the same transit, then $n = o$ or some even integer, as -2 or $+2$; if to opposite transits, $n =$ some odd integer, as -1 or $+1$.

53. The following rules are here collected together for convenience of reference:

Sequence of tides.

$$0° < \mathrm{M}_2° - \mathrm{K}_1° - \mathrm{O}_1° < 180°, \text{ HHW precedes LLW};$$
$$180° < \mathrm{M}_2° - \mathrm{K}_1° - \mathrm{O}°_1 < 360°, \text{ LLW precedes HHW}.$$

Greater height inequality.

$$-90° < \mathrm{M}_2° - \mathrm{K}_1° - \mathrm{O}_1° < 90°, \text{ HWQ} > \text{LWQ};$$
$$90° < \mathrm{M}_2° - \mathrm{K}_1° - \mathrm{O}_1° < 270°, \text{ LWQ} > \text{HWQ}.$$

The transit.

		For HHW use	For LHW use
HWI taken approximately equal to $M_2°/29$ hours.	$-90° < \frac{1}{2}(M_2° - K_1° - O_1°) < 90°$	upper north, or lower south;	lower north, or upper south;
	$90° < \frac{1}{2}(M_2° - K_1° - O_1°) < 270°$	lower north, or upper south.	upper north, or lower south.
		For LLW use	**or HLW use**
LWI taken approximately equal to HWI $+ 6^h 13^m$.	$0° < \frac{1}{2}(M_2° - K_1° - O_1°) < 180°$	upper north, or lower south;	lower north, or upper south;
	$180° < \frac{1}{2}(M_2° - K_1° - O_1°) < 360°$	lower north, or upper south.	upper north, or lower south.

Intervals applied to $\frac{\text{an upper north or a lower south}}{\text{a lower north or an upper south}}$ transit, may be marked $\frac{a}{b}$. At stations where $\frac{\text{HWQ} < 2\,\text{P}_{1}}{\text{LWQ} < 2\,\text{P}_{1}}$, the order of $\frac{\text{high}}{\text{low}}$ waters may be reversed even when the moon is farthest from the equator. See § 21.

The heights.—If the planes of reference at the two stations have the same definition with respect to the tide, corresponding heights at the two stations differ only by a constant factor. This factor is usually taken as the ratio of the mean ranges. Where evanescent tides frequently occur, the ratio of the (great) tropic ranges or of the great diurnal ranges, should be used instead. If the planes of reference have different definitions at the two stations, the heights at one of the stations must be properly reduced before applying the factor.

The height differences are

$$
\begin{aligned}
&\text{Difference for HW} = C\frac{(\text{Mn})_{\prime\prime}}{(\text{Mn})_{\prime}} - D + (\text{Mn})_{\prime\prime} - (\text{Mn})_{\prime},\\
&\text{Difference for LW} = C\frac{(\text{Mn})_{\prime\prime}}{(\text{Mn})_{\prime}} - D,
\end{aligned}
\tag{193}
$$

where D is the depression of the plane of reference below meau low water at the principal station and C the depression below mean low water at the principal station of a plane there having the same definition as the plane of reference at the subordinate station.

When the two stations are far apart, it will generally be necessary to allow for the long period components if *absolute* heights of the tide are required.

54. *Tides comparable by inversion.*

When the tide at a subordinate station is similar to the one at the principal station inverted, the $\frac{\text{HW}}{\text{LW}}$ at the subordinate station is proportional to the $\frac{\text{LW}}{\text{HW}}$ at the principal station, the plane of reference being mean sea level. To ascertain if such a comparison can be made, find the quantities in (a), (b), and (c). The procedure is as in the case of tides *directly* comparable, except that $M_2° - K_1° - O_1°$ and $2\,M_2° - M_4°$ should no longer have the same values at the two stations, but each should be 180° greater or less at the one station than at the other. Using nonharmonic criteria, LWQ at the one station should be proportional to HWQ at the other. The duration of fall at the one should equal the duration of rise at the other.

55. *To infer harmonic constants through the comparison of nonharmonic constants, and conversely.*

Supposing the tides to be similar at two stations, we have for any short period component C

$$(C°)_{\prime\prime} = (C°)_{\prime} + c\,\frac{(\text{HWI})_{\prime\prime} + (\text{LWI})_{\prime\prime} - (\text{HWI})_{\prime} - (\text{LWI})_{\prime}}{2}, \tag{194}$$

$$(C)_{\prime\prime} = (C)_{\prime} \times \frac{(\text{Mn})_{\prime\prime}}{(\text{Mn})_{\prime}}. \tag{195}$$

If the tides are not exactly similar, i. e. if the quantities (a′), (b′), and (c′), § 51, differ slightly at the two stations, then, instead of equation (195), we have

$$(C_2)_{//} = (C_2)_{/} \times \frac{(\mathrm{Mn})_{//}}{(\mathrm{Mn})_{/}} \tag{196}$$

where C_2 denotes any lunar semidiurnal component;

$$(C_2)_{//} = (C_2)_{/} \times \frac{(\mathrm{Sg})_{//} - (\mathrm{Np})_{//}}{(\mathrm{Sg})_{/} - (\mathrm{Np})_{/}} \tag{197}$$

where C_2 denotes any solar semidiurnal component, including lunisolar K_2;

$$(C_1)_{//} = (C_1)_{/} \times \frac{(\mathrm{HWQ})_{//} + (\mathrm{LWQ})_{//}}{(\mathrm{HWQ})_{/} + (\mathrm{LWQ})_{/}} \tag{198}$$

where C_1 denotes any diurnal component;

$$(C_4)_{//} = (C_4)_{/} \times \frac{(\mathrm{LWI})_{//} - (\mathrm{HWI})_{//} \mp 6^{\mathrm{h}}\,13^{\mathrm{m}}}{(\mathrm{LWI})_{/} - (\mathrm{HWI})_{/} \mp 6^{\mathrm{h}}\,13^{\mathrm{m}}} \tag{199}$$

where C_4 denotes any quarter diurnal component; and where the upper or lower sign is used according as LWI is taken greater or less than HWI.

For inferring nonharmonic constants through the harmonic, we have

$$(\mathrm{HWI})_{//} = (\mathrm{HWI})_{/} + \frac{(\mathrm{M_2}^{\circ})_{//} - (\mathrm{M_2}^{\circ})_{/}}{\mathrm{m}_2}, \tag{200}$$

$$(\mathrm{LWI})_{//} = (\mathrm{LWI})_{/} + \frac{(\mathrm{M_2}^{\circ})_{//} - (\mathrm{M_2}^{\circ})_{/}}{\mathrm{m}_2}, \tag{201}$$

$$(\mathrm{Mn})_{//} = (\mathrm{Mn})_{/} \times \frac{(\mathrm{M}_2)_{//}}{(\mathrm{M}_2)_{/}}. \tag{202}$$

If the tides are exactly similar, Mn in the equation last written may be replaced by any other quantity measurable in feet, as HWQ, Sg, etc. If, on the other hand, (a), (b), and (c), § 50, differ somewhat at the two stations, then the following equations should be used:

$$(\mathrm{HWQ})_{//} = (\mathrm{HWQ})_{/} \times \frac{(\mathrm{K}_1 + \mathrm{O}_1)_{//}}{(\mathrm{K}_1 + \mathrm{O}_1)_{/}}, \tag{203}$$

$$(\mathrm{LWQ})_{//} = (\mathrm{LWQ})_{/} \times \frac{(\mathrm{K}_1 + \mathrm{O}_1)_{//}}{(\mathrm{K}_1 + \mathrm{O}_1)_{/}}, \tag{204}$$

$$(\mathrm{Sg})_{//} = (\mathrm{Sg})_{/} \times \frac{(\mathrm{M}_2 + \mathrm{S}_2)_{//}}{(\mathrm{M}_2 + \mathrm{S})_{/}}, \tag{205}$$

$$(\mathrm{Np})_{//} = (\mathrm{Np})_{/} \times \frac{(\mathrm{M}_2 - \mathrm{S}_2)_{//}}{(\mathrm{M}_2 - \mathrm{S}_2)_{/}}. \tag{206}$$

The deviation of a tropic interval from the mean is obtained by multiplying the value of this deviation at the first station by $\frac{(\mathrm{K}_1 + \mathrm{O}_1)_{//}}{(\mathrm{K}_1 + \mathrm{O}_1)_{/}}$. In this connection see § 49.

Long period components are taken from a neighboring station and without alteration.

If the nonharmonic quantities at the second station are not properly corrected for the effect of tidal (not accidental) inequalities, they should be used along with the same quantities obtained simultaneously and in like manner, at the first station, using either observations or predictions. If the nonharmonic quantities are affected with accidental inequalities, it is important that the two stations be situated near each other, and that simultaneous observations be used in making inferences.

56. *Cotidal lines.*

A cotidal line is an assemblage of points on the earth's surface where tides occur at the same absolute time.

The number of each such line is usually taken as the lunar time (i. e. the lunar hour after upper or lower transit) at Greenwich when HW occurs at stations along the cotidal line. If solar hours are used, reckoned, of course, from the time of the moon's transits, each period of cotidal lines will consist of 12·42 hour-lines instead of 12.

The cotidal lunar hour of a place whose west longitude in time equals L is

$$0{\cdot}966\ \mathrm{HWI} + L,$$

or more exactly,

$$0{\cdot}483\ (\mathrm{HWI} + \mathrm{LWI} \mp 6{\cdot}210) + L; \tag{207}$$

while the cotidal solar hour is

$$\mathrm{HWI} + 1{\cdot}035\ L,$$

or more exactly,

$$\frac{\mathrm{HWI} + \mathrm{LWI} \mp 6{\cdot}^{\mathrm{h}}210}{2} + 1{\cdot}035\ L. \tag{208}$$

The $\begin{smallmatrix}\text{upper}\\ \text{lower}\end{smallmatrix}$ sign is used when the low water interval is taken $\begin{smallmatrix}\text{greater}\\ \text{less}\end{smallmatrix}$ than the high water.

The cotidal lines relating to the semidiurnal portion of the tide are quite distinct from those relating to the diurnal.

The tropic high and low water intervals of the diurnal wave are obtained thus (see § 16):

$$\mathrm{HWI}\ (a) = \frac{\mathrm{K_1}^\circ + \mathrm{O_1}^\circ}{29{\cdot}2} \tag{209}$$

$$\mathrm{LWI}\ (a) = \mathrm{HWI}\ (a) \pm 12^{\mathrm{h}}\ 20^{\mathrm{m}}. \tag{210}$$

To change the transits add or subtract $12^{\mathrm{h}}\ 25^{\mathrm{m}}$.

The cotidal lunar and solar hours for the diurnal wave become

$$0{\cdot}966\ \mathrm{HWI}\ (a) + L \tag{211}$$

and

$$\mathrm{HWI}\ (a) + 1{\cdot}035\ L, \tag{212}$$

respectively.

CHAPTER VI.

PREDICTION OF TIDES.

TIDE–PREDICTING MACHINES.

57. *British tide-predicting machines.**

The object of these machines is to continuously sum the series

$$H_o + A \cos(at + \alpha) + B \cos(bt + \beta) + \quad . \quad . \quad . \quad , \tag{213}$$

or to trace the curve

$$y = H_o + A \cos(at + \alpha) + B \cos(bt + \beta) + \quad . \quad . \quad . \quad , \tag{214}$$

thus giving the height of the sea or tide at any particular time.

Among the earlier mechanical devices for summing a series of cosine or sine terms, where the angles vary uniformly with the time t, or with the angle θ, may be mentioned the following: †

In the British Association for the Advancement of Science Report for 1845, Cambridge Meeting, Rev. F. Bashforth mentions a machine for describing the curve

$$\rho = A \cos(a\,\theta + \alpha) + B \cos(b\,\theta + \beta) + \quad . \quad . \quad . \tag{215}$$

where $A, B, C, \quad . \quad . \quad . \quad a, b, c, \quad . \quad . \quad .$ are real quantities, integral or fractional. He applies it to the finding of *real* roots of the algebraic equation

$$p_o x^n + p_1 x^{n-1} + \quad . \quad . \quad . \quad + p_n = 0 \tag{216}$$

where n is a positive integer. By writing $x = \cos\theta$ this equation becomes of the form

$$q_o \cos n\,\theta + q_1 \cos \overline{n-1}\,\theta + \quad . \quad . \quad . \quad + q_n = 0. \tag{217}$$

The values of $\cos\theta$ at the intersections of

$$\rho = k + q_o \cos n\,\theta + q_1 \cos \overline{n-1}\,\theta + \quad . \quad . \quad + q_n$$

and

$$\rho = k$$

are, obviously, the roots required.

In the Proceedings of the Royal Society of London, Vol. 18 (1869), pages 72, 73, Mr. W. H. L. Russell shows how the curve

$$y = A \cos(at + \alpha) + B \cos(bt + \beta) + \quad . \quad . \quad . \tag{218}$$

may be described mechanically. From his diagram it will be noticed that the mechanism is "substantially the same as that of the tide predictor" (No. 2 or 3).

In the Minutes of Proceedings of the Institution of Civil Engineers (London), Vol. 65, page 16, a description is given of a wooden model designed by Sir William Thomson and constructed for him in 1872–73. This model made provision for eight components, and its plan was very like tide predictor No. 1.

---

*For an account of these machines see the Minutes of Proceedings of the Institution of Civil Engineers (London), Vol. 65 (1881), pages 15–72.

For "predictor No. 1" see Thomson and Tait's Natural Philosophy.

For the Indian predictor see also Proceedings of the Royal Society of London, Vol. 29 (1879), pages 198–201, and especially The Engineer (London) of December 19, 1879, where drawings of the machine are given.

†See Popular Lectures and Addresses by Sir William Thomson, Vol. III, page 185.

Tide predictor No. 1.—This predictor, designed by Thomson, was constructed in about 1876. It contains ten components and is described in Thomson and Tait's Natural Philosophy, Part I. It differs from Nos. 2 and 3 in the number of components represented and in having cranks carrying pulleys instead of pins working in slots. This machine is now kept at the South Kensington Museum.

Tide predictor No. 2.—This machine was designed by Mr. E. Roberts and is now worked under his direction in preparing the "Tide tables for Indian ports." The number of components is twenty.

Tide predictor No. 3.—This predictor, designed by Thomson, differs from No. 2 in the number of components, which is fifteen or sixteen instead of twenty, and in the direct character of the gearing. It was constructed in about 1881.

General description.—Upon one or more shafts, driven by hand or by clockwork, are fixed a number of wheels which mesh into other wheels, causing the latter (or wheels moved by them) to revolve with angular velocities having given ratios to the angular velocities of the shafts. These ratios are taken, as nearly as possible, proportional to the speeds of particular tidal components. Rigidly connected to these wheels are cranks carrying pulleys, or pins working in slots, imparting to rods carrying pulleys rectilineal harmonic motions. At one end of the predictor a chain or flexible wire is made fast; thence it is laid alternately over and under the pulleys. To the free end of the wire is attached a marking point, which when moved transversely to the line of motion of the paper roll traces the tidal curve. This machine evidently sums the series (213) or traces the curve (214).

58. *Maxima and minima tide-predicting machine.*

This machine, designed by the late Prof. William Ferrel, of the United States Coast and Geodetic Survey, was constructed in the years 1881–82. A description is given in the Report for 1883, pages 253–272. Most of the predictions published by the Survey since 1885 have been made upon it. The machine provides for nineteen components. The amplitude of the principal lunar component is introduced into the predictor by shortening one of the summation chains about to be mentioned. Its argument is indicated by a hand on the face of the machine making two revolutions for each lunar day. Generally speaking, the speed ratios in the machine are not those of the components themselves, but are the differences between such ratios and the speed ratio of the principal lunar component. The smaller short-period components are each provided with two cranks fixed at right angles to each other, one for cosine terms and one for sine terms. Each set of cranks carries pulleys, and the resultant effects are obtained by means of two summation chains. One chain is made to impart a vertical motion to a slotted horizontal strip of steel, while the other imparts a horizontal motion to a slotted vertical strip. These slots intersect at the center of the machine when all short-period components are set at zero amplitude. In general, two settings of the machine are required, one for times and one for heights. When set for times, the direction of the intersection of the slots from the center of the machine shows (by aid of an oscillating needle) how the lunar hand must be directed * at the time of a high or low water. When set for heights the distance of the intersection from the center † is, when multiplied by the cosine of the angle between the lunar hand and the oscillating needle,‡ the elevation or depression of the sea or tide from mean sea level.

The machine gives the times of high and low water directly; no tidal curve is traced, nor is any computation required. Approximate values of the heights are also given directly, but the determination of their exact values or of the height of the sea at any intermediate time requires a little additional labor, viz., the setting and reading of an extensible cosine scale.

59. *A proposed tide predictor.*

The machine briefly described below will carry on simultaneously four operations, viz.:

1. The tracing of the tidal curve.
2. The marking upon the axis of the curve the times of the maxima and minima.

---

* This angle reckoned counterclockwise from the vertical is the u of § 60.

† This distance is the R of § 60.

‡ This angle is $at + \alpha + z$ of § 60; at times of high or low water it becomes $z - u$. z is always the angle made by the oscillating needle with the vertical reckoned counterclockwise, the machine being set for heights; $at + \alpha$ is the argument or angle of the lunar hand reckoned clockwise, as on the face of the machine.

3. The exhibiting upon the face of the machine the times of high and low water.

4. The exhibiting upon the face of the machine the heights of high and low water, and the height of the sea at any given time.

(1) In regard to the accomplishment of the first of these objects nothing further need be said, for a mechanism similar to that used in the British machines seems quite satisfactory.

(2) Now let there be a set of cranks at right angles to the set used in producing the tide curve; for distinction the former may be called the time cranks and the latter height cranks. Each set has its resultant effect shown by the movement of the free end (or of a mark a constant distance therefrom) of a wire or slender chain. In each case the result might be shown in the form of a continuous curve. This, however, is not ordinarily wanted in the case of the time cranks. A projection upon the wire uniting their effects comes into contact with a pencil placed at mean sea level, causing a short mark to be made every time high or low water occurs.

(3) A mark on the wire at a fixed distance from the projection just referred to passes before an opening in the face of the machine, and is so adjusted that when it crosses a given line or mark the time shown on the face is the time of high or low water according to the direction of the wire's movement.

(4) The heights may be conveniently shown by means of a pointer moving along a circular row of figures outside the time dial.

A clear mode of indicating the time would be to have the day of the month and the hour of the day show through two small openings, while a single hand points out the minutes.

In order to warn the operator that a high or low water is close at hand, a bell may be made to ring just before each tide.*

Imaginary roots of equations, etc.†—A purely mathematical use for a machine of this kind is the finding of imaginary roots of equations.

Let

$$Z = A\,z^a + B\,z^b + \quad . \quad . \quad . \quad = 0 \tag{219}$$

where the coefficients are general but the exponents are real quantities. Now, as z describes a circle whose radius is k, Z will for each value of arg z or θ have a certain affix

$$\begin{aligned} &\text{mod}\,A\,k^a\,[\cos(a\,\theta + \alpha) + i\sin(a\,\theta + \alpha)] \\ + \quad &\text{mod}\,B\,k^b\,[\cos(b\,\theta + \beta) + i\sin(b\,\theta + \beta)] \\ + \quad &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned} \tag{220}$$

The locus of Z will be mechanically described by a point whose ordinate and abscissa represent the resultant effects of the height and time cranks each set to the same amplitudes.

Such values of k and θ as will make both sums zero give roots of the equation.

60. *Equations relating to tide-predicting machines.*

The height of the sea or tide at any given time t may be expressed in various ways:

$$y = H_0 + A\cos(at + \alpha) + B\cos(bt + \beta) + \quad . \quad . \quad . \tag{221}$$

$$\begin{aligned} &= H_0 + \bar{A}\cos at + \bar{B}\cos bt + \quad . \quad . \quad . \\ &\qquad + \bar{\bar{A}}\sin at + \bar{\bar{B}}\sin bt + \quad . \quad . \quad . \end{aligned} \tag{222}$$

where

$$\bar{A} = A\cos\alpha\,,\ \bar{\bar{A}} = -A\sin\alpha\,;$$

$$\therefore A = (\bar{A}^2 + \bar{\bar{A}}^2)^{\frac{1}{2}},\ \tan\alpha = -\frac{\bar{\bar{A}}}{\bar{A}}. \tag{223}$$

$$y = H_0 + M\cos(at + \alpha) - N\sin(at + \alpha) \tag{224}$$

where

$$M = A + B\cos(\overline{b - a}t + \beta - \alpha) + C\cos(\overline{c - a}t + \gamma - \alpha) + \quad . \quad . \quad . \quad , \tag{225}$$

$$N = 0 + B\sin(\overline{b - a}t + \beta - \alpha) + C\sin(\overline{c - a}t + \gamma - \alpha) + \quad . \quad . \quad . \quad ; \tag{226}$$

$$y = H_0 + R\cos(at + \alpha + z) \tag{227}$$

where

$$R = (M^2 + N^2)^{\frac{1}{2}},\ \tan z = \frac{N}{M}. \tag{228}$$

z is evidently the angle of a right-angled triangle, whose sides are M, N, lying opposite N.

[* This survey has undertaken the designing and construction of a machine which will carry on the above operations. When completed, a full description of the machine will be published.—July, 1895.]

† The notation here employed is independent of the notation used elsewhere in this paper.

The times of high and low water are roots of the derived equation

$$-\frac{dy}{at} = A\,a \sin(at + \alpha) + B\,b \sin(bt + \beta) + \quad . \quad . \quad . \quad = 0 \tag{229}$$

$$= \bar{A}\,a \sin at + \bar{B}\,b \sin bt + \quad . \quad . \quad .$$
$$- (\bar{\bar{A}}\,a \cos at + \bar{\bar{B}}\,b \cos bt + \quad . \quad . \quad .) \tag{230}$$

where

$$\bar{A} = A \cos \alpha,\ \bar{\bar{A}} = -A \sin \alpha;$$
$$\therefore A = (\bar{A}^2 + \bar{\bar{A}})^{\frac{1}{2}},\ \tan \alpha = -\frac{\bar{\bar{A}}}{\bar{A}}. \tag{231}$$

$$-\frac{dy}{at} = a\,M' \sin(at + \alpha) + a\,N' \cos at + \alpha) = 0 \tag{232}$$

where

$$M' = A + \frac{B\,b}{a} \cos(\overline{b - a}t + \beta - \alpha) + \frac{C\,c}{a} \cos(\overline{c - a}t + \gamma - \alpha) + \quad . \quad . \quad . \quad . \, , \tag{233}$$

$$N' = 0 + \frac{B\,b}{a} \sin(\overline{b - a}t + \beta - \alpha) + \frac{C\,c}{a} \sin(\overline{c - a}t + \gamma - \alpha) + \quad . \quad . \quad . \quad . \, ; \tag{234}$$

$$-\frac{dy}{at} = R' \sin(at + \alpha + u) = 0 \tag{235}$$

where

$$R' = (M'^2 + N'^2)^{\frac{1}{2}},\ \tan u = \frac{N'}{M'}. \tag{236}$$

The angle u shows how much (in A degrees) the high or low water is accelerated because of components other than A. The times of the tide are therefore

$$t' = \frac{n\,\pi - \alpha - u}{a} \tag{237}$$

where n is any integer, even for a high and odd for a low water. v and w of § 1 are values of u for these two cases.

A GRAPHIC METHOD OF PREDICTING TIDES FROM HARMONIC CONSTANTS.

61. The prediction of tides, as already stated, implies the combining of a series of waves into one for the purpose of ascertaining the height of the tide at a given time or of determining the times and heights of high and low water. The computation of these quantities, while not difficult, is extremely laborious; on this account mechanical tide predictors have been constructed. As these instruments are too expensive for general use, a method is here proposed whereby predictions may be made, without computation, from the constants used in setting a machine—in other words, from the harmonic tidal constants, the initial equilibrium arguments, and certain factors (f of Table 10).

The apparatus consists, first, of a smooth board (about 3 by 4 feet), one section of which is divided into twenty-four equal spaces, representing the hours of the day; second, of a set of pairs of movable curves, drawn once for all for a given station, each pair provided with a "calendar" good for all stations and all time; and third, of two stationary cleats for holding the curves in place, and a number of movable cleats for indices—one for each pair of curves.

62. The equation of any component with its harmonics may be written

$$\begin{aligned} y = {} & A_1 \cos(a_1 t + \arg_0 A_1 - A_1{}^{\circ}) \\ & + A_2 \cos(2\,a_1 t + \arg_0 A_2 - A_2{}^{\circ}) \\ & + A_3 \cos(3\,a_1 t + \arg_0 A_3 - A_3{}^{\circ}) \\ & + \quad . \quad . \quad . \end{aligned} \tag{238}$$

where A_1 is the oscillation of longest period, or the fundamental, not necessarily a diurnal component. Of course, a curve representing this compound wave can always be replaced by as many simple cosine curves as there are terms in y. In fact, the only instances where it is generally

advantageous to use other than simple curves are M_2 and S_1, with their respective harmonics. Strictly speaking, the amplitude of the harmonics of M_2 vary faster on account of the change in the inclination of the lunar orbit to the earth's equator than does the amplitude of M_2; but as the amount of this change is small in small quantities, no great inaccuracy will be introduced by making the whole M wave vary as M_2 varies.

It is customary to put

$$\arg_0 A_2 = 2 \arg_0 A_1$$
$$\arg_0 A_3 = 3 \arg_0 A_1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad (239)$$

The above equation assumes that one has the value of the initial arguments at the time $t = 0$, local time, midnight, say. In practice it is convenient to take the origin of time at the instant of midnight on a certain standard time meridian, S, and to have the initial arguments made out for the meridian of Greenwich; then one might alter the arguments in such a way as to adapt them to the longitude of the locality, L, and to the standard time midnight. But if, instead of altering the Greenwich arguments, one alter the epochs by the same amounts, with opposite signs, the same results will be attained and with the great advantage of being done once for all. The epoch of any component A_p thus modified is

$$\mathfrak{A}_p{}^\circ = A_p{}^\circ + (15\,p - a_p)\,L + a_p\,(L - S), = A_p{}^\circ + 15\,pL - apS \qquad (240)$$

where p is such an integer that $15\,p$ is nearly equal to a_1. When A_1 denotes a diurnal component, then $p = 1, 2, \quad . \quad . \quad .$ according as A_p is diurnal, semidiurnal, etc.

63. *Construction of a height curve.*—For convenience in drawing the curve, suppose $\arg_0 A_1$ to be zero; then its equation with modified epochs becomes

$$y = A_1 \cos(a_1 t - \mathfrak{A}_1{}^\circ) + A_2 \cos(2\,a_1 t - \mathfrak{A}_2{}^\circ) + A_3 \cos(3\,a_1 t - \mathfrak{A}_3{}^\circ) + \quad . \quad . \quad . \quad . \qquad (241)$$

The position of the maxima of A_1 are given by the equation

$$t = \frac{\mathfrak{A}_1{}^\circ}{a_1} + \frac{2\,n\,\pi}{a_1} \qquad (242)$$

where n is zero or a positive or negative integer. At this time the phases of the harmonics are

$$2\,\mathfrak{A}_1{}^\circ - \mathfrak{A}_2{}^\circ,\ 3\,\mathfrak{A}_1{}^\circ - \mathfrak{A}_3{}^\circ, \quad . \quad . \quad . \quad , \qquad (243)$$

while the positions of their respective maxima are given by the equations

$$t = \frac{\mathfrak{A}_2{}^\circ}{2\,a_1} + \frac{2\,n\,\pi}{2\,a_1},\quad t = \frac{\mathfrak{A}_3{}^\circ}{3\,a_1} + \frac{2\,n\,\pi}{3\,a_1}, \quad . \quad . \quad . \quad . \quad . \qquad (244)$$

The beginning of the curve should be $\mathfrak{A}_1{}^\circ$ degrees or $\mathfrak{A}_1{}^\circ/a_1$ hours before the first maximum. It should extend over a trifle more than two or three days according as A_1 is a semidiurnal or a diurnal component. From the beginning and along the axis, generally lay off two periods, leaving the first blank and dividing the second into degrees from zero to 360. The number equivalent to $\arg_0 A_1$ when brought over the stationary zero hour line fixes the position of the curve for the day.

Construction of a time curve.—The equation of the derived curve corresponding to (241) is

$$-y = A_1\,a_1 \sin(a_1 t - \mathfrak{A}_1{}^\circ) + 2\,A_2\,a_1 \sin(2\,a_1 - \mathfrak{A}_2{}^\circ) + 3\,A_3\,a_1 \sin(3\,a_1 - \mathfrak{A}_3{}^\circ) + \quad . \quad . \quad . \quad . \qquad (245)$$

It will be noticed that the $\begin{smallmatrix}\text{descending}\\\text{ascending}\end{smallmatrix}$ nodes of the derived curve will coincide in time with the $\begin{smallmatrix}\text{maxima}\\\text{minima}\end{smallmatrix}$ of the height curve. This is of course equally true for each element or component of the curve.

In ascertaining the magnitude of the coefficients $A_1\,a_1$, $2\,A_2\,a_1$, $3\,A_3\,a_1 \quad . \quad . \quad .$ relative to $A_1, A_2, A_3, \quad . \quad . \quad . \quad .$, it is usually convenient to take the speed of M_2 as unity, expressing other speeds in term of this unit. The derived curve is used in ascertaining the times of tide, and is drawn in a broken line upon the movable strip upon which the height curve has already been drawn.

Construction of a "calendar."—It will now be supposed that $\arg_0 A_2$ is given for only one day of each year, January 1. The change in the given $\arg_0 A_2$ necessary to adapt it to any other

day of the year may be assumed to be uniform, and the same for all years. This change is introduced by means of a "calendar" peculiar to the component and its harmonics.

Anywhere at the left of the curves, a section of the strip one period in length is set apart for the days of the year. The positions of the dates are given by Table 4. January 1 will be located at the right or left edge of the period according as the speed of the component is less or greater than that of the solar component of approximately the same speed.

It is sometimes more convenient to set apart two or more periods for the calendar. This will necessitate two or more blank periods of the curve, instead of one, before the period used in laying down $\arg_0 A_1$. This means, of course, an increase in the length of the curve.

Lunar nodal components, etc.—In order to avoid numerous small components, especially those dependent upon the longitude of the moon's node, the lunar tides are made to undergo small changes in argument which are not directly proportional to the time, and also small changes of amplitude. The former have been accounted for in Table 3 by the u of $V_0 + u$, or $\arg_0$, where u has an irregular variation, but so slow that for the purpose of prediction it may be regarded as constant during a year.

The variations of amplitude are given by the f's of Table 10, and can be shown upon the curves by means of a scale of proportional parts of the mean values of the amplitude and ordinates of each component. But as the f's vary quite uniformly with I (the inclination of the lunar orbit to the equator, Tables 1 and 13) it is generally sufficient to mark upon certain ordinates of the curve their increased or decreased values corresponding to changes in I.

The time scale.—This scale consists of twenty-five equidistant lines drawn upon the stationary board underneath all of the curves. The strips upon which the curves are drawn should be separated from one another by narrow spaces, so that the time scale, or rather the hour lines, may be seen between them.

64. *Directions for predicting.*

Find in Table 3 the value of $V_0 + u$ of each component for the given year and bring this number of the degree scale over the zero hour line. Set the various indices over the date January 1. For any other day move the scales until the indices, as already set, come over the required date upon each scale.

Suppose, in the first instance, that the height of the sea or tide is required at any particular hour of this day.

Along the edge of a piece of paper made to coincide with the given hour line, mark the positive ordinates of the full curves (slightly modified to account for the varying obliquity of the moon's orbit to the equator) so that all such ordinates shall be added together. Along the same edge, but on the opposite surface of the paper, mark the negative ordinates, beginning at the height upon the paper slip reached by the positive ordinates and going in an opposite direction. The resulting difference is the height of the tide above or below mean sea level at the stated hour. If a lower plane of reference be preferred, its depression below mean sea level should be laid off upon the paper slip before marking the ordinates upon it.

To find the time of high or low water.—By glancing at the dotted curves of the principal components one can usually tell roughly where the algebraic sum of their ordinates approach zero. If a high water time is sought, select a descending node; if a low water, an ascending.

At two adjacent hours near the high or low water, lay off upon a slip of paper the positive and negative ordinates of the dotted curves, properly modified for the varying obliquity of the moon's orbit to the equator.

Now take any scale about three hours in length, divided to single minutes. From this lay off the two resulting heights, one hour apart. A straightedge will show where a line joining the two points just laid down will cut the time scale. The time thus determined is the time of high or low water with reference to either hour line.

65. It is obvious that the foregoing method is applicable to periodic phenomena, although the constituent periodic curves may have no resemblance to simple cosine curves. Such curves are constructed by means of component hourly ordinates obtained from the observations. These remarks have special reference to the prediction of tidal currents.

APPROXIMATE PREDICTION OF HIGH AND LOW WATER.*

66. By means of the following method predictions can be made without resorting to graphic processes, and without access to a tide-predicting machine.

Moon's transits, phases, etc.—If the transits used are for the meridian of Greenwich (E) and expressed in civil time, they may be adapted to another meridian (L) and expressed in standard or prediction time (S) by adding

$$L - S + 0{\cdot}035\,(L - E) \tag{246}$$

§ 27. The times of the moon's phases, distances, and declinations are expressed in standard time by adding

$$E - S$$

to their Greenwich civil times. A further increase of τ hours, the age of the particular inequality, gives the standard or prediction time of the conspiring or interfering to which the inequality is due. See sheet headed "Ephemeris."

The tidal constants.—The constants are as follows:

$$\text{HWI, LWI, } \triangle_2,\ \mathrm{M_2},\ \mathrm{S_2},\ \tau\,(\mathrm{S_2};\ \mathrm{M_2}),\ \mathrm{N_2},\ \tau\,(\mathrm{N_2};\ \mathrm{M_2})\ \mathrm{K_1},\ \mathfrak{K}_1{}^\circ,\ \mathrm{O_1},\ \tau\,(\mathrm{O_1};\ \mathrm{K_1}),$$

$$\tau_2 = \frac{12\,\tau\,(\mathrm{N_2};\ \mathrm{M_2}) + 5\,\tau\,(\mathrm{S_2};\ \mathrm{M_2})}{17 \times 24{\cdot}84\ (= 422{\cdot}3)}. \tag{247}$$

$\triangle_2$ is the mean amplitude of the tide with the effects of the diurnal components excluded. $\mathfrak{K}_1{}^\circ$ is the modified form of $\mathrm{K_1}{}^\circ$, defined in § 62. HWI and LWI are so taken that their sum is zero, as nearly as possible. $\triangle_2$, $\mathrm{M_2}$, $\mathrm{S_2}$, $\mathrm{N_2}$, $\mathrm{K_1}$, $\mathrm{O_1}$ are reduced to their values for the year of the predictions by the factors $f\,(\mathrm{M_2})$, $f\,(\mathrm{M_2})$, 1, $f\,(\mathrm{M_2})$, $f\,(\mathrm{K_1})$, $f\,(\mathrm{O_1})$, respectively, found in Table 10.

67. The "Ephemeris" and the sheet following it indicate the work preparatory to making predictions. The process of prediction indicated below consists of three steps, the determination of the semidiurnal wave, of the diurnal, and of combining the two thus determined. Port Townsend, Washington, is taken as an illustration because of its large diurnal components.

Ephemeris.

		h.	m.
Port Townsend, Wash., 1895.	Longitude of ephemeris (E)	0	00
	Longitude of time meridian (S)	8	00
	$E-S=$	-8	00

Moon.	Greenwich civil time.			$E-S+\tau$	Prediction time of conspiring or interfering.				
		d.	*h.*	*h. h. h.*		*d.*	*h.*		
☾ Last quarter	February	16	13	$-8+21=13$	Feb.	17	2		
● New moon	"	24	17	"	"	25	6		
☽ First quarter	March	4	13	"	Mar.	5	2		
Apogee	February	22	19	$-8+51=43$	Feb.	24	14		
(Midtime)	March	2	10	"	Mar.	4	5		
Perigee	"	10	1	"					
							(for $\mathrm{P_1}$.)		
							h.	*d.*	*h.*
Farthest S.	February	19	1	$-8+16=\ 8$	Feb.	19	$9+15=20$		0
On equator	"	26	14	"	"	26	$22+14=27$		12
Farthest N.	March	5	15	"	Mar.	5	$23+12=\ 6$		11

* For other modes of predicting high and low water from harmonic constants, see an article entitled "On tidal prediction," by Prof. G. H. Darwin, Phil. Trans., Vol. 182 (1891), A, pages 159–229; and, by the same author, a portion of the article on "Tides" in the Manual of Scientific Enquiry (fifth edition), pages 75–90. See also Gezeitentafeln (Berlin), 1894. The equations for passing from the $\triangle$'s of the components in these tables to their epochs are

$$B_2{}^\circ = \mathrm{M_2}{}^\circ - \triangle\,B_2{}^\circ + (b_2 - \mathrm{m_2})\,(0{\cdot}0345\ \mathrm{M_2}{}^\circ + L).$$
$$\mathrm{K_1}{}^\circ = -\triangle\,\mathrm{K_1}{}^\circ + (\mathrm{k_1} - \mathrm{m_1})\,(0{\cdot}345\ \mathrm{M_2}{}^\circ + L),$$
$$B_1{}^\circ = \mathrm{K_1}{}^\circ - \triangle\,B_1{}^\circ + (b_1 - \mathrm{k_1})\,(0{\cdot}0345\ \mathrm{M_2}{}^\circ + L),$$

where the small letters denote speeds, and $\triangle\,B^\circ$, $\triangle\,\mathrm{K_1}{}^\circ$ are written instead of $\triangle\,b$, $\triangle\,\mathrm{k_1}$ of the Gezeitentafeln.

The standard time (120° W.) of the moon's transits across the meridian of Port Townsend (122° 45′) can be obtained from the table of transits, Pacific Coast Tide Tables, by adding 11^m.

PORT TOWNSEND, WASHINGTON, 1895.

CONSTANTS.

HWI	LWI	$\triangle_2$,	M_2,	S_2,	$\tau(S_2;\ M_2)$,	N_2,	$\tau(N_2;\ M_2)$,	K_1,	$\mathfrak{K}_1°$,	O_1,	$\tau(O_1;\ K_1)$.	τ_2
3 53 (*a*)	−2 44 (*a*)	2·40	2·24	0·55	21^h	$0^{ft}·46$	51^h	$2^{ft}·47$	151°·5	$1^{ft}·41$	16	1·70 lunar days.

Factors for the year 1895.

$f(K_2)$,	$f(M_2)$,	$F(K_1)$,	$f(K_1)$,	$f(O_1)$.
1·308	0·964	0·901	1·110	1·179.

CONSTANTS.

Amplitudes for the year 1895.

$\triangle_2$,	M_2,	S_2,	N_2,	K_1,	O_1.
2·31	2·16	0·55	0·44	2·74	1·66

Amplitudes for obtaining height corrections.

February 20	S_2 (with K_2 and T_2)	1·25 × 0·55 = 0·69,	Table 31 or 33
March 2	"	1·32 × 0·55 = 0·73	
February 22	N_2 (with ν_2)	1·16 × 0·44 = 0·51,	Table 34
March 10	"	1·17 × 0·44 = 0·51	
February 20	K_1 (with P_1)	0·90 × 2·74 = 2·47,	Table 31
March 2	"	0·80 × 2·74 = 2·19	
February 21	O_1 (with Q_1)	0·81 × 1·66 = 1·34,	Table 26
22	"	0·81 × " = 1·34	
23	"	0·81 × " = 1·34	
24	"	0·83 × " = 1·38	
25	"	0·86 × " = 1·43	
26	"	0·88 × " = 1·46	
27	"	0·89 × " = 1·48	
28	"	0·93 × " = 1·54	

Amplitude for obtaining time corrections.

February 20	S_2 (with T_2 and solar K_2)	1·09 × 0·55 = 0·60.	S_2/M_2 = 0·27,	Table 31
March 2	"	1·10 × " = 0·61.	" = 0·28	

Acceleration.

			$\tau(O_1;\ K_1)$ increased.	
February 20	K_1 by P_1	−17° = −68^m.	$-17(-\frac{10}{11}) = +15^h$,	Table 31
March 2	"	−14 = −56.	$-14(-\frac{10}{11}) = +13$	"

Time of K_1 *tides.*

$\text{Arg}_0\ K_1$ (Feb. 21) = 62°·55 =	4^h	09^m	Table 3
$\mathfrak{K}_1°$ = 151 · 5 =	10	04	
$-[\text{arg}_0\ K_1 - \mathfrak{K}_1°]$ =	5	55	Feb. 21 = K_1 HW
½ of a K_1 day =	11	58	
	17	53	Feb. 21 = K_1 LW

AUXILIARY TABLES

FOR THE

REDUCTION AND PREDICTION

OF

TIDES.

AUXILIARY TABLES

FOR THE

REDUCTION AND PREDICTION

OF

TIDES.

189

TABLE 1.—*The principal harmonic components,*

Symbol.	Name of component, etc.	Speed per mean solar hour. Formula.	Speed per mean solar hour. c	Speed ratios. $\frac{c}{s_1}$	Speed ratios. $\frac{s_1}{c}$	Speed ratios. $\frac{c}{m_2}$	Speed ratios. $\frac{c}{k_1}$	Synodic period.* $15/(m\sim c)$
			°					d.
J_1	Smaller lunar elliptic diurnal	$\gamma+\sigma-\varpi$	15·5854433	1·03903	0·96244	0·53772	1·03619	13·71879
	Lunar elliptic diurnal, second order	$\gamma-4\sigma+2\varpi$	12·8542862	0·85695	1·16693	0·44349	0·85461	9·15882
	Larger lunar evectional diurnal	$\gamma-3\sigma-\varpi+2\eta$	13·4715144	0·89810	1·11346	0·46479	0·89565	14·69813
$[K_1]$	Lunar diurnal	γ	15·0410686	1·00274	0·99727	0·51894	1·00000	27·32158
$[K_1]$	Solar diurnal	γ	15·0410686	1·00274	0·99727	0·51894	1·00000	27·32158
K_1	Luni-solar diurnal	γ	15·0410686	1·00274	0·99727	0·51894	1·00000	27·32158
$[K_2]$	Lunar semidiurnal	2γ	30·0821372	2·00548	0·49863	1·03788	2·00000	13·66079
$[K_2]$	Solar semidiurnal	2γ	30·0821372	2·00548	0·49863	1·03788	2·00000	13·66079
K_2	Luni-solar semidiurnal	2γ	30·0821372	2·00548	0·49863	1·03788	2·00000	13·66079
L_2	Smaller lunar elliptic semidiurnal	$\left\{\begin{matrix}2\gamma-\sigma-\varpi \text{ and}\\ 2\gamma-\sigma+\varpi\end{matrix}\right\}$	29·5284788	1·96857	0·50798	1·01878	1·96319	27·55456
$[L_2]$	[Ferrel's L_2]	$2\gamma-\sigma-\varpi$	29·5284788	1·96857	0·50798	1·01878	1·96319	27·55456
		$2\gamma+\sigma-\varpi$	30·6265120	2·04177	0·48977	1·05667	2·03619	9·13293
M_1	Smaller lunar elliptic diurnal	$\left\{\begin{matrix}\gamma-\sigma-\varpi \text{ and}\\ \gamma-\sigma+\varpi\dagger\end{matrix}\right\}$	14·4920521	0·96614	1·03505	0·50000	0·96350	
$[M_1]$	[Ferrel's m_1 or Q_1']	$\gamma-\sigma+\varpi$	14·4966939	0·96645	1·03472	0·50016	0·96381	
M_2	Principal lunar series	$2(\gamma-\sigma)$	28·9841042	1·93227	0·51753	1·00000	1·92700	
M_3	Principal lunar series	$3(\gamma-\sigma)$	43·4761563	2·89841	0·34502	1·50000	2·89050	
M_4	Principal lunar series	$4(\gamma-\sigma)$	57·9682084	3·86455	0·25876	2·00000	3·85400	
M_6	Principal lunar series	$6(\gamma-\sigma)$	86·9523126	5·79682	0·17251	3·00000	5·78099	
M_8	Principal lunar series	$8(\gamma-\sigma)$	115·9364168	7·72909	0·12938	4·00000	7·70799	
N_2	Larger lunar elliptic, semidiurnal	$2\gamma-3\sigma+\varpi$	28·4397296	1·89598	0·52743	0·98122	1·89081	27·55456
2 N	Lunar elliptic semidiurnal, second order	$2\gamma-4\sigma+2\varpi$	27·8953548	1·85969	0·53772	0·96244	1·85461	13·77728
O_1	Lunar diurnal	$\gamma-2\sigma$	13·9430356	0·92954	1·07581	0·48106	0·92700	27·32158
OO	Lunar diurnal, second order	$\gamma+2\sigma$	16·1391016	1·07594	0·92942	0·55683	1·07300	9·10719
P_1	Solar diurnal	$\gamma-2\eta$	14·9589314	0·99726	1·00275	0·51611	0·99454	32·12822
Q_1	Larger lunar elliptic diurnal	$\gamma-3\sigma+\varpi$	13·3986609	0·89324	1·11951	0·46228	0·89081	13·71879
R_2	Smaller solar elliptic	$2\gamma-\eta$	30·0410686	2·00274	0·49932	1·03647	1·99727	14·19158
S_1	Principal solar series	$\gamma-\eta$	15·0000000	1·00000	1·00000	0·51753	0·99727	29·53059
S_2	Principal solar series	$2(\gamma-\eta)$	30·0000000	2·00000	0·50000	1·03505	1·99454	14·76529
S_3	Principal solar series	$3(\gamma-\eta)$	45·0000000	3·00000	0·33333	1·55257	2·99181	
S_4	Principal solar series	$4(\gamma-\eta)$	60·0000000	4·00000	0·25000	2·07010	3·98908	
T_2	Larger solar elliptic	$2\gamma-3\eta$	29·9589314	1·99726	0.50069	1·03363	1·99181	15·38734
λ_2	Smaller lunar evectional	$2\gamma-\sigma+\varpi-2\eta$	29·4556254	1·96371	0·50924	1·01627	1·95835	31·81193
μ_2	Variational	$2\gamma-4\sigma+2\eta$	27·9682084	1·86455	0·53632	0·96495	1·85946	14·76529
ν_2	Larger lunar evectional	$2\gamma-3\sigma-\varpi+2\eta$	28·5125830	1·90084	0·52608	0·98373	1·89565	31·81193

*The diurnal components are synodic with M_1, and the semidiurnal components with M_2.
†The first of these components is Ferrel's n_1; the second his m_1 or Q_1', and is the same as $[M_1]$. U. S. C. and G. S. Report, 1878, App. 11.

with their speeds, coefficients, etc.

Symbol.	Coefficients. Formula.	Mean value.	Coeff. ratios. $\frac{C}{M_2}$	$\frac{C}{K_1}$	Factors for reduction. F	Equilibrium arguments. V	u
J_1	$\frac{3}{8} e \sin 2I$	·01485	0·03269	0·05600	$F(J_1) = \frac{0·72147}{\sin 2I}$	$t+h+s-p+90°$	$-\nu$
	$\frac{17}{4} e^2 \sin I \cos^2 \frac{1}{2} I$	·00487	0·01072	0·01836	$F = F(O_1)$	$t+h-4s+2p-90°$	$2\xi-\nu$
	$\frac{105}{32} me \sin I \cos^2 \frac{1}{2} I$	·00512* ·00708	0·01127 0·01559	0·01930 0·02669	$F = F(O_1)$	$t+3h-3s-p-90°$	$2\xi-\nu$
$[K_1]$	$(\frac{1}{4}+\frac{3}{8} e^2) \sin 2I$	·18115	0·39878	0·68302	$F([K_1]) = \frac{0·72147}{\sin 2I} = F(J_1)$	$t+h+90°$	$-\nu$
$[K_1]$	$(\frac{1}{4}+\frac{3}{8} e_1) \frac{\tau_1}{\tau^2} \sin 2\omega$	·08407	0·18507	0·31698	$F([K_1]) =$ unity	$t+h+90°$	Zero
K_1	$(\frac{1}{4}+\frac{3}{8} e^2) \sin 2I + (\frac{1}{4}+\frac{3}{8} e_1^2) \frac{\tau_1}{} \sin 2\omega$	·26522	0·58385	1·00000	$F(K_1) = \frac{1·05628}{(\sin^2 2I + 0·66962 \cos\nu \sin 2I + 0.11210)^{\frac{1}{2}}}$	$t+h+90°$	$-\nu'$
$[K_2]$	$(\frac{1}{4}+\frac{3}{8} e^2) \sin^2 I$	·03929	0·08649	0·14814	$F([K_2]) = \frac{0·15652}{\sin^2 I}$	$2t+2h$	-2ν
$[K_2]$	$(\frac{1}{4}+\frac{3}{8} e_1^2) \frac{\tau_1}{\tau} \sin^2 \omega$	·01823	0·04013	0·06874	$F([K_2]) =$ unity.	$2t+2h$	Zero
K_2	$(\frac{1}{4}+\frac{3}{8} e^2) \sin^2 I + (\frac{1}{4}+\frac{3}{8} e_1^2) \frac{\tau_1}{\tau} \sin^2 \omega$	·05752	0·12662	0·21688	$F(K_2) = \frac{0·22915}{(\sin^4 I + 0·14527 \cos 2\nu \sin^2 I + 0·00528)^{\frac{1}{2}}}$	$2t+2h$	$-2\nu''$
L_2	$\frac{1}{4} e \left\{1-12 \tan^2 \frac{1}{2} I \cos 2P\right\}^{\frac{1}{2}} \cos^4 \frac{1}{2} I$†	·01257	0·02767	0·04739	$F(L_2) = F(M_2) \times R'$	$2t+2h-s-p+180°$	$2\xi-2\nu-R$
$[L_2]$	$\frac{1}{4} e \cos^4 \frac{1}{2} I$	·01257	0·02767	0·04739	$F([L_2]) = \frac{0·91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$2t+2h-s-p+180°$	$2\xi-2\nu$
	$\frac{3}{8} e \sin^2 I$	·00323	0·00711	0·01218	$F = \frac{0·15652}{\sin^2 I}$	$2t+2h+s-p$	-2ν
M_1	$\frac{1}{2} e \left\{\frac{5}{2}+\frac{3}{2} \cos 2P\right\}^{\frac{1}{2}} \sin I \cos^2 \frac{1}{2} I$†	·00522‡ ·01649	0·01149 0·03630	0·01968 0·06217	$F(M_1) = F(O_1) \times Q'$	$t+h-s+90°$	$\xi-\nu+Q$
$[M_1]$	$\frac{3}{8} e \sin 2I$	·01485	0·03269	0·05599	$F([M_1]) = \frac{0.72147}{\sin 2I} = F(J_1)$	$t+h-s+p+90°$	$-\nu$
M_2	$(\frac{1}{2}-\frac{5}{4} e^2) \cos^4 \frac{1}{2} I$	·45426	1·00000	1·71277	$F(M_2) = \frac{0·91538}{\cos^4 \frac{1}{2} I}$	$2t+2h-2s$	$2\xi-2\nu$
M_3	$\frac{5}{12} d \cos^6 \frac{1}{2} I$, (and shallow water)	·00599	0·01319	·02259	$F(M_3) = \frac{0·87579}{\cos^6 \frac{1}{2} I} = F^{\frac{3}{2}}(M_2)$	$3t+3h-3s+180°$	$3\xi-3\nu$
M_4	(Shallow water)				$F(M_4) = \frac{0·83792}{\cos^8 \frac{1}{2} I} = F^2(M_2)$	$4t+4h-4s$	$4\xi-4\nu$
M_6	(Shallow water)				$F(M_6) = \frac{0·76701}{\cos^{12} \frac{1}{2} I} = F^3(M_2)$	$6t+6h-6s$	$6\xi-6\nu$
M_8	(Shallow water)				$F(M_8) = \frac{0·70210}{\cos^{16} \frac{1}{2} I} = F^4(M_2)$	$8t+8h-8s$	$8\xi-8\nu$
N_2	$\frac{7}{4} e \cos^4 \frac{1}{2} I$	·08796	0·19363	0·33165	$F(N_2) = \frac{0·91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$2t+2h-3s+p$	$2\xi-2\nu$
$2N$	$\frac{17}{4} e^2 \cos^4 \frac{1}{2} I$	·01173	0·02582	0·04423	$F(2N) = \frac{0·91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$2t+2h-4s+2p$	$2\xi-2\nu$
O_1	$(\frac{1}{2}-\frac{5}{4} e^2) \sin I \cos^2 \frac{1}{2} I$	·18856	0·41509	0·71096	$F(O_1) = \frac{0·38005}{\sin I \cos^2 \frac{1}{2} I}$	$t+h-2s-90°$	$2\xi-\nu$
OO	$(\frac{1}{2}-\frac{5}{4} e^2) \sin I \sin^2 \frac{1}{2} I$	·00812	0·01788	0·03062	$F(OO) = \frac{0·01638}{\sin I \sin^2 \frac{1}{2} I}$	$t+h+2s+90°$	$-2\xi-\nu$
P_1	$(\frac{1}{2}-\frac{5}{4} e_1^2) \frac{\tau_1}{\tau} \sin \omega \cos^2 \frac{1}{2} \omega$	·08775	0·19317	0·33086	$F(P_1) =$ unity	$t-h-90°$	Zero
Q_1	$\frac{7}{4} e \sin I \cos^2 \frac{1}{2} I$	·03651	0·08037	0·13766	$F(Q_1) = \frac{0.38005}{\sin I \cos^2 \frac{1}{2} I} = F(O_1)$	$t+h-3s+p-90°$	$2\xi-\nu$
R_2	$\frac{1}{4} e_1 \frac{\tau_1}{\tau} \cos^4 \frac{1}{2} \omega$	·00178	0·00392	0·00671	$F(R_2) =$ unity	$2t+h-p_1$	Zero
S_1	(Chiefly meteorological)				$F(S_1) =$ unity	$t+180°$	Zero
S_2	$(\frac{1}{2}-\frac{5}{4} e_1^2) \frac{\tau_1}{\tau} \cos^4 \frac{1}{2} \omega$	·21137	0·46531	0·79696	$F(S_2) =$ unity	$2t$	Zero
S_3	(Chiefly shallow water)				$F(S_3) =$ unity	$3t+180°$	Zero
S_4	(Shallow water), $(S_2/M_2)^2 M_4$				$F(S_4) =$ unity	$4t$	Zero
T_2	$\frac{7}{4} e_1 \frac{\tau_1}{\tau} \cos^4 \frac{1}{2} \omega$	·01243	0·02736	0·04687	$F(T_2) =$ unity	$2t-h+p_1$	Zero
λ_2	$\frac{15}{32} me \cos^4 \frac{1}{2} I$	·00176* ·00330	0·00387 0·00726	0·00664 0·01244	$F(\lambda_2) = \frac{0·91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$2t-s+p+180°$	$2\xi-2\nu$
μ_2	$\frac{23}{16} m^2 \cos^4 \frac{1}{2} I$	·00736* ·01094	0·01620 0·02408	0·02775 0·04125	$F(\mu_2) = \frac{0·91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$2t+4h-4s$	$2\xi-2\nu$
ν_2	$\frac{105}{32} me \cos^4 \frac{1}{2} I$	·01234* ·01706	0·02717 0·03756	0·04653 0·06432	$F(\nu_2) = \frac{0·91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$2t+4h-3s-p$	$2\xi-2\nu$

* The lower of these two figures gives the value when the coefficients in the evection and variation have their full values as derived from Lunar Theory.

† The coefficients of L_2 and M_1 are approximately expressed by the given formulæ; the true mean values are 0·01278 and 0·01531, respectively.

‡ The first of these two numbers is the mean value of the coefficients of the tide $\gamma-\sigma-\varpi$; the second applies to the tide M_1, compounded from $\gamma-\sigma-\varpi$ and $\gamma-\sigma+\varpi$.

TABLE 1.—*The principal harmonic components,*

Symbol.	Name of component, etc.	Speed per mean solar hour.		Speed ratios.				Synodic period.* $15/(m \sim c)$
		Formula.	c	$\frac{c}{s_1}$	$\frac{s_1}{c}$	$\frac{c}{m_2}$	$\frac{c}{k_1}$	
			°					d.
MK	Compound tides	$3\gamma-2\sigma$	44·0251728	2·93501	0·34071	1·51894	2·92700	
2 MK		$3\gamma-4\sigma$	42·9271398	2·86181	0·34943	1·48106	2·85400	
MN		$4\gamma-5\sigma+\varpi$	57·4238338	3·82826	0·26122	1·98122	3·81780	
MS †		$4\gamma-2\sigma-2\eta$	58·9841042	3·93227	0·25431	2·03505	3·92154	
2 MS		$2\gamma-4\sigma+2\eta$	27·9682084	1·86455	0·53632	0·96495	1·85946	
2 SM		$2\gamma+2\sigma-4\eta$	31·0158958	2·06773	0·48362	1·07010	2·06208	
Mf	Lunar fortnightly	2σ	1·0980330	0·07320	13·66079	0·03788	0·07300	
MSf †	Luni-solar synodic fortnightly	$2(\sigma-\eta)$	1·0158958	0·06773	14·76529	0·03505	0·06754	
Mm	Lunar monthly	$\sigma-\varpi$	0·5443747	0·03629	27·55455	0·01878	0·03619	
Sa	Solar annual	η	0·0410686	0·00274	365·24219	0·00142	0·00273	
Ssa	Solar semiannual	2η	0·0821372	0·00548	182·62109	0·00283	0·00546	
	Right ascension of local meridian	γ	15·04106864	1·00274	0·99727	0·51894	1·00000	
h	Mean longitude of sun	η	0·04106864	0·00274	365·24219	0·00142	0·00273	
s	Mean longitude of moon	σ	0·54901653	0·03660	27·32158	0·01894	0·03650	
p	Mean longitude of lunar perigee	ϖ	0·00464183	0·00031	3231·48	0·00016	0·00031	
p_1	Mean longitude of solar perigee		0·00000196	0·00000	....	0·00000	0·00000	
t	Time in hours after midnight, or do. multiplied by 15	$\gamma-\eta$	15·00000000	1·00000	1·00000	0·51753	0·99727	

c = speed of any component.
c_1 = speed of any diurnal component.
k_1 = speed of the diurnal component K_1.
m_1 = speed of the diurnal component M_1.
m_2 = speed of the semidiurnal component $M_2 = 2\,m_1$.
s_1 = speed of the diurnal component $S_1 = 15°$ per hour.

* The diurnal components are synodic with M_1, and the semidiurnal components with M_2.

† Suggested by Helmholtz's theory of compound sounds. B. A. A. S. Report 1869, p. 504. MS has been called the Helmholtz luni-solar tide. B. A. A. S. Report 1870, p. 149. Ferrel designated it by $(MS)_4$.

with their speeds, coefficients, etc.—Continued.

Symbol.	Coefficients.		Coeff. ratios.		Factors for reduction.	Equilibrium arguments.	
	Formula.	Mean value.	$\frac{C}{M_2}$	$\frac{C}{K_1}$	F	V	u
MK	(Shallow water)				$F(MK) = F(M_2) \times F(K_1)$	$3t+3h-2s+90°$	$2\xi-2\nu-\nu'$
2 MK	(Shallow water)				$F(2\,MK) = F(M_4) \times F(K_1)$	$3t+3h-4s-90°$	$4\xi-4\nu+\nu'$
MN	(Shallow water)				$F(MN) = \frac{0{\cdot}83792}{\cos^8 \frac{1}{2} I} = F(M_4)$	$4t+4h-5s+p$	$4\xi-4\nu$
MS	(Shallow water), $2\,(S_2/M_2)\,M_4$				$F(MS) = \frac{0{\cdot}91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$4t+2h-2s$	$2\xi-2\nu$
2 MS	(Shallow water)				$F(2\,MS) = \frac{0{\cdot}83792}{\cos^8 \frac{1}{2} I} = (F\,M_4)$	$2t+4h-4s$	$4\xi-4\nu$
2 SM	(Shallow water)				$F(2\,SM) = \frac{0{\cdot}91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$2t-2h+2s$	$-2\xi+2\nu$
Mf	$(\frac{1}{2}-\frac{5}{4} e^2) \sin^2 I$	·07827	·17230	0·29511	$F(Mf) = \frac{0{\cdot}15779}{\sin^2 I}$	$2s$	-2ξ
MSf	$m^2 (1-\frac{3}{2} \sin^2 I)$, (and shallow water)	·00422* ·00621	·00929 ·01367	0·01591 0·02341	$F(MSf) = \frac{0{\cdot}91538}{\cos^4 \frac{1}{2} I} = F(M_2)$	$-2h+2s$	$-2\xi+2\nu$
Mm	$e (1-\frac{3}{2} \sin^2 I)$	·04136	·09105	0·15595	$F(Mm) = \frac{0{\cdot}75316}{1-1{\cdot}5 \sin^2 I}$	$s-p$	Zero
Sa	(Chiefly meteorological)				$F(Sa)$ = unity	h	Zero
Ssa	$(\frac{1}{2}-\frac{5}{4} e_1^2) \frac{\tau_1}{\tau} \sin^2 \omega$	·03643	·08020	0·13736	$F(Ssa)$ = unity	$2h$	Zero

C = Coefficient or theoretical amplitude of any component.
M_2 = Coefficient or theoretical amplitude of the component M_2.
K_1 = Coefficient or theoretical amplitude of the component K_1.

$$m = \frac{\text{mean motion of sun}}{\text{mean motion of moon}} = 0{\cdot}07480 = \frac{1}{13{\cdot}369}.$$

R is such that $\tan R = \frac{\sin 2P}{\frac{1}{6}\cot^2 \frac{1}{2} I - \cos 2P}$; see Table 8.

$R' = \left(\frac{1}{1-12 \tan^2 \frac{1}{2} I \cos 2P}\right)^{\frac{1}{2}}$; see Table 11.

ν' is such that $\tan \nu' = \frac{\sin \nu \sin 2I}{0{\cdot}334811 + \cos \nu \sin 2I}$; see Table 7.

e = eccentricity of moon's orbit = 0·0549.
e_1 = eccentricity of earth's orbit = 0·0168.
ω = obliquity of the ecliptic = 23° 27′·3.

$$\frac{\tau_1}{\tau} = \frac{\text{mass of sun}}{\text{mass of moon}} \times \left(\frac{\text{mean dist. of moon}}{\text{mean dist. of sun}}\right)^3 = 0{\cdot}46035 = \frac{1}{2{\cdot}17223}.$$

Q is such that $\tan Q = \frac{1}{2} \tan P$; see Table 9.

$Q' = \left(\frac{1}{2{\cdot}5 + 1{\cdot}5 \cos 2P}\right)^{\frac{1}{2}}$; see Table 12.

ν'' is such that $\tan 2\nu'' = \frac{\sin 2\nu \sin^2 I}{0{\cdot}072634 + \cos 2\nu \sin^2 I}$; see Table 7.

I = inclination of lunar orbit to the plane of the earth's equator; it varies between $\omega-5° 8'{\cdot}8$ and $\omega+5° 8'{\cdot}8$, i. e. from 18° 18′·5 to 28° 36′·1. See Tables 6 and 7.

P = mean longitude of lunar perigee measured from the intersection of moon's orbit with the plane of the earth's equator. See Table 6.

d = earth's radius divided by moon's mean distance, i. e. the lunar parallax expressed in radians $= 0{\cdot}01659 = \frac{1}{60{\cdot}27}$.

ξ = longitude in moon's orbit of the intersection of the lunar orbit with the plane of the earth's equator. See Table 7.
ν = right ascension of the intersection of the lunar orbit with the plane of the earth's equator. See Table 7.

The semidiurnals have a general coefficient $\cos^2 \lambda$; the diurnals, $\sin 2\lambda$; those of long period, $\frac{1}{2}-\frac{3}{2}\sin^2\lambda$; and M_3, $\cos^3 \lambda$. λ denotes the latitude.

* The lower of these two figures gives the value when the coefficients in the evection and variation have their full values as derived from Lunar Theory.

TABLE 2.—*Dependence of component speeds upon certain astronomical quantities (epoch 1900).*

$$\text{Solar day} = 24^{h}\cdot000\ 0000 = \frac{2 \times 360}{s_2};\quad s_2 = \frac{2 \times 360}{\text{solar day}} = 30^{\circ}\cdot000\ 0000$$

$$\text{Lunar day} = 24\cdot841\ 2024 = \frac{2 \times 360}{m_2};\quad m_2 = \frac{2 \times 360}{\text{lunar day}} = 28\cdot984\ 1042$$

$$\text{Tropical day} = 23\cdot934\ 4696 = \frac{2 \times 360}{k_2};\quad k_2 = \frac{2 \times 360}{\text{tropical day}} = 30\cdot082\ 1373$$

If c denote the speed of any given component C, then a component whose speed is

$$c \pm \frac{360}{\Pi}$$

has with C a synodic period Π hours in length. If C be the principal one of the two components, then the effect of the other is to introduce an inequality into C of a period equal to Π. The known motions of the moon and sun suggest what tidal inequalities may exist corresponding to periodic irregularities in the motions of these bodies, or in the motions of the elements of their orbits. The following tabulation shows that it is often necessary to use more than one component to account for what may be regarded as an irregularity in the principal component:

	Π		Speed.		
			Formula.		Symbol.
	Name.	Value (hours, days).	$c+\frac{360}{\Pi}$	$c-\frac{360}{\Pi}$	
$c = m_2 = 28^{\circ}\cdot9841042$	Half synodic month	½ (708·734115) h. ½ (29·5305881) d.	° 30·0000000	° 27·9682084	S_2 μ_2
	Half tropical month	½ (655·717960) h. ½ (27·3215817) d.	30·0821373		k_2
	Anomalistic month	661·309206 h. 27·5545503 d.	29·5284789	 28·4397295	l_2 n_2
	Half anomalistic month	½ (661·309206) h. ½ (27·5545503) d.		27·8953549	$2n$
	Evectional period	763·486534 h. 31·8119389 d.	29·4556253	 28·5125831	λ_2 ν_2
$c = s_2 = 30^{\circ}$	Half tropical year	½ (8765·812722) h. ½ (365·2421968) d.	30·0821373		k_2
	Tropical year *	8765·812722 h. 365·2421968 d.	30·0410686	 29·9589314	r_2 t_2
$c = k_1 = 15^{\circ}\cdot0410686$	Half tropical month	½ (655·717960) h. ½ (27·3215817) d.	16·1391017	 13·9430356	OO O_1
	Anomalistic month	661·309206 h. 27·5545503 d.	15·5854433		J_1
	Half tropical year	½ (8765·812722) h. ½ (365·2421968) d.		14·9589314	P_1
$c = 0$	Anomalistic month	661·309206 h. 27·5545503 d.		13·3986609	Q_1
$c = l_2$ $c = n_2$	Half evectional period in moon's parallax	½ (9882·831893) h. ½ (411·7846622) d.	 28·5125831	29·4556253	λ_2 ν_2

* The tropical year is taken for convenience instead of the anomalistic year (= 8766·2295 h. or 365·25956 d.).

This table is based in part upon values given in Harkness' paper on the solar parallax, App. III, Washington Observations for 1885.

TABLE 3.—*Equilibrium arguments* ($V_0 + u$) *at the midnight preceding January 1 of each year from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them.*

Component.	1850	1851	1852	1853	1854	1855	1856	1857	1858	1859
	°	°	°	°	°	°	°	°	°	°
J_1	29·66	116·00	204·03	307·53	38·07	129·39	221·21	327·32	59·40	151·25
K_1	3·47	1·64	0·92	2·20	3·31	5·03	7·13	10·45	12·78	14·92
K_2	187·75	183·76	181·83	183·99	186·04	189·52	193·95	200·87	205·84	210·36
L_2	155·70	350·62	161·48	330·94	177·72	26·05	194·57	353·33	204·19	54·37
$[L_2]$	149·54	338·59	167·84	346·01	175·74	5·69	195·78	14·65	204·84	34·96
M_1	6·83	273·54	165·21	50·56	346·93	274·81	170·80	55·36	347·23	281·70
$[M_1]$	330·16	239·05	149·63	49·56	322·66	236·54	150·91	53·45	328·08	242·49
M_2	299·79	40·11	140·64	217·03	318·04	59·26	160·63	237·71	339·18	80·58
M_3	269·69	240·17	210·96	145·54	117·05	88·89	60·95	356·57	328·78	300·86
M_4	239·58	80·22	281·28	74·06	276·07	118·52	321·26	115·43	318·37	161·15
M_6	179·38	120·34	61·93	291·08	234·11	177·77	121·89	353·14	297·55	241·73
M_8	119·17	160·45	202·57	148·11	192·14	237·03	282·52	230·86	270·74	322·30
N_2	270·04	281·64	293·45	268·05	280·33	292·83	305·48	280·78	293·52	306·19
2 N	240·29	163·17	86·25	319·06	242·63	166·40	90·33	323·84	247·86	171·81
O_1	299·88	42·59	143·81	218·48	317·67	56·29	154·57	227·33	325·45	63·71
OO	239·95	132·46	29·85	318·61	223·07	129·63	37·54	333·44	242·03	150·04
P_1	349·70	349·94	350·18	349·43	349·67	349·91	350·15	349·40	349·64	349·88
Q_1	270·13	284·11	296·62	269·50	279·96	289·87	299·42	270·39	279·79	289·32
R_2	359·93	359·67	359·42	0·15	359·89	359·63	359·38	0·11	359·85	359·60
$S_{1,3}$	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00
$S_{2,4,6}$	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
T_2	0·07	0·33	0·58	359·85	0·11	0·37	0·62	359·89	0·15	0·40
λ_2	148·79	59·63	330·69	228·92	140·45	52·19	324·09	223·02	135·01	46·93
μ_2	241·04	82·12	283·40	76·16	277·92	119·89	322·02	115·47	317·70	159·84.
ν_2	270·79	200·59	130·60	25·14	315·62	246·32	177·17	72·41	3·36	294·22
MK	303·26	41·75	141·56	219·23	321·35	64·29	167·77	248·16	351·96	95·49
2 MK	236·12	78·59	280·37	71·86	272·76	113·49	314·13	104·98	305·59	146·23
MN	209·83	321·75	74·09	125·07	238·37	352·09	106·11	158·49	272·71	26·77
MS	299·79	40·11	140·64	217·03	318·04	59·26	160·63	237·71	339·18	80·58
2 MS	239·58	80·22	281·28	74·06	276·07	118·52	321·26	115·43	318·37	161·15
2 SM	60·21	319·89	219·36	142·97	41·96	300·74	199·37	122·29	20 82	279·42
Mf	240·03	134·94	33·02	320·06	222·70	126·67	31·48	323·06	228·29	133·17
MSf	60·21	319·89	219·36	142·97	41·96	300·74	199·37	122·29	20·82	279·42
Mm	29·75	118·47	207·20	308·98	37·71	126·43	215·15	316·94	45·66	134·38
Sa	280·30	280·06	279·82	280·57	280·33	280·09	279·85	280·60	280·36	280·12
Ssa	200·60	200·12	199·64	201·13	200·66	200·18	199·70	201·20	200·72	200·24
Elements.	Values at Greenwich, midnight beginning each year.									
h	280·30	280·06	279·82	280·57	280·33	280·09	279·85	280·60	280·36	280·12
s	129·67	259·06	28·44	171·00	300·39	69·77	199·16	341·72	111·10	240·49
p	99·92	140·58	181·25	222·02	262·68	303·35	344·01	24·78	65·44	106·11
	Values at the middle of each year, or for July 2 at Greenwich mean noon for common years, and at preceding midnight for leap years.									
p_1	280·37	280·39	280·40	280·42	280·44	280·46	280·47	280·49	280·51	280·52
P	110·60	149·33	189·70	231·38	273·98	317·24	0·98	44·92	88·82	132·53
N	136·54	117·21	97·85	78·50	59·17	39·84	20·49	1·13	341·81	322·48
I	20·02	21·57	23·28	24·97	26·44	27·59	28·33	28·60	28·39	27·71
Q	126·93	163·48	184·89	212·04	277·91	335·19	0·49	26·51	87·63	151·41
R	353·84	347·96	6·36	15·08	358·02	339·64	1·21	21·32	0·66	345·41
ξ	9·65	11·59	11·93	10·97	9·04	6·44	3·42	0·19	356·96	353·91
ν	10·39	12·54	12·99	12·02	9·96	7·13	3·79	0·21	356·62	353·25
ν'	6·83	8·42	8·90	8·37	7·02	5·06	2·72	0·15	357·58	355·20
$2\nu''$	12·85	16·36	17·81	17·14	14·62	10·67	5·75	0·32	354·88	349·88

In making analyses or predictions it is not desirable to modify the values of $V_0 + u$ as here given either on account of the longitude of the station or the longitude of the time meridian. Such alteration should be made once for all in the epoch (C^o) of the particular component (C) as will adapt it to the tabular $V_0 + u$.

TABLE 3.—*Equilibrium arguments* (V_0+u) *at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1860	1861	1862	1863	1864	1865	1866	1867	1868	1869
	°	°	°	°	°	°	°	°	°	°
J_1	242·65	347·35	76·95	165·17	251·71	350·55	74·00	157·03	240·76	340·06
K_1	16·70	18·88	19·28	18·69	16·99	15·22	11·71	7·97	4·62	3·11
K_2	213·98	218·19	218·62	216·95	213·19	209·68	203·16	196·37	190·05	186·95
L_2	229·00	26·04	228·97	72·20	259·39	62·53	249·41	86·02	283·89	100·40
$[L_2]$	224·92	43·36	232·88	62·16	251·23	68·81	257·60	86·37	275·18	92·79
M_1	178·86	61·04	340·98	280·05	179·71	56·79	311·75	235·51	162·59	47·19
$[M_1]$	156·44	57·57	329·72	240·49	149·59	44·86	310·86	216·44	122·73	18·46
M_2	181·82	258·47	359·27	99·83	200·17	275·96	16·03	116·07	216·16	291·99
M_3	272·72	207·70	178·90	149·74	120·25	53·94	24·05	354·11	324·25	257·99
M_4	3·63	156·94	358·53	199·65	40·33	191·92	32·06	232·15	72·33	223·99
M_6	185·45	55·40	357·80	299·48	240·50	107·88	48·10	348·22	288·49	155·98
M_8	7·26	313·87	357·06	39·30	80·66	23·84	64·13	104·30	144·66	87·98
N_2	318·71	293·58	305·65	317·49	329·11	303·11	314·46	325·78	337·15	311·19
2 N	95·61	328·68	252·04	175·15	98·05	330·27	252·89	175·49	98·14	330·39
O_1	162·27	236·01	335·93	77·00	179·52	258·25	3·60	109·34	214·42	292·74
OO	56·81	348·91	250·74	148·66	41·77	317·16	201·26	84·13	329·07	245·78
P_1	350·12	349·37	349·61	349·85	350·09	349·34	349·58	349·82	350·06	349·31
Q_1	299·17	271·12	282·32	294·66	308·46	285·41	302·03	319·05	335·40	311·93
R_2	359·34	0·07	359·82	359·56	359·30	0·03	359·78	359·52	359·27	359·99
$S_{1,3}$	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00
$S_{2,4,6}$	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
T_2	0·66	359·93	0·18	0·44	0·70	359·97	0·22	0·48	0·73	0·01
λ_2	318·69	217·19	128·51	39·59	310·46	208·10	118·69	29·26	299·87	197·55
μ_2	1·83	154·86	356·41	197·72	38·81	190·98	31·80	232·60	73·44	225·64
ν_2	224·94	119·75	50·02	340·06	269·87	163·82	93·37	22·89	312·45	206·44
MK	198·51	277·35	18·54	118·51	217·15	291·18	27·74	124·04	220·78	295·10
2 MK	346·93	138·06	339·25	180·96	23·34	176·70	20·36	224·18	67·71	220·88
MN	140·53	192·04	304·92	57·31	169·27	219·07	330·49	81·86	193·31	243·19
MS	181·82	258·47	359·27	99·83	200·17	275·96	16·03	116·07	216·16	291·99
2 MS	3·63	156·94	358·53	199·65	40·33	191·92	32·06	232·15	72·33	223·99
2 SM	178·18	101·53	0·73	260·17	159·83	84·04	343·97	243·93	143·84	68·01
Mf	37·27	326·45	227·40	125·83	21·13	299·45	188·83	77·40	327·33	246·52
MSf	178·18	101·53	0·73	260·17	159·83	84·04	343·97	243·93	143·84	68·01
Mm	223·11	324·09	53·61	142·34	231·06	332·85	61·57	150·29	239·01	340·80
Sa	279·88	280·63	280·39	280·15	279·91	280·66	280·42	280·18	279·94	280·69
Ssa	199·76	201·26	200·78	200·30	199·83	201·32	200·84	200·37	199·89	201·38
Elements.	Values at Greenwich, midnight beginning each year.									
h	279·88	280·63	280·39	280·15	279·91	280·66	280·42	280·18	279·94	280·69
s	9·87	152·43	281·82	51·21	180·59	323·15	92·54	221·92	351·31	133·87
p	146·77	187·54	228·21	268·87	309·53	350·31	30·97	71·63	112·29	153·07
	Values at the middle of each year, or for July 2, at Greenwich mean noon for common years and at preceding midnight for leap years.									
p_1	280·54	280·56	280·57	280·59	280·61	280·63	280·64	280·66	280·68	280·69
P	175·91	218·66	260·42	300·91	339·89	17·21	53·18	88·74	125·04	162·79
N	303·12	283·77	264·44	245·11	225·76	206·40	187·07	167·75	148·39	129·03
I	26·60	25·15	23·49	21·77	20·19	18·98	18·36	18·46	19·25	20·58
Q	177·95	201·80	251·35	320·13	349·62	8·81	33·73	87·48	144·51	171·19
R	355·92	17·32	3·91	349·96	351·83	6·27	8·19	0·34	351·29	352·39
ξ	351·24	349·21	348·12	348·29	350·03	353·43	358·12	3·22	7·64	10·61
ν	350·34	348·17	347·06	347·32	349·27	352·95	357·99	3·45	8·20	11·43
ν'	353·19	351·75	351·11	351·46	352·92	355·44	358·71	2·21	5·33	7·58
$2\nu''$	345·78	343·07	342·16	343·35	346·64	351·64	357·68	4·00	9·84	14·43

From § 62 we have for the modified epoch ($\mathfrak{C}^\circ$)

$$\mathfrak{C}^\circ = C^\circ + 15\,p\,L - c\,S$$

where p is to be put equal 1, 2, . . . according as C is diurnal, semidiurnal, etc.; c is the hourly speed of C; L, S denote the west longitude in hours of the station and of the time meridian used. The values of L and S should always accompany the work of analysis or prediction, thus enabling one to pass from $\mathfrak{C}^\circ$ to C° or C° to $\mathfrak{C}^\circ$ as the case may be.

TABLE 3.—*Equilibrium arguments* (V_0+u) *at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1870	1871	1872	1873	1874	1875	1876	1877	1878	1879
	°	°	°	°	°	°	°	°	°	°
J_1	67·07	155·69	245·61	350·54	82·08	174·02	266·11	12·19	103·89	195·03
K_1	1·71	1·41	2·04	4·40	6·29	8·51	10·86	14·12	16·15	17·73
K_2	183·73	182·64	183·58	188·21	192·12	196·81	201·81	208·64	212·86	216·00
L_2	270·95	100·60	307·85	139·35	296·92	121·83	334·74	170·64	331·40	151·77
[L_2]	281·92	111·26	300·85	119·34	309·36	139·50	329·70	148·55	338·61	168·50
M_1	297·71	201·00	145·32	47·99	301·76	202·48	145·46	55·40	309·99	207·85
[M_1]	288·03	199·20	111·67	13·03	287·13	201·62	116·27	18·77	293·03	206·72
M_2	32·39	133·02	233·88	310·59	51·88	153·30	254·77	331·84	73·18	174·35
M_3	228·59	199·52	170·81	105·88	77·82	49·95	22·16	317·76	289·77	261·52
M_4	64·79	266·03	107·75	261·18	103·76	306·60	149·55	303·68	146·36	348·70
M_6	97·18	39·05	341·63	211·76	155·63	99·90	44·32	275·53	219·55	163·04
M_8	129·58	172·06	215·50	162·35	207·51	253·20	299·10	247·37	292·73	337·39
N_2	322·87	334·77	346·91	321·82	334·40	347·10	359·85	335·13	347·75	0·19
2 N	253·35	176·52	99·94	333·08	256·92	180·90	104·93	338·42	262·32	186·04
O_1	34·84	135·57	235·24	308·80	47·27	145·48	243·58	316·37	54·72	153·47
OO	140·26	39·33	302·04	234·76	141·93	50·16	318·80	254·59	162·20	68·28
P_1	349·55	349·79	350·03	349·28	349·52	349·75	349·99	349·25	349·49	349·73
Q_1	325·32	337·32	348·27	320·05	329·80	339·28	348·66	319·65	329·29	339·32
R_2	359·74	359·49	359·23	359·96	359·70	359·45	359·19	359·92	359·67	359·41
$S_{1,\,3}$	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00
$S_{2,\,4,\,6}$	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
T_2	0·26	0·51	0·77	0·04	0·30	0·55	0·81	0·08	0·33	0·59
λ_2	108·47	19·62	291·00	189·56	101·37	13·32	285·32	184·23	96·09	7·78
μ_2	66·79	268·17	109·78	262·86	104·91	307·08	149·31	302·75	144·84	346·76
ν_2	136·32	66·42	356·75	251·62	182·39	113·28	44·23	299·46	230·27	160·91
MK	34·11	134·42	235·92	314·99	58·17	161·81	265·63	345·97	89·33	192·07
2 MK	63·07	264·62	105·71	256·78	97·46	298·09	138·69	289·56	130·21	330·97
MN	355·27	107·79	220·78	272·42	26·28	140·40	254·63	306·97	60·93	174·54
MS	32·39	133·02	233·88	310·59	51·88	153·30	254·77	331·84	73·18	174·35
2 MS	64·79	266·03	107·75	261·18	103·76	306·60	149·55	303·68	146·36	348·70
2 SM	327·61	226·98	126·12	49·41	308·12	206·70	105·23	28·16	286·82	185·65
Mf	142·71	41·88	303·40	232·98	137·33	42·34	307·61	239·11	143·74	47·41
MSf	327·61	226·98	126·12	49·41	308·12	206·70	105·23	28·16	286·82	185·65
Mm	69·52	158·25	246·97	348·75	77·48	166·20	254·92	356·71	85·43	174·15
Sa	280·45	280·21	279·97	280·72	280·48	280·25	280·01	280·75	280·51	280·27
Ssa	200·90	200·43	199·95	201·44	200·97	200·49	200·01	201·50	201·03	200·55
Elements.	Values at Greenwich, midnight beginning each year.									
h	280·45	280·21	279·97	280·72	280·48	280·25	280·01	280·75	280·51	280·27
s	263·25	32·64	162·02	304·58	73·97	203·35	332·74	115·30	244·68	14·07
p	193·73	234·39	275·05	315·83	356·49	37·15	77·81	118·59	159·25	199·91
	Values at the middle of each year, or for July 2, at Greenwich mean noon for common years and at preceding midnight for leap years.									
p_1	280·71	280·73	280·75	280·76	280·78	280·80	280·81	280·83	280·85	280·87
P	202·16	243·03	285·12	328·07	11·52	55·30	99·27	143·18	186·77	229·88
N	109·71	90·38	71·03	51·67	32·34	13·01	353·66	334·30	314·97	295·65
I	22·22	23·05	25·57	26·03	27·03	28·40	28·57	28·18	27·32	26·07

TABLE 3.—*Equilibrium arguments* ($V_0 + u$) *at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1880	1881	1882	1883	1884	1885	1886	1887	1888	1889
	°	°	°	°	°	°	°	°	°	°
J_1	285·32	28·47	116·06	201·92	286·12	23·31	106·49	190·75	276·68	18·39
K_1	18·64	19·66	18·65	16·53	13·43	10·74	7·07	4·01	1·94	1·96
K_2	217·59	219·20	216·70	212·21	206·24	201·48	194·79	188·87	184·45	184·02
L_2	359·19	189·85	8·80	184·51	17·70	204·53	38·89	219·81	35·38	219·58
$[L_2]$	358·17	176·28	5·47	194·47	23·31	200·77	29·54	218·40	47·40	225·28
M_1	142·24	58·16	312·82	203·44	108·15	31·14	301·37	192·56	83·72	344·54
$[M_1]$	119·56	19·14	289·29	197·71	104·45	358·07	263·80	170·62	79·11	337·25
M_2	275·29	351·62	92·08	192·36	292.48	8·15	108·20	208·33	308·61	24·71
M_3	232·94	167·42	138·13	108·54	78·72	12·22	342·30	312·50	282·91	217·06
M_4	190·58	343·23	184·17	24·72	224·96	16·30	216·40	56·67	257·22	49·42
M_6	105·88	334·85	276·25	217·07	157.44	24·44	324·61	265·00	205·83	72·12
M_8	21·17	326·46	8·34	49·43	89·92	32·59	72·81	113·34	154·44	98·83
N_2	12·41	346·95	358·70	10·25	21·65	355·53	6·86	18·27	29·82	4·13
2 N	109·54	342·29	265·31	188·14	110·82	342·91	265·52	188·21	111·04	343·56
O_1	252·85	327·81	69·41	172·54	277·19	357·46	103·06	207·65	310·72	26·89
OO	332·03	259·80	155·95	47·08	293·37	203·93	87·23	333·72	225·07	148·75
P_1	349·96	349·22	349·46	349·69	349·93	349·19	349·43	349·66	349·90	349·16
Q_1	349·97	323·15	336·02	350·43	6·36	344·84	1·72	17·59	31·93	6·31
R_2	359·15	359·88	359·63	359·37	359·11	359·84	359·59	359·33	359·08	359·81
$S_{1,\,3}$	180·00	180·00	180.00	180·00	180·00	180·00	180.00	180·00	180·00	180·00
$S_{2,\,4,\,6}$	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
T_2	0·85	0·12	0·37	0·63	0·89	0·16	0·41	0·67	0·92	0·19
λ_2	279·25	177·42	88·41	359.21	269·85	167·37	77·94	348·60	259·40	157·34
μ_2	188·46	341·15	182·37	23·40	224·27	16·31	217·12	58·00	259·03	51·50
ν_2	91·33	345·82	275·76	205·51	135·11	28·93	318·46	248·07	177·82	72·08
MK	293·93	11·28	110.73	208·89	305·91	18·88	115·28	212·34	310·55	26·67
2MK	171·95	323·57	165·52	8·18	211·53	5·56	209·33	52·66	255·28	47·46
MN	287·71	338·57	90·78	202·61	314·13	3·68	115·06	226·60	338·43	28·84
MS	275·29	351·62	92·08	192·36	292·48	8·15	108·20	208·33	308·61	24·71
2MS	190·58	343·23	184·17	24·72	224·96	16·30	216·40	56·67	257·22	49·42
2SM	84·71	8·38	267·92	167·64	67·52	351·85	251·80	151·67	51·39	335·29
Mf	309·59	236·00	133·27	27·27	278·09	193·23	82·09	333·04	227·17	150·93
MSf	84·71	8·38	267·92	167·64	67·52	351·85	251·80	151·67	51·39	335·29
Mm	262·88	4·66	93·39	182·11	270·83	12·62	101·34	190·06	278·79	20·57
Sa	280·04	280·78	280·54	280·31	280·07	280·81	280·57	280·34	280·10	280·84
Ssa	200·07	201·57	201·09	200·61	200·13	201·63	201·15	200·67	200·20	201.69
Elements.	Values at Greenwich, midnight beginning each year.									
h	280·04	280·78	280·54	280·31	280·07	280·81	280·57	280·34	280·10	280·84
s	143·45	286·01	55·40	184·79	314·17	96·73	226·12	355·50	124·89	267·45
p	240·58	281·35	322·01	2·68	43·34	84·11	124·77	165·44	206·10	246·87
	Values at the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years.									
p_1	280·88	280·90	280·92	280·93	280·95	280·97	280·99	281·00	281·02	281·04
P	272·30	313·67	353·58	31·86	68·60	104·33	140·03	176·79	215·19	255·22
N	276·29	256·94	237·61	218·28	198·93	179·57	160·24	140·91	121·56	102·21
I	24·53	22·82	21·12	19·66	18·66	18·31	18·69	19·71	21·19	22·89
Q	274·60	332·35	356·78	17·26	51·91	117·06	157·27	178·39	199·42	242·19
R	358·98	346·43	356·67	9·96	5·61	356·24	350·65	358·58	12·01	5·71
ξ	348·66	348·02	348·77	351·15	355·13	0·11	5·07	8·98	11·30	11·98
ν	347·60	346·98	347·87	350·49	354·78	0·12	5·43	9·65	12·21	13·03
ν'	351·40	351·12	351·89	353·77	356·64	·08	3·50	6·33	8·16	8·89
$2\nu''$	342·48	342·37	344·39	348·40	353·90	0·14	6·36	11·80	15·74	17·67

Table 3.—*Equilibrium arguments ($V_0 + u$) at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1890	1891	1892	1893	1894	1895	1896	1897	1898	1899
	°	°	°	°	°	°	°	°	°	°
J_1	107·55	197·88	289·05	34·83	126·86	218·95	310·88	56·45	147·29	237·16
K_1	2·04	2·98	4·58	7·61	9·90	12·24	14·45	17·31	18·65	19·25
K_2	183·75	185·40	188·60	194·35	199·71	204·70	209·37	215·21	217·80	218·67
L_2	65·75	260·29	55·86	240·32	93·98	295·30	86·65	268·40	118·43	317·10
$[L_2]$	54·72	244·40	74·29	253·04	83·22	273·42	103·56	282·25	112·05	301·63
M_1	287·73	192·32	85·29	340·37	288·59	199·79	93·36	344·85	289·57	205·07
$[M_1]$	248·96	161·84	75·57	337·77	252·35	167·00	81·48	343·48	256·88	169·30
M_2	125·42	226·38	327·55	44·52	145·97	247·45	348·86	65·76	166·84	267·70
M_3	188·14	159·57	131·33	66·77	38·95	11·17	343·29	278·64	250·26	221·55
M_4	250·85	92·76	295·10	89·03	291·94	134·89	337·72	131·52	333·68	175·40
M_6	16·27	319·14	262·66	133·55	77·91	22·34	326·58	197·29	140·53	83·09
M_8	141·70	185·52	230·21	178·06	223·88	269·78	315·44	263·05	307·37	350·79
N_2	16·13	28·36	40·81	15·99	28·72	41·47	54·16	29·28	41·64	53·77
2 N	266·83	190·34	114·07	347·46	271·47	195·50	119·47	352·80	276·43	199·84
O_1	127·17	226·52	325·24	38·21	136·37	234·47	332·68	45·80	144·76	244·47
OO	49·35	313·21	219·38	154·39	62·85	331·47	239·68	174·14	79·41	341·99
P_1	349·39	349·63	349·87	349·13	349·36	349·60	349·84	349·09	349·33	349·57
Q_1	17·87	28·50	38·50	9·69	19·11	28·50	37·98	9·31	19·55	30·54
R_2	359·55	359·30	359·04	359·77	359·51	359·26	359·00	359·73	359·48	359·22
$S_{1,3}$	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00
$S_{2,4,6}$	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
T_2	0·45	0·70	0·96	0·23	0·49	0·74	1·00	0·27	0·52	0·78
λ_2	68·58	340·06	251·76	150·56	62·54	334·54	246·48	145·23	56·83	328·21
μ_2	252·97	94·68	296·60	89·94	292·15	134·38	336·54	129·82	331·65	173·26
ν_2	2·27	292·70	223·35	118·47	49·40	340·35	271·24	166·30	96·85	27·19
MK	127·46	229·36	332·13	52·13	155·87	259·68	3·31	83·07	185·50	286·95
2MK	248·81	89·78	290·53	81·42	282·04	122·65	323·27	114·21	315·03	156·14
MN	141·55	254·74	8·36	60·50	174·69	288·92	43·02	95·04	208·48	321·47
MS	125·42	226·38	327·55	44·52	145·97	247·45	348·86	65·76	166·84	267·70
2MS	250·85	92·76	295·10	89·03	291·94	134·89	337·72	131·52	333·68	175·40
2SM	234·58	133·62	32·45	315·48	214·03	112·55	11·14	294·24	193·16	92·30
Mf	51·09	313·34	217·07	148·09	53·24	318·50	223·50	154·17	57·33	318·76
MSf	234·58	133·62	32·45	315·48	214·03	112·55	11·14	294·24	193·16	92·30
Mm	109·30	198·02	286·74	28·53	117·25	205·97	294·70	36·48	125·21	213·93
Sa	280·61	280·37	280·13	280·87	280·64	280·40	280·16	280·91	280·67	280·43
Ssa	201·21	200·74	200·26	201·75	201·27	200·80	200·32	201·82	201·33	200·86
Elements.	Values at Greenwich, midnight beginning each year.									
h	280·61	280·37	280·13	280·87	280·64	280·40	280·16	280·91	280·67	280·43
s	36·83	166·22	295·60	78·16	207·55	336·93	106·32	248·88	18·26	147·65
p	287·54	328·20	8·86	49·63	90·30	130·96	171·62	212·40	253·06	293·72
	Values at the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years.									
p	281·05	281·07	281·09	281·11	281·12	281·14	281·16	281·17	281·19	281·21
P	296·58	338·99	22·18	65·85	109·70	153·61	197·44	240·93	283·79	325·79
N	82·88	63·55	44·19	24·84	5·51	346·18	326·83	307·47	288·15	268·82
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	28·58	28·48	27·00	26·88	25·50	23·88

TABLE 3.—*Equilibrium arguments ($V_o + u$) at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909
	°	°	°	°	°	°	°	°	°	°
J_1	325·72	52·66	137·84	221·53	304·55	42·08	126·95	213·56	301·84	45·54
K_1	18·90	17·46	14·93	11·56	7·81	5·30	2·59	0·93	0·39	1·81
K_2	217·49	214·18	209·05	202·71	195·90	191·37	185·96	182·27	180·69	183·17
L_2	127·09	309·02	147·11	345·00	174·12	339·64	162·47	1·20	204·78	17·54
$[L_2]$	130·96	320·07	149·00	337·81	166·58	344·05	172·95	2·03	191·32	9·53
M_1	97·46	351·64	271·67	203·30	101·59	338·11	232·12	158·08	90·59	336·18
$[M_1]$	80·41	349·91	257·65	163·90	69·47	323·42	230·85	140·02	50·85	310·97
M_2	8·31	108·70	208·90	309·00	49·04	124·72	224·90	325·26	65·83	142·25
M_3	192·46	163·04	133·35	103·49	73·55	7·09	337·36	307·89	278·74	213·37
M_4	16·62	217·39	57·80	257·99	98·07	249·45	89·81	290·52	131·65	284·50
M_6	24·92	326·09	266·71	206·99	147·11	14·17	314·71	255·77	197·48	66·74
M_8	33·23	74·78	115·61	155·98	196·14	138·90	179·62	221·03	263·30	208·99
N_2	65·66	77·32	88·81	100·18	111·50	85·40	96·85	108·49	120·33	94·97
2 N	123·01	45·95	328·71	251·36	173·96	46·07	328·81	251·72	174·84	47·69
O_1	345·24	87·41	191·16	296·28	42·03	121·97	226·01	328·47	69·48	143·99
OO	240·89	135·15	24·33	269·15	151·99	63·53	311·79	205·11	103·20	32·52
P_1	349·81	350·05	350·29	350·53	350·76	350·02	350·26	350·49	350·73	349·99
Q_1	42·59	56·04	71·06	87·46	104·49	82·65	97·96	111·70	123·99	96·71
R_2	358·97	358·71	358·45	358·20	357·94	358·67	358·41	358·16	357·90	358·63
$S_{1,3}$	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00
$S_{2,4,6}$	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
T_2	1·03	1·29	1·55	1·80	2·06	1·33	1·59	1·84	2·10	1·37
λ_2	239·35	150·26	60·99	331·61	242·17	139·70	50·41	321· 9	232·38	130·64
μ_2	14·62	215·76	56·72	257·57	98·36	250·42	91·35	292·46	133·78	286·57
ν_2	317·27	247·13	176·81	106·39	35·90	289·75	219·40	149·23	79·27	333·85
MK	27·21	126·16	223·83	320·55	56·84	130·02	227·49	326·19	66·21	144·06
2 MK	357·71	199·93	42·87	246·44	90·26	244·15	87·22	289·58	131·27	282·69
MN	73·97	186·02	297·71	49·17	160·53	210·12	321·76	73·75	186·16	237·21
MS	8·31	108·70	208·90	309·00	49·04	124·72	224·90	325·26	65·83	142·25
2 MS	16·62	217·39	57·80	257·99	98·07	249·45	89·81	290·52	131·65	284·50
2 SM	351·69	251·30	151·10	51·00	310·96	235·28	135·10	34·74	294·17	217·75
Mf	217·82	113·87	6·59	256·44	144·98	60·78	312·89	208·32	106·86	34·26
MSf	351·69	251·30	151·10	51·00	310·96	235·28	135·10	34·74	294·17	217·75
Mm	302·65	31·37	120·09	208·82	297·54	39·33	128·05	216·77	305·49	47·28
Sa	280·19	279·95	279·71	279·47	279·24	279·98	279·74	279·51	279·27	280·01
Ssa	200·38	199·90	199·42	198·95	198·47	199·96	199·49	199·01	198·53	200·03
Elements.	Values at Greenwich, midnight beginning each year.									
h	280·19	279·95	279·71	279·47	279·24	279·98	279·74	279·51	279·27	280·01
s	277·03	46·42	175·80	305·19	74·57	217·13	346·52	115·90	245·29	27·85
p	334·38	15·05	55·71	96·37	137·03	177·81	218·47	259·13	299·80	340·57
	Values at the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years.									
p_1	281·23	281·24	281·26	281·28	281·29	281·31	281·33	281·35	281·36	281·38
P	6·59	45·89	83·53	119·73	155·34	191·39	228·73	267·72	308·32	350·18
N	249·49	230·16	210·83	191·51	172·15	152·80	133·47	114·14	94·79	75·43
I	22·15	20·52	19·21	18·44	18·37	19·01	20·24	21·83	23·56	25·22
Q	3·31	27·29	77·22	138·80	167·07	185·75	209·67	265·45	327·68	355·06
R	3·87	11·05	1·89	352·82	352·45	4·41	10·48	0·83	346·54	351·99
ξ	348·12	349·48	352·51	356·97	2·08	6·75	10·07	11·74	11·86	10·72
ν	347·13	348·67	351·97	356·76	2·23	7·23	10·85	12·72	12·92	11·76
ν'	351·29	352·49	354·78	357·92	1·43	4·68	7·16	8·57	8·88	8·21
$2\nu''$	342·89	345·72	350·37	356·24	2·57	8·59	13·53	16·73	17·84	16·85

Table 3.—*Equilibrium arguments* ($V_0 + u$) *at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1910	1911	1912	1913	1914	1915	1916	1917	1918	1919
	°	°	°	°	°	°	°	°	°	°
J_1	136.22	227.63	319.51	65.64	157.69	249.49	340.79	85.35	174.75	262.72
K_1	3.03	4.82	6.99	10.30	12.61	14.72	16.41	18.49	18.74	17.98
K_2	185.47	189.13	193.69	200.64	205.56	209.97	213.40	217.36	217.46	215.46
L_2	179.69	22.00	235.00	55.42	206.55	48.99	260.18	81.51	245.33	75.84
$[L_2]$	199.29	29.27	219.39	38.26	228.44	58.54	248.48	66.88	256.36	85.61
M_1	229.57	146.18	92.11	343.35	237.05	149.20	96.33	350.12	242.01	142.58
$[M_1]$	224.21	138.19	52.61	315.17	229.78	144.14	57.99	318.97	230.93	141.45
M_2	243.29	344.54	85.94	163.03	264.49	5.86	107.07	183.69	284.45	24.97
M_3	184.94	156.81	128.91	64.54	36.73	8.79	340.61	275.53	246.67	217.46
M_4	126.58	329.08	171.88	326.06	168.97	11.72	214.15	7.38	208.89	49.94
M_6	9.88	313.63	257.82	129.08	73.46	17.59	321.22	191.07	133.34	74.92
M_8	253.17	298.17	343.76	292.11	337.94	23.45	68.30	14.76	57.78	99.89
N_2	107.29	119.82	132.49	107.79	120.53	133.18	145.67	120.50	132.53	144.34
2 N	331.29	255.09	179.04	52.56	336.57	260.50	184.27	57.31	340.62	263.71
O_1	243.07	341.63	79.88	152.62	250.75	349.05	87.69	161.53	261.61	2.89
OO	297.36	204.19	112.24	48.19	316.73	224.58	131.09	62.79	324.07	221.28
P_1	350.23	350.47	350.70	349.96	350.20	350.43	350.67	349.93	350.17	350.40
Q_1	107.07	116.91	126.43	97.39	106.79	116.37	126.28	98.34	109.70	122.26
R_2	358.38	358.12	357.87	358.60	358.34	358.08	357.83	358.56	358.30	358.05
$S_{1,3}$	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00
$S_{2,4,6}$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
T_2	1.62	1.88	2.13	1.40	1.66	1.92	2.17	1.44	1.70	1.95
λ_2	42.21	313.99	225.91	124.84	36.83	308.72	220.46	118.92	30.20	301.25
μ_2	128.37	330.37	172.52	325.98	168.19	10.32	212.29	5.27	206.78	48.06
ν_2	264.37	195.10	125.97	21.22	312.15	243.00	173.69	68.46	358.69	288.69
MK	246.32	349.37	92.93	173.33	277.10	20.58	123.49	202.17	303.18	42.95
2 MK	123.55	324.26	164.89	315.75	156.36	357.01	197.73	348.89	190.15	31.97
MN	350.58	104.36	218.43	270.82	25.01	139.05	252.75	304.19	56.98	169.31
MS	243.29	344.54	85.94	160.03	264.49	5.86	107.07	183.69	284.45	24.97
2 MS	126.58	329.08	171.88	326.06	168.97	11.72	214.15	7.38	208.89	49.94
2 SM	116.71	15.46	274.06	196.97	95.51	354.14	352.93	176.31	75.55	335.03
Mf	297.14	201.28	106.18	37.79	302.99	207.77	111.70	40.63	301.23	199.19
MSf	116.71	15.46	274.06	196.97	95.51	354.14	252.93	176.31	75.55	335.03
Mm	136.00	224.73	313.45	55.23	143.96	232.68	321.40	63.19	151.91	240.63
Sa	279.77	279.53	279.30	280.04	279.80	279.57	279.33	280.07	279.83	279.60
Ssa	199.55	199.07	198.59	200.09	199.61	199.13	198.66	200.15	199.67	199.19
Elements.	Values at Greenwich, midnight beginning each year.									
h	279.77	279.53	279.30	280.04	279.80	279.57	279.33	280.07	279.83	279.60
s	157.24	286.62	56.01	198.57	327.95	97.34	226.72	9.28	138.67	268.05
p	21.23	61.89	102.56	143.33	183.99	224.66	265.32	306.09	346.76	27.42
	Values at the middle of each year, or for July 2, at Greenwich mean noon for common years and at preceding midnight for leap years.									
p_1	281.40	281.41	281.43	281.45	281.47	281.48	281.50	281.52	281.53	281.55
P	22.90	76.25	120.03	163.99	207.87	251.54	294.84	337.46	19.03	59.29
N	56.10	36.77	17.42	358.06	338.74	319.41	300.05	280.70	261.37	242.04
I	26.65	27.74	28.40	28.60	28.31	27.55	26.39	24.90	23.21	21.50

TABLE 3.—*Equilibrium arguments ($V_0 + u$) at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1920	1921	1922	1923	1924	1925	1926	1927	1928	1929
	°	°	°	°	°	°	°	°	°	°
J_1	348.98	87.57	170.87	253.94	337.87	77.45	164.74	253.58	343.67	88.73
K_1	16.11	14.19	10.60	6.88	3.63	2.29	1.08	0.93	1.70	4.15
K_2	211.39	207.66	201.05	194.28	188.11	185.27	182.37	181.62	182.87	187.73
L_2	277.13	101.57	284.24	101.71	291.23	119.89	319.09	133.44	306.72	141.37
$[L_2]$	274.64	92.19	280.98	109.75	298.58	116.22	305.37	134.75	324.38	142.91
M_1	78.36	344.03	235.99	124.59	25.11	309.86	226.49	119.97	15.31	292.00
$[M_1]$	50.27	305.28	211.14	116.77	23.25	279.26	189.10	100.50	13.15	274.63
M_2	125.28	201.05	301.11	41.16	141.27	217.12	317.55	58.21	159.11	235.86
M_3	187.93	121.58	91.67	61.74	31.90	325.68	296.33	267.32	238.67	173.79
M_4	250.57	42.10	242.22	82.32	282.53	74.24	275.10	116.42	318.23	111.72
M_6	15.85	243.16	183.33	123.48	63.80	291.37	232.65	174.64	117.34	347.57
M_8	141.14	84.21	124.44	164.64	205.06	148.49	190.20	232.85	276.46	223.43
N_2	155.93	129.91	141.24	152.57	163.96	138.02	149.73	161.67	173.85	148.81
2 N	186.57	58.77	341.38	263.99	186.65	58.93	341.91	265.13	188.59	61.75
O_1	105.66	184.63	290.10	35.81	140.70	218.77	320.63	61.17	160.71	234.18
OO	113.58	28.22	271.91	154.90	40.43	317.94	213.20	112.91	16.11	309.18
P_1	350.64	349.89	350.13	350.37	350.61	349.86	350.10	350.34	350.58	349.83
Q_1	136.31	113.49	130.24	147.22	163.39	139.67	152.81	164.63	175.44	147.13
R_2	357.79	358.52	358.26	358.01	357.75	358.48	358.23	357.97	357.72	358.45
$S_{1,3}$	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00
$S_{2,4,6}$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
T_2	2.21	1.48	1.74	1.99	2.25	1.52	1.77	2.03	2.28	1.55
λ_2	212.08	109.70	20.28	290.85	201.48	99.18	10.14	281.32	192.75	91.33
μ_2	249.13	41.26	242.07	82.88	283.74	65.96	277.14	118.56	320.22	113.33
ν_2	218.48	112.41	41.94	331.47	261.05	155.06	84.96	15.10	305.48	200.38
MK	141.39	215.24	311.71	48.04	144.90	219.42	318.63	59.15	160.81	240.01
2 MK	234.46	27.91	231.62	75.44	278.90	71.95	274.02	115.49	316.53	107.56
MN	281.21	330.96	82.35	193.73	305.22	355.15	107.28	219.88	332.96	24.66
MS	125.28	201.05	301.11	41.16	141.27	217.12	317.55	58.21	159.11	235.86
2 MS	250.57	42.10	242.22	82.32	282.53	74.24	275.10	116.42	318.23	111.72
2 SM	234.72	158.95	58.89	318.84	218.73	142.88	42.45	301.79	200.89	124.14
Mf	93.96	11.80	260.90	149.54	39.86	319.59	216.28	115.87	17.70	307.50
MSf	234.72	158.95	58.89	318.84	218.73	142.88	42.45	301.79	200.89	124.14
Mm	329.36	71.14	159.87	248.59	337.31	79.10	167.82	256.54	345.26	87.05
Sa	279.36	280.11	279.87	279.63	279.39	280.14	279.90	279.66	279.42	280.17
Ssa	198.72	200.21	199.73	199.26	198.78	200.27	199.79	199.32	198.84	200.33
Elements.	Values at Greenwich, midnight beginning each year.									
h	279.36	280.11	279.87	279.63	279.39	280.14	279.90	279.66	279.42	280.17
s	37.44	180.00	309.38	78.77	208.15	350.71	120.10	249.48	18.87	161.43
p	68.08	108.86	149.52	190.18	230.84	271.62	312.28	352.94	33.61	74.38
	Values at the middle of each year, or for July 2, at Greenwich, mean noon for common years and at preceding midnight for leap years.									
p_1	281.57	281.59	281.60	281.62	281.64	281.65	281.67	281.69	281.70	281.72
P	98.01	135.09	170.92	206.52	243.01	281.03	320.65	1.73	43.97	87.03
Λ	222.69	203.33	184.00	164.68	145.32	125.97	106.64	87.31	67.96	48.60
I	19.96	18.83	18.32	18.54	19.43	20.83	22.50	24.22	25.81	27.12
Q	105.72	153.50	175.43	194.01	224.47	291.29	337.71	0.87	25.75	84.07
R	357.51	350.62	356.73	8.04	7.35	356.33	346.28	1.32	17.66	1.54
ξ	350.46	354.10	358.93	4.00	8.22	10.92	11.96	11.55	10.02	7.68
ν	349.74	353.68	358.86	4.28	8.83	11.78	12.98	12.62	11.01	8.49
ν'	353.25	355.92	359.27	2.75	5.75	7.84	8.82	8.73	7.72	6.01
$2\nu''$	347.33	352.55	358.68	4.98	10.67	15.01	17.42	17.69	15.97	12.61

Table 3. - *Equilibrium arguments* (V_0+u) *at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1930	1931	1932	1933	1934	1935	1936	1937	1938	1939
	°	°	°	°	°	°	°	°	°	°
J_1	180·35	272·33	4·43	110·47	202·10	293·12	23·24	126·18	213·51	299·09
K_1	6·11	8·36	10·71	13·95	15·92	17·40	18·19	19·06	17·87	15·58
K_2	191·79	196·56	201·57	208·33	212·41	215·33	216·64	217·91	215·07	210·31
L_2	352·21	170·08	330·77	167·92	18·80	201·72	5·95	193·11	35·89	227·40
[L_2]	332·94	163·10	353·30	172·15	2·18	192·04	21·67	199·74	28·90	217·87
M_1	228·28	126·48	21·61	293·28	234·44	134·12	26·99	285·68	227·36	132·55
[M_1]	188·80	103·33	17·99	280·46	194·64	108·22	20·90	280·26	190·14	98·28
M_2	337·17	78·60	180·08	257·14	358·45	99·59	200·50	276·78	17·21	117·46
M_3	145·75	117·91	90·13	25·71	357·68	329·38	300·74	235·17	205·82	176·19
M_4	314·34	157·21	0·17	154·28	356·90	199·18	40·99	193·56	34·43	234·92
M_6	291·51	235·81	180·25	51·42	355·36	298·76	241·49	110·34	51·64	352·39
M	268·68	314·42	0·34	308·56	353·81	38·35	81·98	27·12	68·86	109·85
N_2	161·40	174·11	186·87	162·13	174·72	187·14	199·32	173·82	185·53	197·06
2 N	345·62	269·61	193·65	67·13	351·00	274·69	198·15	70·86	353·85	276·65
O_1	332·59	70·77	168·87	241·68	340·08	78·92	178·43	253·56	355·39	98·78
OO	216·56	124·89	33·55	329·25	236·66	142·43	45·71	332·87	228·26	118·58
P_1	350·07	350·31	350·55	349·80	350·04	350·28	350·52	349·77	350·01	350·25
Q_1	156·82	166·28	175·65	146·67	156·35	166·47	177·25	150·60	163·71	178·38
R_2	358·19	357·93	357·68	358·41	358·15	357·90	357·64	358·37	358·11	357·87
S_1,	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00	180·00
$S_{2,4,6}$	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
T_2	1·81	2·07	2·32	1·59	1·85	2·10	2·36	1·63	1·89	2·14
λ_2	3·17	275·13	187·13	86·03	357·87	269·53	180·96	79·09	350·05	260·82
μ_2	315·39	157·58	359·82	153·24	355·30	197·20	38·86	191·51	32·70	233·70
ν_2	131·17	62·08	353·03	248·25	179·03	109·65	40·03	294·47	224·38	154·10
MK	343·28	86·96	190·79	271·09	14·37	116·99	218·69	295·84	35·09	133·04
2 MK	308·23	148·85	349·46	140·33	340·99	181·77	22·80	174·50	16·56	219·35
MN	138·57	252·71	6·95	59·27	173·18	286·73	39·82	90·60	202·75	314·52
MS	337·17	78·60	180·08	257·14	358·45	99·59	200·50	276·78	17·21	117·46
2 MS	314·34	157·21	0·17	154·28	356·90	199·18	40·99	193·56	34·43	234·92
2 SM	22·83	281·40	179·92	102·86	1·55	260·41	159·50	83·22	342·79	242·54
Mf	211·98	117·06	22·34	313·78	218·29	121·76	23·64	309·66	206·43	99·90
MSf	22·83	281·40	179·92	102·86	1·55	260·41	159·50	83·22	342·79	242·54
Mm	175·77	264·49	353·22	95·01	183·73	272·45	1·17	102·96	191·68	280·40
Sa	279·93	279·69	279·45	280·20	279·96	279·72	279·48	280·23	279·99	279·75
Ssa	199·86	199·38	198·90	200·40	199·92	199·44	198·96	200·46	199·98	199·50
Elements.	Values at Greenwich, midnight beginning each year.									
h	279·93	279·69	279·45	280·20	279·96	279·72	279·48	280·23	279·99	279·75
s	290·82	60·20	189·59	332·15	101·53	230·92	0·30	142·86	272·25	41·63
p	115·04	155·70	196·37	237·14	277·80	318·47	359·13	39·90	80·57	121·23
	Values at the middle of each year, or for July 2, at Greenwich mean noon for common years and at preceding midnight for leap years.									
p_1	281·74	281·76	281·77	281·79	281·81	281·82	281·84	281·86	281·88	281·89
P	130·55	174·37	218·33	262·22	305·75	348·76	31·03	72·20	111·87	149·88
N	29·27	9·94	350·59	331·23	311·91	292·58	273·22	253·87	234·54	215·21
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	25·85	24·26	22·54	20·87	19·47

TABLE 3.—*Equilibrium arguments* (V_0+u) *at the midnight preceding January 1 of each year, from 1850 to 1950, for the meridian of Greenwich, together with the elements used in computing them*—Continued.

Component.	1940	1941	1942	1943	1944	1945	1946	1947	1948	1949	1950
	°	°	°	°	°	°	°	°	°	°	°
J_1	23.06	120.18	203.46	287.96	14.18	116.15	205.52	296.00	27.28	133.12	225.17
K_1	12.35	9.62	6.02	3.09	1.19	1.39	1.62	2.68	4.37	7.45	9.75
K_2	204.17	199.37	192.74	187.02	182.90	182.80	182.85	184.79	188.19	194.55	199.47
L_2	44.29	215.18	49.56	248.99	82.00	240.19	64.35	271.82	118.73	270.72	86.64
$[L_2]$	46.69	224.15	52.93	241.80	70.83	248.75	78.23	267.94	97.87	276.64	106.82
M_1	21.99	260.90	173.39	109.54	12.89	251.95	151.20	91.30	14.50	257.36	155.34
$[M_1]$	4.80	258.35	164.19	71.25	340.01	238.41	150.34	63.37	337.20	239.47	154.08
M_2	217.57	293.23	33.29	133.44	233.75	309.88	50.63	151.63	252.83	329.82	71.28
M_3	146.35	79.85	49.93	20.16	350.62	284.82	255.95	227.44	199.25	134.73	106.92
M_4	75.14	226.46	66.58	266.88	107.50	259.76	101.27	303.25	145.67	299.64	142.56
M_6	292.70	159.70	99.87	40.33	341.24	209.65	151.90	94.88	38.50	269.46	213.83
M_8	150.27	92.93	133.16	173.77	214.99	159.53	202.54	246.50	291.34	239.28	285.11
N_2	208.44	182.32	193.65	205.08	216.67	191.01	203.04	215.31	227.80	203.00	215.73
2 N	199.31	71.40	354.02	276.73	199.59	72.14	355.45	279.00	202.76	76.17	0.19
O_1	203.64	283.97	29.47	133.84	236.65	312.60	52.71	151.94	250.59	323.52	61.65
OO	4.22	274.55	158.19	45.38	297.54	221.96	123.15	27.42	293.89	229.07	137.59
P_1	350.49	349.74	349.98	350.22	350.46	349.71	349.95	350.19	350.43	349.68	349.92
Q_1	194.51	173.06	189.83	205.48	219.57	193.73	205.12	215.63	225.56	196.70	206.11
R_2	357.60	358.33	358.08	357.82	357.57	358.29	358.04	357.78	357.53	358.26	358.00
$S_{1,3}$	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00	180.00
$S_{2,4,6}$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
T_2	2.40	1.67	1.92	2.18	2.43	1.71	1.96	2.22	2.47	1.74	2.00
λ_2	171.45	68.96	339.54	250.22	161.05	59.02	330.30	241.82	153.55	52.38	324.36
μ_2	74.56	226.59	67.40	268.31	109.37	261.87	103.38	305.12	147.08	300.44	142.65
ν_2	83.69	337.51	267.04	196.67	126.45	20.74	310.97	241.43	172.12	67.26	358.19
MK	229.92	302.85	39.31	136.53	234.94	311.27	52.25	154.31	257.20	337.27	81.03
2 MK	62.78	216.84	60.56	263.79	106.30	258.38	99.65	300.57	141.30	292.19	132.80
MN	66.01	115.55	226.94	338.53	90.41	140.89	253.68	6.94	120.63	272.82	287.01
MS	217.57	293.23	33.29	133.44	233.75	309.88	50.63	151.63	252.83	329.82	71.28
2 MS	75.14	226.46	66.58	266.88	107.50	259.76	101.27	303.25	145.67	299.64	142.56
2 SM	142.43	66.77	326.71	226.56	126.25	50.12	309.37	208.37	107.17	30.18	288.72
Mf	350.29	265.29	154.36	45.77	300.44	224.68	125.22	27.74	291.65	222.77	127.97
MSf	142.43	66.77	326.71	226.56	126.25	50.12	309.37	208.37	107.17	30.18	288.72
Mm	9.13	110.91	199.64	288.36	17.08	118.87	207.59	296.31	25.04	126.82	215.55
Sa	279.51	280.26	280.02	279.78	279.54	280.29	280.05	279.81	279.57	280.32	280.08
Ssa	199.02	200.52	200.04	199.56	199.09	200.58	200.10	199.62	199.15	200.64	200.16
Elements.	Values at Greenwich, midnight beginning each year.										
h	279.51	280.26	280.02	279.78	279.54	280.29	280.05	279.81	279.57	280.32	280.08
s	171.02	313.58	82.96	212.35	341.73	124.29	253.68	23.06	152.45	295.01	64.40
p	161.89	202.66	243.33	283.99	324.65	5.43	46.09	86.75	127.41	168.19	208.85
	Values at the middle of each year, or for July 2, at Greenwich mean noon for common years and at preceding midnight for leap years.										
p_1	281.91	281.93	281.94	281.96	281.98	281.99	282.01	282.03	282.05	282.06	282.08
P	186.41	222.06	257.87	294.86	333.53	13.80	55.35	97.89	141.17	184.89	228.77
N	195.86	176.50	157.17	137.85	118.49	99.14	79.81	60.48	41.13	21.77	2.44
I	18.55	18.32	18.81	19.92	21.45	23.17	24.86	26.35	27.53	28.29	28.60
Q	183.21	204.29	246.75	312.82	346.02	7.01	35.89	105.49	158.08	182.45	209.71
R	2.41	8.96	3.36	352.81	348.82	8.56	13.88	356.12	339.14	5.92	20.18
ξ	355.87	0.93	5.78	9.46	11.51	11.95	11.07	9.19	6.63	3.63	0.41
ν	355.58	1.00	6.19	10.17	12.45	13.01	12.12	10.13	7.33	4.03	0.46
ν'	357.16	0.64	4.00	6.69	8.35	8.90	8.43	7.13	5.21	2.87	0.33
$2\nu''$	354.85	1.15	7.30	12.54	16.19	17.78	17.25	14.84	10.96	6.09	0.70

where T is the numb.r of Julian years of 365¼ mean solar days; D, the number of mean solar days; H, the number of mean solar hours after Greenwich mean noon, January 1, 1880. On account of the slowness of the secular changes in the coefficients o T, D, or H, the epoch of this table may be regarded as 1900. See Hansen's Tables de la Lune, p. 15, from which these formulæ may be obtained by putting $t = 80$. Newcomb's corrections (Washington Observations, Vol. 22 (1875), App. II, pp. 268, 274) are not of sufficient magnitude to affect the values in Table 3.

TABLE 4.—*For adapting the uniformly varying portion (V_0) of the equilibrium arguments of Table 3 to Greenwich midnight, beginning any day throughout the year.*

Months.		K_1	K_2	L_2	M_2	M_3	M_4	M_6	M_8	N_2	O_1	P_1	Q_1
Jan. 1		0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°	0°
Feb. 1		30·56	61·11	9·19	324·17	306·26	288·35	252·52	216·69	279·16	293·62	329·44	248·60
Mar. 1	Com. yr.	58·15	116·31	52·33	1·49	2·24	2·98	4·48	5·97	310·66	303·34	301·85	252·50
	Leap yr.	59·14	118·28	41·01	337·11	325·66	314·22	291·33	268·44	273·21	277·97	300·86	214·07
Apr. 1	Com. yr.	88·71	177·42	61·54	325·66	308·50	291·33	257·00	222·66	229·82	236·96	271·29	141·11
	Leap yr.	89·69	179·39	50·20	301·28	271·93	242·57	183·85	125·13	192·37	211·59	270·31	102·68
May 1	Com. yr.	118·28	236·56	82·02	314·22	291·33	268·44	222·66	176·88	186·42	195·94	241·72	68·14
	Leap yr.	119·26	238·53	70·70	289·84	254·76	219·68	149·52	79·36	148·98	170·58	240·74	29·71
June 1	Com. yr.	148·83	297·66	91·21	278·39	237·59	196·79	115·18	33·58	105·58	129·56	211·17	316·75
	Leap yr.	149·82	299·64	79·89	254·01	201·02	148·02	42·04	296·05	68·13	104·19	210·18	278·32
July 1	Com. yr.	178·40	356·80	111·71	266·95	220·42	173·90	80·85	347·80	62·18	88·55	181·60	243·78
	Leap yr.	179·39	358·78	100·40	242·57	183·85	125·14	7·70	250·27	24·74	63·18	180·61	205·35
Aug. 1	Com. yr.	208·96	57·91	120·90	231·12	166·68	102·24	333·37	204·49	341·34	22·16	151·04	132·39
	Leap yr.	209·94	59·89	109·58	206·74	130·11	53·48	260·22	106·96	303·90	356·80	150·06	93·96
Sept. 1	Com. yr.	239·51	119·02	130·09	195·30	112·94	30·59	225·89	61·18	260·50	315·78	120·49	20·99
	Leap yr.	240·50	121·00	118·77	170·92	76·37	341·83	152·74	323·66	223·06	290·42	119·50	342·56
Oct. 1	Com. yr.	269·08	178·16	150·59	183·85	95·78	7·70	191·55	15·40	217·11	274·77	90·92	308·03
	Leap yr.	270·07	180·13	139·28	159·47	59·20	318·94	118·41	277·88	179·66	249·40	89·93	269·59
Nov. 1	Com. yr.	299·64	239·27	159·78	148·02	42·04	296·05	84·07	232·10	136·27	208·39	60·36	196·63
	Leap yr.	300·62	241·24	148·47	123·64	5·46	247·29	10·93	134·57	98·82	183·02	59·38	158·20
Dec. 1	Com. yr.	329·21	298·41	180·29	136·58	24·87	273·16	49·74	186·32	92·87	167·37	30·79	123·67
	Leap yr.	330·19	300·38	168·97	112·20	348·30	224·40	336·59	88·79	55·43	142·01	29·81	85·23
Dec. 32	Com. yr.	359·76	359·52	189·48	100·75	331·13	201·51	302·26	43·01	12·03	100·99	0·24	12·27
	Leap yr.	0·75	1·49	178·16	76·37	294·56	152·74	229·11	305·49	334·58	75·62	359·25	333·84
Day of month													
1		0	0	0	0	0	0	0	0	0	0	0	0
2		0·99	1·97	348·68	335·62	323·43	311·24	286·86	262·47	322·55	334·63	− 0·99	321·57
3		1·97	3·94	337·37	311·24	286·86	262·47	213·71	164·95	285·11	309·27	− 1·97	283·14
4		2·96	5·91	326·05	286·86	250·28	213·71	140·57	67·42	247·66	283·90	− 2·96	244·70
5		3·94	7·88	314·73	262·47	213·71	164·95	67·42	329·90	210·21	258·53	− 3·94	206·27
6		4·93	9·86	303·42	238·09	177·14	116·18	354·28	232·37	172·77	233·16	− 4·93	167·84
7		5·91	11·83	292·10	213·71	140·57	67·42	281·13	134·84	135·32	207·80	− 5·91	129·41
8		6·90	13·80	280·78	189·33	103·99	18·66	207·99	37·32	97·88	182·43	− 6·90	90·98
9		7·88	15·77	269·47	164·95	67·42	329·90	134·84	299·79	60·43	157·06	− 7·88	52·54
10		8·87	17·74	258·15	140·57	30·85	281·13	61·70	202·27	22·98	131·70	− 8·87	14·11
11		9·86	19·71	246·84	116·18	354·28	232·37	348·56	104·74	345·54	106·33	− 9·86	335·68
12		10·84	21·68	235·52	91·80	317·70	183·61	275·41	7·21	308·09	80·96	−10·84	297·25
13		11·83	23·66	224·20	67·42	281·13	134·84	202·27	269·69	270·64	55·59	−11·83	258·81
14		12·81	25·63	212·88	43·04	244·56	86·08	129·12	172·16	233·20	30·23	−12·81	220·38
15		13·80	27·60	201·57	18·66	207·99	37·32	55·98	74·64	195·75	4·86	−13·80	181·95
16		14·78	29·57	190·25	354·28	171·42	348·56	342·83	337·11	158·30	339·49	−14·78	143·52
17		15·77	31·54	178·94	329·90	134·84	299·79	269·69	239·58	120·86	314·13	−15·77	105·09
18		16·76	33·51	167·62	305·52	98·27	251·03	196·54	142·06	83·41	288·76	−16·76	66·65
19		17·74	35·48	156·30	281·13	61·70	202·27	123·40	44·53	45·96	263·39	−17·74	28·22
20		18·73	37·46	144·99	256·75	25·13	153·50	50·26	307·01	8·52	238·02	−18·73	349·79
21		19·71	39·43	133·67	232·37	348·56	104·74	337·11	209·48	331·07	212·66	−19·71	311·36
22		20·70	41·40	122·35	207·99	311·98	55·98	263·97	111·95	293·62	187·29	−20·70	272·92
23		21·68	43·37	111·04	183·61	275·41	7·21	190·82	14·43	256·18	161·92	−21·68	234·49
24		22·67	45·34	99·72	159·23	238·84	318·45	117·68	276·90	218·73	136·56	−22·67	196·06
25		23·66	47·31	88·40	134·84	202·27	269·69	44·53	179·38	181·28	111·19	−23·66	157·63
26		24·64	49·28	77·09	110·46	165·69	220·92	331·39	81·85	143·84	85·82	−24·64	119·20
27		25·63	51·25	65·77	86·08	129·12	172·16	258·24	344·32	106·39	60·45	−25·63	80·76
28		26·61	53·22	54·45	61·70	92·55	123·40	185·10	246·80	68·94	35·09	−26·61	42·33
29		27·60	55·20	43·14	37·32	55·98	74·64	111·95	149·27	31·50	9·72	−27·60	3·90
30		28·58	57·17	31·82	12·94	19·40	25·87	38·81	51·75	354·05	344·35	−28·58	325·47
31		29·57	59·14	20·50	348·56	342·83	337·11	325·66	314·22	316·60	318·99	−29·57	287·04

The upper line for each month is for common years, and the lower line for leap years. For longitude corrections see Table 5, and for the portion (u) of the equilibrium arguments of Table 3, which depend upon the longitude of the moon's node, see Tables 6 and 7. The changes for other components may be found from those above, as follows:

$J_1 = L_2 - O_1$	$[L_2] = L_2$	$M_1 = h - s$	$[M_1] = K_1 + \lambda_2$	$2N = N_2 + \lambda_2$	$OO = K_2 - O_1$	$R_2 = K_1$	$S_{1,3}$ 0
$S_{2,4,6} = 0$	$T_2 = P_1$	$MN = M_2 + N_2$	$MS = M_2$	$2MS = M_4$	$2SM = MSf$	$Sa = K_1$	$Ssa = K_2$

TABLE 4.—*For adapting the uniformly varying portion (V_0) of the equilibrium arguments of Table 3 to Greenwich midnight, beginning any day throughout the year*—Continued.

Months.		λ_2	μ_2	ν_2	M K	2 M K	M Sf	M f	M m	h	s	p	N*
Jan. 1		° 0	° 0	° 0	° 0	° 0	° 0	° 0	° 0	° 0	° 0	° 0	° C.+9·66 L.+9·69
Feb. 1		314·98	288·35	333·36	354·73	257·79	35·83	96·94	45·02	30·56	48·47	3·45	+8·02 +8·05
Mar. 1	Com. yr.	309·16	2·98	53·82	59·64	304·83	358·51	114·82	50·84	58·15	57·41	6·57	+6·54
	Leap yr.	296·10	314·22	18·12	36·25	255·08	22·89	141·17	63·90	59·14	70·58	6·68	+6·51
Apr. 1	Com. yr.	264·15	291·33	27·18	54·37	202·62	34·34	211·75	95·85	88·71	105·88	10·03	+4·90
	Leap yr.	251·09	242·57	351·48	30·98	152·87	58·72	238·10	108·91	89·69	119·05	10·14	+4·87
May 1	Com. yr.	232·20	268·44	36·24	72·50	150·16	45·78	282·34	127·80	118·28	141·17	13·37	+3·31
	Leap yr.	219·14	219·68	0·54	49·10	100·41	70·16	308·69	140·86	119·26	154·34	13·48	+3·28
June 1	Com. yr.	187·19	196·79	9·60	67·23	47·96	81·61	19·27	172·81	148·83	189·64	16·82	+1·67
	Leap yr.	174·12	148·02	333·90	43·83	358·21	105·99	45·62	185·88	149·82	202·81	16·93	+1·64
July 1	Com. yr.	155·24	173·90	18·66	85·35	355·50	93·05	89·86	204·76	178·40	224·93	20·16	+0·08
	Leap yr.	142·17	125·14	342·96	61·96	305·75	117·43	116·21	217·83	179·39	238·10	20·28	+0·05
Aug. 1	Com. yr.	110·22	102·24	352·02	80·08	253·29	128·88	186·79	249·78	208·96	273·40	23·62	—1·56
	Leap yr.	97·16	53·48	316·32	56·68	203·54	153·26	213·14	262·84	209·94	286·57	23·73	—1·59
Sept. 1	Com. yr.	65·21	30·59	325·38	74·81	151·08	164·70	283·73	294·79	239·51	321·86	27·07	—3·20
	Leap yr.	52·14	341·83	289·69	51·41	101·33	189·08	310·08	307·86	240·50	335·04	27·18	—3·23
Oct. 1	Com. yr.	33·26	7·70	334·44	92·93	98·62	176·15	354·31	326·74	269·08	357·16	30·41	—4·79
	Leap yr.	20·19	318·94	298·75	69·54	48·88	200·53	20·66	339·81	270·07	10·33	30·52	—4·82
Nov. 1	Com. yr.	348·24	296·05	307·81	87·66	356·41	211·98	91·25	11·76	299·64	45·62	33·87	—6·43
	Leap yr.	335·18	247·29	272·11	64·26	306·66	236·36	117·60	24·82	300·62	58·80	33·98	—6·46
Dec. 1	Com. yr.	316·29	273·16	316·87	105·79	303·95	223·42	161·83	43·71	329·21	80·92	37·21	—8·02
	Leap yr.	303·23	224·40	281·17	82·39	254·20	247·80	188·19	56·77	330·19	94·09	37·32	– 8·05
Dec. 32	Com. yr.	271·28	201·51	290·23	100·51	201·74	259·25	258·77	88·72	359·76	129·38	40·66	—9·66
	Leap yr.	258·21	152·74	254·53	77·12	152·00	283·63	285·12	101·79	0·75	142·56	40·77	—9·69
Day of month.													
1		°	°	°	°	°	°	°	°	°	°	°	°
2		346·93	311·24	324·30	336·60	310.25	24·38	26·35	13·07	0·99	13·18	0·11	—0·05
3		333·87	262·47	288·60	313·21	260·50	48·76	52·71	26·13	1·97	26·35	0·22	—0·11
4		320·80	213·71	252·91	289·81	210·75	73·14	79·06	39·20	2·96	39·53	0·33	—0·16
5		307·74	164·95	217·21	266·42	161·00	97·53	105·41	52·26	3·94	52·71	0·45	—0·21
6		294·68	116·18	181·51	243·02	111·26	121·91	131·76	65·32	4·93	65·88	0·56	—0·26
7		281·61	67·42	145·81	219·62	61·51	146·29	158·12	78·39	5·91	79·06	0·67	—0·32
8		268·54	18·66	110·11	196·23	11·76	170·67	184·47	91·46	6·90	92·23	0·78	—0·37
9		255·48	329·90	74·42	172·83	322·01	195·05	210·82	104·52	7·88	105·41	0·89	—0·42
10		242·42	281·13	38·72	149·44	272·26	219·43	237·18	117·58	8·87	118·59	1·00	—0·48
11		229·35	232·37	3·02	126·04	222·51	243·82	263·53	130·65	9·86	131·76	1·11	—0·53
12		216·28	183·61	327·32	102·65	172·76	268·20	289·88	143·72	10·84	144·94	1·23	—0·58
13		203·22	134·84	291·62	79·25	123·02	292·58	316·23	156·78	11·83	158·12	1·34	—0·64
14		190·16	86·08	255·93	55·85	73·27	316·96	342·59	169·84	12·81	171·29	1·45	—0·69
15		177·09	37·32	220·23	32·46	23·52	341·34	8·94	182·91	13·80	184·47	1·56	—0·74
16		164·02	348·56	184·53	9·06	333·77	5·72	35·29	195·98	14·78	197·65	1·67	—0·79
17		150·96	299·79	148·83	345·67	284·02	30·10	61·64	209·04	15·77	210·82	1·78	—0·85
18		137·90	251·03	113·13	322·27	234·27	54·48	88·00	222·10	16·76	224·00	1·89	—0·90
19		124·83	202·27	77·44	298·88	184·52	78·87	114·35	235·17	17·74	237·18	2·01	—0·95
20		111·76	153·50	41·74	275·48	134·78	103·25	140·70	248·24	18·73	250·35	2·12	—1·01
21		98·70	104·74	6·04	252·08	85·03	127·63	167·06	261·30	19·71	263·53	2·23	—1·06
22		85·64	55·98	330·34	228·69	35·28	152·01	193·41	274·36	20·70	276·70	2·34	—1·11
23		72·57	7·21	294·64	205·29	345·53	176·39	219·76	287·43	21·68	289·88	2·45	—1·16
24		59·50	318·45	258·95	181·90	295·78	200·77	246·11	300·50	22·67	303·06	2·56	—1·22
25		46·44	269·69	223·25	158·50	246·03	225·16	272·47	313·56	23·66	316·23	2·67	—1·27
26		33·38	220·92	187·55	135·10	196·28	249·54	298·82	326·62	24·64	329·41	2·79	—1·32
27		20·31	172·16	151·85	111·71	146·54	273·92	325·17	339·69	25·63	342·59	2·90	—1·38
28		7·24	123·40	116·15	88·31	96·79	298·30	351·52	352·76	26·61	355·76	3·01	—1·43
29		354·18	74·64	80·46	64·92	47·04	322·68	17·88	5·82	27·60	8·94	3·12	—1·48
30		341·12	25·87	44·76	41·52	357·29	347·06	44·23	18·88	28·58	22·12	3·23	—1·54
31		328·05	337·11	9·06	18·12	307·54	11·44	70·58	31·95	29·57	35·29	3·34	—1·59

*This column gives the longitude of the moon's ascending node for any day when applied to the N of Table 3.

The change in the dial reading of any component (except M_2 and Sa) on the Ferrel machine = ± (tabular value for component − tabular value for M_2), the $^{\text{upper}}_{\text{lower}}$ sign being used when the speed of the component is $^{\text{greater}}_{\text{less}}$ than that of M_2. The changes for M_2 and Sa are given directly by this table.

TABLE 5.—*For adapting the uniformly varying portion (V_0) of the equilibrium arguments of Table 3 to local midnight for any degree of west longitude.*

West longitude.	K_1	K_2	L_2	M_2	M_3	M_4	M_6	M_8	N_2	O_1	P_1	Q_1
	°	°	°	°	°	°	°	°	°	°	°	°
¼	0·00	0·00	−0·01	−0·02	−0·03	−0·03	−0·05	−0·07	−0·03	−0·02	0·00	−0·03
½	0·00	0·00	−0·02	−0·03	−0·05	−0·07	−0·10	−0·14	−0·05	−0·04	0·00	−0·05
¾	0·00	0·00	−0·02	−0·05	−0·08	−0·10	−0·15	−0·20	−0·08	−0·05	0·00	−0·08
1	0·00	0·01	−0·03	−0·07	−0·10	−0·14	−0·20	−0·27	−0·10	−0·07	0·00	−0·11
2	0·01	0·01	−0·06	−0·14	−0·20	−0·27	−0·41	−0·54	−0·21	−0·14	−0·01	−0·21
3	0·01	0·02	−0·09	−0·20	−0·30	−0·41	−0·61	−0·81	−0·31	−0·21	−0·01	−0·32
4	0·01	0·02	−0·13	−0·27	−0·41	−0·54	−0·81	−1·08	−0·42	−0·28	−0·01	−0·43
5	0·01	0·03	−0·16	−0·34	−0·51	−0·68	−1·02	−1·35	−0·52	−0·35	−0·01	−0·53
6	0·02	0·03	−0·19	−0·41	−0·61	−0·81	−1·22	−1·63	−0·62	−0·42	−0·02	−0·64
7	0·02	0·04	−0·22	−0·47	−0·71	−0·95	−1·42	−1·90	−0·73	−0·49	−0·02	−0·75
8	0·02	0·04	−0·25	−0·54	−0·81	−1·08	−1·63	−2·17	−0·83	−0·56	−0·02	−0·85
9	0·02	0·05	−0·28	−0·61	−0·91	−1·22	−1·83	−2·44	−0·94	−0·63	−0·02	−0·96
10	0·03	0·05	−0·31	−0·68	−1·02	−1·35	−2·03	−2·71	−1·04	−0·70	−0·03	−1·07
20	0·05	0·11	−0·63	−1·35	−2·03	−2·71	−4·06	−5·42	−2·08	−1·41	−0·05	−2·14
30	0·08	0·16	−0·94	−2·03	−3·05	−4·06	−6·10	−8·13	−3·12	−2·11	−0·08	−3·20
40	0·11	0·22	−1·26	−2·71	−4·06	−5·42	−8·13	−10·84	−4·16	−2·82	−0·11	−4·27
50	0·14	0·27	−1·57	−3·39	−5·08	−6·77	−10·16	−13·55	−5·20	−3·52	−0·14	−5·34
60	0·16	0·33	−1·89	−4·06	−6·10	−8·13	−12·19	−16·25	−6·24	−4·23	−0·16	−6·41
70	0·19	0·38	−2·20	−4·74	−7·11	−9·48	−14·22	−18·96	−7·28	−4·93	−0·19	−7·47
80	0·22	0·44	−2·51	−5·42	−8·13	−10·84	−16·25	−21·67	−8·32	−5·64	−0·22	−8·54
90	0·25	0·49	−2·83	−6·10	−9·14	−12·19	−18·29	−24·38	−9·36	−6·34	−0·25	−9·61
100	0·27	0·55	−3·14	−6·77	−10·16	−13·55	−20·32	−27·09	−10·40	−7·05	−0·27	−10·68
110	0·30	0·60	−3·46	−7·45	−11·17	−14·90	−22·35	−29·80	−11·44	−7·75	−0·30	−11·74
120	0·33	0·66	−3·77	−8·13	−12·19	−16·25	−24·38	−32·51	−12·48	−8·46	−0·33	−12·81
130	0·36	0·71	−4·09	−8·80	−13·21	−17·61	−26·41	−35·22	−13·52	−9·16	−0·36	−13·88
140	0·38	0·77	−4·40	−9·48	−14·22	−18·96	−28·45	−37·93	−14·56	−9·87	−0·38	−14·95
150	0·41	0·82	−4·72	−10·16	−15·24	−20·32	−30·48	−40·64	−15·60	−10·57	−0·41	−16·01
160	0·44	0·88	−5·03	−10·84	−16·25	−21·67	−32·51	−43·34	−16·64	−11·27	−0·44	−17·08
170	0·47	0·93	−5·34	−11·51	−17·27	−23·03	−34·54	−46·05	−17·68	−11·98	−0·47	−18·15
180	0·49	0·99	−5·66	−12·19	−18·29	−24·38	−36·57	−48·76	−18·72	−12·68	−0·49	−19·22
190	0·52	1·04	−5·97	−12·87	−19·30	−25·74	−38·60	−51·47	−19·76	−13·39	−0·52	−20·28
200	0·55	1·10	−6·29	−13·55	−20·32	−27·09	−40·64	−54·18	−20·80	−14·09	−0·55	−21·35
210	0·57	1·15	−6·60	−14·22	−21·33	−28·45	−42·67	−56·89	−21·84	−14·80	−0·57	−22·42
220	0·60	1·20	−6·92	−14·90	−22·35	−29·80	−44·70	−59·60	−22·88	−15·50	−0·60	−23·49
230	0·63	1·26	−7·23	−15·58	−23·37	−31·15	−46·73	−62·31	−23·92	−16·21	−0·63	−24·55
240	0·66	1·31	−7·54	−16·25	−24·38	−32·51	−48·76	−65·02	−24·96	−16·91	−0·66	−25·62
250	0·68	1·37	−7·86	−16·93	−25·40	−33·86	−50·79	−67·73	−26·00	−17·62	−0·68	−26·69
260	0·71	1·42	−8·17	−17·61	−26·41	−35·22	−52·83	−70·44	−27·04	−18·32	−0·71	−27·76
270	0·74	1·48	−8·49	−18·29	−27·43	−36·57	−54·86	−73·14	−28·08	−19·03	−0·74	−28·82
280	0·77	1·53	−8·80	−18·96	−28·45	−37·93	−56·89	−75·85	−29·13	−19·73	−0·77	−29·89
290	0·79	1·59	−9·12	−19·64	−29·46	−39·28	−58·92	−78·56	−30·17	−20·43	−0·79	−30·96
300	0·82	1·64	−9·43	−20·32	−30·48	−40·64	−60·95	−81·27	−31·21	−21·14	−0·82	−32·03
310	0·85	1·70	−9·74	−21·00	−31·49	−41·99	−62·99	−83·98	−32·25	−21·84	−0·85	−33·09
320	0·88	1·75	−10·06	−21·67	−32·51	−43·34	−65·02	−86·69	−33·29	−22·55	−0·88	−34·16
330	0·90	1·81	−10·37	−22·35	−33·52	−44·70	−67·05	−89·40	−34·33	−23·25	−0·90	−35·23
340	0·93	1·86	−10·69	−23·03	−34·54	−46·05	−69·08	−92·11	−35·37	−23·96	−0·93	−36·30
350	0·96	1·92	−11·00	−23·70	−35·56	−47·41	−71·11	−94·82	−36·41	−24·66	−0·96	−37·36
360	0·99	1·97	−11·32	−24·38	−36·57	−48·76	−73·14	−97·53	−37·45	−25·37	−0·99	−38·43

The changes for other components may be found from those above as follows:

$J_1 = L_2 - O_1$	$[L_2] = L_2$	$M_1 = h - s$	$[M_1] = K_1 + \lambda_2$	$2N = N_2 + \lambda_2$	$OO = K_2 - O_1$	$R_2 = K_1$	$S_{1,3} = 0$
$S_{2,4,6} = 0$	$T_2 = P_1$	$MN = M_2 + N_2$	$MS = M_2$	$2MS = M_4$	$2SM = MSf$	$Sa = K_1$	$Ssa = K_2$

TABLE 5.—*For adapting the uniformly varying portion (V_0) of the equilibrium arguments of Table 3 to local midnight for any degree of west longitude*—Continued.

West longitude.	λ_2	μ_2	ν_2	MK	2MK	MSf	Mf	Mm	h	s	p	N
	°	°	°	°	°	°	°	°	°	°	°	°
¼	−0.01	−0.03	−0.02	−0.02	−0.03	0.02	0.02	0.01	0.00	0.01	0.00	0.00
½	−0.02	−0.07	−0.05	−0.03	−0.07	0.03	0.04	0.02	0.00	0.02	0.00	0.00
¾	−0.03	−0.10	−0.07	−0.05	−0.10	0.05	0.05	0.03	0.00	0.03	0.00	0.00
1	−0.04	−0.14	−0.10	−0.06	−0.14	0.07	0.07	0.04	0.00	0.04	0.00	0.00
2	−0.07	−0.27	−0.20	−0.13	−0.28	0.14	0.15	0.07	0.01	0.07	0.00	0.00
3	−0.11	−0.41	−0.30	−0.19	−0.41	0.20	0.22	0.11	0.01	0.11	0.00	0.00
4	−0.15	−0.54	−0.40	−0.26	−0.55	0.27	0.29	0.15	0.01	0.15	0.00	0.00
5	−0.18	−0.68	−0.50	−0.32	−0.69	0.34	0.37	0.18	0.01	0.18	0.00	0.00
6	−0.22	−0.81	−0.59	−0.39	−0.83	0.41	0.44	0.22	0.02	0.22	0.00	0.00
7	−0.25	−0.95	−0.69	−0.45	−0.97	0.47	0.51	0.25	0.02	0.26	0.00	0.00
8	−0.29	−1.08	−0.79	−0.52	−1.11	0.54	0.59	0.29	0.02	0.29	0.00	0.00
9	−0.33	−1.22	−0.89	−0.58	−1.24	0.61	0.66	0.33	0.02	0.33	0.00	0.00
10	−0.36	−1.35	−0.99	−0.65	−1.38	0.68	0.73	0.36	0.03	0.37	0.00	0.00
20	−0.73	−2.71	−1.98	−1.30	−2.76	1.35	1.46	0.73	0.05	0.73	0.01	0.00
30	−1.09	−4.06	−2.97	−1.95	−4.15	2.03	2.20	1.09	0.08	1.10	0.01	0.00
40	−1.45	−5.42	−3.97	−2.60	−5.53	2.71	2.93	1.45	0.11	1.46	0.01	−0.01
50	−1.81	−6.77	−4.96	−3.25	−6.91	3.39	3.66	1.81	0.14	1.83	0.02	−0.01
60	−2.18	−8.13	−5.95	−3.90	−8.29	4.06	4.39	2.18	0.16	2.20	0.02	−0.01
70	−2.54	−9.48	−6.94	−4.55	−9.67	4.74	5.12	2.54	0.19	2.56	0.02	−0.01
80	−2.90	−10.84	−7.93	−5.20	−11.06	5.42	5.86	2.90	0.22	2.93	0.02	−0.01
90	−3.27	−12.19	−8.92	−5.85	−12.44	6.10	6.59	3.27	0.25	3.29	0.03	−0.01
100	−3.63	−13.55	−9.92	−6.50	−13.82	6.77	7.32	3.63	0.27	3.66	0.03	−0.01
110	−3.99	−14.90	−10.91	−7.15	−15.20	7.45	8.05	3.99	0.30	4.03	0.03	−0.02
120	−4.36	−16.25	−11.90	−7.80	−16.58	8.13	8.78	4.36	0.33	4.39	0.04	−0.02
130	−4.72	−17.61	−12.89	−8.45	−17.96	8.80	9.52	4.72	0.36	4.76	0.04	−0.02
140	−5.08	−18.96	−13.88	−9.10	−19.35	9.48	10.25	5.08	0.38	5.12	0.04	−0.02
150	−5.44	−20.32	−14.87	−9.75	−20.73	10.16	10.98	5.44	0.41	5.49	0.05	−0.02
160	−5.81	−21.67	−15.87	−10.40	−22.11	10.84	11.71	5.81	0.44	5.86	0.05	−0.02
170	−6.17	−23.03	−16.86	−11.05	−23.49	11.51	12.44	6.17	0.47	6.22	0.05	−0.03
180	−6.53	−24.38	−17.85	−11.70	−24.87	12.19	13.18	6.53	0.49	6.59	0.06	−0.03
190	−6.90	−25.74	−18.84	−12.35	−26.26	12.87	13.91	6.90	0.52	6.95	0.06	−0.03
200	−7.26	−27.09	−19.83	−13.00	−27.64	13.55	14.64	7.26	0.55	7.32	0.06	−0.03
210	−7.62	−28.45	−20.82	−13.65	−29.02	14.22	15.37	7.62	0.57	7.69	0.06	−0.03
220	−7.98	−29.80	−21.82	−14.30	−30.40	14.99	16.10	7.98	0.60	8.05	0.07	−0.03
230	−8.35	−31.15	−22.81	−14.95	−31.78	15.58	16.84	8.35	0.63	8.42	0.07	−0.03
240	−8.71	−32.51	−23.80	−15.60	−33.17	16.25	17.57	8.71	0.66	8.78	0.07	−0.04
250	−9.07	−33.86	−24.79	−16.25	−34.55	16.93	18.30	9.07	0.68	9.15	0.08	−0.04
260	−9.44	−35.22	−25.78	−16.90	−35.93	17.61	19.03	9.44	0.71	9.52	0.08	−0.04
270	−9.80	−36.57	−26.77	−17.55	−37.31	18.29	19.76	9.80	0.74	9.88	0.08	−0.04
280	−10.16	−37.93	−27.77	−18.20	−38.69	18.96	20.50	10.16	0.77	10.25	0.09	−0.04
290	−10.52	−39.28	−28.76	−18.85	−40.08	19.64	21.23	10.52	0.79	10.61	0.09	−0.04
300	−10.89	−40.64	−29.75	−19.50	−41.46	20.32	21.96	10.89	0.82	10.98	0.09	−0.04
310	−11.25	−41.99	−30.74	−20.15	−42.84	21.00	22.69	11.25	0.85	11.35	0.10	−0.05
320	−11.61	−43.34	−31.73	−20.80	−44.22	21.67	23.42	11.61	0.88	11.71	0.10	−0.05
330	−11.98	−44.70	−32.72	−21.45	−45.60	22.35	24.16	11.98	0.90	12.08	0.10	−0.05
340	−12.34	−46.05	−33.71	−22.10	−46.98	23.03	24.89	12.34	0.93	12.44	0.11	−0.05
350	−12.70	−47.41	−34.71	−22.75	−48.37	23.70	25.62	12.70	0.96	12.81	0.11	−0.05
360	−13.07	−48.76	−35.70	−23.40	−49.75	24.38	26.35	13.07	0.99	13.18	0.11	−0.05

TABLE 6.—*Values of N, I, and P for Greenwich midnight, beginning each month, from 1850 to 1949.*

Month.	N	I	P	N	I	P	N	I	P	N	I	P
		1850			1855			1860			1865	
	°	°	°	°	°	°	°	°	°	°	°	°
Jan. 1	146.201	19.379	91.861	49.506	27.066	295.540	312.812	27.201	154.257	216.065	19.518	358.775
Feb. 1	144.559	19.479	95.015	47.865	27.162	299.217	311.171	27.105	157.936	214.424	19.416	1.933
Mar. 1	143.076	19.572	97.875	46.382	27.247	302.542	309.635	27.014	161.374	212.941	19.329	4.777
Apr. 1	141.435	19.679	101.049	44.741	27.339	306.226	307.994	26.913	165.045	211.300	19.234	7.913
May 1	139.846	19.785	104.135	43.152	27.424	309.794	306.405	26.813	168.594	209.711	19.146	10.938
June 1	138.205	19.899	107.335	41.510	27.509	313.485	304.763	26.707	172.258	208.069	19.060	14.054
July 1	136.616	20.011	110.442	39.922	27.589	317.059	303.175	26.603	175.797	206.481	18.980	17.061
Aug. 1	134.974	20.130	113.664	38.280	27.669	320.756	301.533	26.491	179.452	204.839	18.902	20.157
Sept. 1	133.333	20.253	116.900	36.639	27.746	324.456	299.892	26.378	183.102	203.198	18.828	23.246
Oct. 1	131.744	20.373	120.041	35.050	27.817	328.038	298.303	26.266	186.629	201.609	18.761	26.223
Nov. 1	130.103	20.500	123.301	33.408	27.888	331.744	296.661	26.149	190.269	199.967	18.696	29.294
Dec. 1	128.514	20.625	126.468	31.820	27.952	335.331	295.073	26.032	193.787	198.379	18.638	32.258
		1851			1856			1861			1866	
Jan. 1	126.872	20.755	129.753	30.178	28.017	339.040	293.431	25.911	197.415	196.737	18.583	35.312
Feb. 1	125.231	20.889	133.047	28.537	28.078	342.752	291.790	25.787	201.038	195.096	18.532	38.360
Mar. 1	123.748	21.011	136.034	27.001	28.132	346.226	290.307	25.673	204.309	193.613	18.490	41.108
Apr. 1	122.107	21.148	139.353	25.360	28.187	349.941	288.666	25.545	207.919	191.972	18.449	44.144
May 1	120.518	21.281	142.575	23.771	28.237	353.539	287.077	25.419	211.410	190.383	18.415	47.078
June 1	118.876	21.421	145.918	22.129	28.285	357.258	285.435	25.288	215.011	188.741	18.383	50.106
July 1	117.288	21.558	149.164	20.541	28.329	0.858	283.847	25.159	218.489	187.153	18.359	53.031
Aug. 1	115.646	21.700	152.528	18.899	28.370	4.580	282.205	25.025	222.076	185.511	18.338	56.051
Sept. 1	114.005	21.845	155.905	17.258	28.408	8.303	280.564	24.888	225.656	183.870	18.323	59.070
Oct. 1	112.416	21.984	159.183	15.669	28.443	11.909	278.975	24.755	229.115	182.281	18.314	61.989
Nov. 1	110.774	22.129	162.582	14.027	28.474	15.633	277.333	24.617	232.682	180.639	18.309	65.004
Dec. 1	109.186	22.270	165.882	12.439	28.501	19.239	275.745	24.481	236.126	179.051	18.309	67.923
		1852			1857			1862			1867	
Jan. 1	107.544	22.416	169.301	10.797	28.526	22.968	274.103	24.340	239.678	177.409	18.315	70.939
Feb. 1	105.903	22.563	172.732	9.156	28.547	26.695	272.462	24.197	243.221	175.768	18.326	74.955
Mar. 1	104.367	22.701	175.951	7.673	28.563	30.064	270.979	24.068	246.415	174.285	18.341	76.682
Apr. 1	102.726	22.848	179.401	6.032	28.578	33.793	269.338	23.924	249.943	172.644	18.362	79.676
May 1	101.137	22.991	182.750	4.443	28.589	37.402	267.749	23.783	253.349	171.054	18.387	82.628
June 1	99.495	23.137	186.221	2.801	28.597	41.132	266.107	23.638	256.860	169.413	18.419	85.657
July 1	97.907	23.279	189.588	1.213	28.601	44.741	264.519	23.497	260.249	167.825	18.454	88.591
Aug. 1	96.265	23.426	193.076	359.571	28.602	48.471	262.877	23.350	263.742	166.183	18.496	91.628
Sept. 1	94.624	23.573	196.574	357.930	28.599	52.202	261.236	23.203	267.226	164.542	18.542	94.671
Oct. 1	93.035	23.714	199.968	356.341	28.593	55.842	259.647	23.060	270.589	162.953	18.592	97.623
Nov. 1	91.393	23.859	203.484	354.699	28.583	59.542	258.005	22.914	274.054	161.311	18.649	100.679
Dec. 1	89.805	23.999	206.895	353.111	28.571	63.150	256.417	22.771	277.398	159.723	18.707	103.644
		1853			1858			1863			1868	
Jan. 1	88.163	24.143	210.426	351.469	28.555	66.879	254.774	22.624	280.842	158.080	18.774	106.715
Feb. 1	86.522	24.285	213.966	349.828	28.535	70.607	253.133	22.477	284.277	156.439	18.844	109.795
Mar. 1	85.039	24.414	217.172	348.345	28.514	73.975	251.650	22.345	287.371	154.903	18.914	112.686
Apr. 1	83.397	24.555	220.727	346.704	28.487	77.701	250.009	22.199	290.785	153.262	18.992	115.783
May 1	81.809	24.689	224.174	345.115	28.458	81.307	248.420	22.058	294.079	151.673	19.073	118.791
June 1	80.167	24.827	227.746	343.473	28.424	85.032	246.778	21.913	297.473	150.031	19.160	121.908
July 1	78.578	24.960	231.207	341.885	28.389	88.636	245.190	21.774	300.746	148.443	19.248	124.935
Aug. 1	76.937	25.095	234.790	340.243	28.349	92.358	243.548	21.631	304.117	146.801	19.343	128.074
Sept. 1	75.295	25.229	238.382	338.602	28.306	96.079	241.907	21.489	307.476	145.160	19.442	131.224
Oct. 1	73.707	25.357	241.863	337.013	28.260	99.679	240.318	21.353	310.716	143.571	19.541	134.283
Nov. 1	72.065	25.488	245.467	335.371	28.211	103.398	238.676	21.214	314.052	141.929	19.647	137.456
Dec. 1	70.476	25.612	248.960	333.783	28.159	106.994	237.088	21.081	317.270	140.341	19.752	140.538
		1854			1859			1864			1869	
Jan. 1	68.834	25.746	252.587	332.140	28.102	110.709	235.446	20.945	320.582	138.699	19.864	143.733
Feb. 1	67.193	25.864	256.198	330.499	28.043	114.421	233.805	20.810	323.883	137.058	19.980	146.940
Mar. 1	65.710	25.974	259.475	329.016	27.986	117.773	232.269	20.687	326.960	135.575	20.087	149.848
Apr. 1	64.069	26.096	263.107	327.375	27.920	121.481	230.628	20.556	330.237	133.934	20.208	153.078
May 1	62.480	26.210	266.626	325.786	27.853	125.067	229.039	20.433	333.397	132.345	20.328	156.216
June 1	60.838	26.328	270.281	324.144	27.781	128.771	227.397	20.307	336.650	130.703	20.453	159.472
July 1	59.250	26.438	273.799	322.556	27.709	132.353	225.809	20.189	339.786	129.115	20.577	162.633
Aug. 1	57.608	26.550	277.451	320.914	27.629	136.050	224.167	20.068	343.015	127.473	20.707	165.911
Sept. 1	55.967	26.659	281.107	319.273	27.549	139.746	222.526	19.951	346.232	125.832	20.840	169.205
Oct. 1	54.378	26.763	284.651	317.684	27.468	143.319	220.937	19.840	349.332	124.243	20.970	172.399
Nov. 1	52.736	26.867	288.316	316.042	27.380	147.008	219.295	19.728	352.526	122.601	21.106	175.715
Dec. 1	51.148	26.966	291.867	314.454	27.293	150.575	217.707	19.623	355.605	121.013	21.240	178.934

N = the mean longitude of the moon's ascending node. I = the inclination of the lunar orbit to the plane of the earth's equator. P = the mean longitude of the lunar perigee measured from the intersection of moon's orbit with the plane of the earth's equator.

TABLE 6.—*Values of N, I, and P for Greenwich midnight, beginning each month, from 1850 to 1949*—Continued.

Month.	N	I	P	N	I	P	N	I	P	N	I	P
	1870			1875			1880			1885		
	°	°	°	°	°	°	°	°	°	°	°	°
Jan. 1	119.371	21.379	182.274	22.677	28.270	33.381	285.983	25.332	251.173	189.236	18.393	86.557
Feb. 1	117.730	21.520	185.623	21.036	28.316	37.100	284.342	25.200	254.770	187.595	18.366	89.582
Mar. 1	116.247	21.649	188.660	19.553	28.355	40.462	282.806	25.074	258.128	186.112	18.345	92.312
Apr. 1	114.606	21.793	192.032	17.912	28.394	44.184	281.165	24.939	261.710	184.471	18.327	95.329
May 1	113.017	21.930	195.306	16.323	28.429	47.788	279.576	24.806	265.171	182.882	18.317	98.249
June 1	111.375	22.076	198.703	14.681	28.462	51.514	277.934	24.667	268.741	181.240	18.309	101.265
July 1	109.787	22.217	201.997	13.093	28.490	55.120	276.346	24.532	272.188	179.652	18.309	104.183
Aug. 1	108.145	22.363	205.414	11.451	28.517	58.846	274.704	24.392	275.742	178.010	18.312	107.198
Sept. 1	106.504	22.510	208.839	9.810	28.539	62.574	273.063	24.249	279.289	176.369	18.321	110.214
Oct. 1	104.915	22.652	212.166	8.221	28.558	66.183	271.474	24.111	282.711	174.780	18.335	113.135
Nov. 1	103.273	22.798	215.613	6.579	28.573	69.913	269.832	23.967	286.244	173.138	18.355	116.155
Dec. 1	101.685	22.941	218.958	4.991	28.585	73.521	268.244	23.828	289.652	171.550	18.379	119.080
	1871			1876			1881			1886		
Jan. 1	100.043	23.088	222.425	3.349	28.595	77.249	266.602	23.682	293.166	169.908	18.409	122.107
Feb. 1	98.402	23.236	225.901	1.708	28.600	80.981	264.961	23.536	296.671	168.267	18.444	125.138
Mar. 1	96.919	23.368	229.050	0.172	28.602	84.471	263.478	23.403	299.829	166.784	18.480	128.881
Apr. 1	95.278	23.515	232.543	358.531	28.600	88.203	261.837	23.257	303.316	165.143	18.525	130.920
May 1	93.689	23.656	235.934	356.942	28.596	91.811	260.248	23.114	306.682	163.554	18.574	133.867
June 1	92.047	23.802	239.446	355.300	28.587	95.541	258.606	22.968	310.151	161.912	18.627	136.923
July 1	90.459	23.942	242.853	353.712	28.576	99.149	257.018	22.823	313.498	160.324	18.685	139.886
Aug. 1	88.817	24.086	246.382	352.070	28.561	102.878	255.376	22.678	316.946	158.682	18.749	142.954
Sept. 1	87.176	24.229	249.920	350.429	28.542	106.608	253.735	22.531	320.386	157.041	18.817	146.031
Oct. 1	85.587	24.367	253.350	348.840	28.521	110.215	252.146	22.389	323.704	155.452	18.888	149.017
Nov. 1	83.945	24.508	256.904	347.198	28.495	113.943	250.504	22.242	327.123	153.810	18.965	152.113
Dec. 1	82.357	24.643	260.349	345.610	28.467	117.549	248.916	22.102	330.419	152.222	19.044	155.116
	1872			1877			1882			1887		
Jan. 1	80.715	24.782	263.916	343.967	28.435	121.273	247.273	21.956	333.815	150.579	19.131	158.230
Feb. 1	79.074	24.919	267.492	342.326	28.399	124.997	245.632	21.813	337.200	148.938	19.220	161.355
Mar. 1	77.538	25.046	271.843	340.843	28.364	128.361	244.149	21.683	340.249	147.455	19.305	164.187
Apr. 1	75.897	25.180	274.431	339.202	28.322	132.082	242.508	21.541	343.612	145.814	19.463	167.332
May 1	74.308	25.309	277.910	337.613	28.278	135.682	240.919	21.404	346.856	144.225	19.500	170.387
June 1	72.666	25.440	281.512	335.971	28.229	139.401	239.277	21.264	350.197	142.583	19.604	173.554
July 1	71.078	25.565	285.003	334.383	28.179	142.998	237.689	21.131	353.418	140.995	19.708	176.631
Aug. 1	69.436	25.693	288.615	332.741	28.123	146.708	236.047	20.994	356.735	139.353	19.819	179.822
Sept. 1	67.795	25.819	292.236	331.100	28.065	150.426	234.406	20.859	0.040	137.712	19.934	183.025
Oct. 1	66.206	25.938	295.744	329.511	28.005	154.017	232.817	20.731	3.227	136.123	20.047	186.136
Nov. 1	64.564	26.059	299.375	327.869	27.940	157.727	231.175	20.600	6.509	134.481	20.167	189.363
Dec. 1	62.976	26.175	302.893	326.281	27.874	161.314	229.587	20.475	9.672	132.893	20.307	192.461
	1873			1878			1883			1888		
Jan. 1	61.333	26.292	305.534	324.639	27.803	165.019	227.945	20.349	12.930	131.251	20.411	195.747
Feb. 1	59.692	26.408	310.180	322.998	27.729	168.720	226.304	20.226	16.174	129.610	20.538	199.021
Mar. 1	58.209	26.509	313.478	321.515	27.659	172.062	224.821	20.116	19.095	128.074	20.660	202.075
Apr. 1	56.568	26.620	317.133	319.874	27.579	175.757	223.180	19.997	22.316	126.433	20.791	205.361
May 1	54.979	26.724	321.674	318.285	27.501	179.331	221.591	19.884	25.422	124.844	20.921	208.553
June 1	53.337	26.829	324.338	316.643	27.413	183.020	219.949	19.772	28.620	123.202	21.056	211.864
July 1	51.749	26.929	327.889	315.055	27.327	186.590	218.361	19.666	31.704	121.614	21.189	215.081
Aug. 1	50.107	27.030	331.560	313.413	27.235	190.273	216.719	19.559	34.878	119.972	21.328	218.413
Sept. 1	48.466	27.127	335.236	311.772	27.141	193.954	215.078	19.457	38.041	118.331	21.468	221.760
Oct. 1	46.877	27.219	338.796	310.183	27.047	197.511	213.489	19.360	41.091	116.742	21.606	225.009
Nov. 1	45.235	27.311	342.480	308.541	26.947	201.185	211.847	19.264	44.231	115.100	21.748	228.378
Dec. 1	43.647	27.397	346.047	306.953	26.848	204.735	210.259	19.176	47.261	113.512	21.887	231.649
	1874			1879			1884			1889		
Jan. 1	42.005	27.484	349.736	305.311	26.743	208.399	208.617	19.088	50.379	111.870	22.032	235.040
Feb. 1	40.364	27.567	353.429	303.670	26.635	212.059	206.976	19.004	53.489	110.229	22.177	238.442
Mar. 1	38.881	27.642	356.768	302.187	26.536	215.361	205.440	18.930	56.390	108.746	22.309	241.524
Apr. 1	37.240	27.718	0.465	300.546	26.424	219.012	203.799	18.854	59.480	107.105	22.455	244.947
May 1	35.651	27.790	4.047	298.957	26.313	222.542	202.210	18.786	62.462	105.516	22.598	248.269
June 1	34.009	27.862	7.752	297.315	26.196	226.184	200.568	18.719	65.535	103.874	22.744	251.713
July 1	32.421	27.928	11.338	295.727	26.081	229.703	198.980	18.659	68.501	102.286	22.887	255.055
Aug. 1	30.779	27.994	15.046	294.085	25.959	233.334	197.338	18.602	71.558	100.644	23.034	258.518
Sept. 1	29.138	28.056	18.757	292.444	25.837	236.961	195.697	18.548	74.610	99.003	23.182	261.991
Oct. 1	27.549	28.113	22.349	290.855	25.715	240.464	194.108	18.504	77.555	97.414	23.324	265.361
Nov. 1	25.907	28.169	26.066	289.213	25.588	244.080	192.474	18.461	80.594	95.772	23.470	268.853
Dec. 1	24.319	28.220	29.663	287.625	25.463	247.571	190.878	18.425	83.530	94.184	23.612	272.241

TABLE 6.—*Values of N, I, and P for Greenwich midnight, beginning each month, from 1850 to 1949*—Continued.

Month.	N	I	P	N	I	P	N	I	P	N	I	P
		1890			1895			1900			1905	
	°	°	°	°	°	°	°	°	°	°	°	°
Jan. 1	92·542	23·757	275·751	355·848	28·590	131·659	259·154	23·017	346·357	162·460	18·608	173·267
Feb. 1	90·901	23·903	279·267	354·207	28·580	135·388	257·513	22·869	349·819	160·819	18·666	176·328
Mar. 1	89·418	24·033	282·454	352·724	28·568	138·754	256·030	22·737	352·938	159·336	18·723	179·055
Apr. 1	87·777	24·176	285·988	351·083	28·550	142·486	254·389	22·589	356·380	157·695	18·790	182·168
May 1	86·188	24·314	289·415	349·494	28·530	146·104	252·800	22·447	359·702	156·106	18·858	185·151
June 1	84·546	24·456	292·966	347·852	28·506	149·822	251·158	22·301	3·124	154·464	18·935	188·242
July 1	82·958	24·593	296·409	346·264	28·479	153·428	249·570	22·159	6·426	152·876	19·011	191·243
Aug. 1	81·316	24·730	299·974	344·622	28·448	157·153	247·928	22·015	9·826	151·234	19·096	194·353
Sept. 1	79·675	24·868	303·547	342·981	28·414	160·878	246·287	21·869	13·217	149·593	19·184	197·474
Oct. 1	78·086	25·001	307·010	341·392	28·377	164·482	244·698	21·730	16·487	148·004	19·273	200·504
Nov. 1	76·444	25·136	310·597	339·750	28·337	168·204	243·056	21·588	19·854	146·362	19·369	203·645
Dec. 1	74·856	25·264	313·074	338·162	28·294	171·805	241·468	21·451	23·102	144·774	19·466	206·696
		1891			1896			1901			1906	
Jan. 1	73·214	25·396	317·673	336·519	28·246	175·523	239·825	21·311	26·447	143·131	19·568	209·861
Feb. 1	71·573	25·526	321·278	334·878	28·195	179·240	238·184	21·172	29·779	141·490	19·676	213·035
Mar. 1	70·090	25·642	324·540	333·342	28·144	182·717	236·701	21·048	32·780	140·007	19·775	215·915
Apr. 1	68·449	25·769	328·157	331·701	28·086	186·430	235·060	20·913	36·089	138·366	19·887	219·113
May 1	66·860	25·889	331·663	330·112	28·028	190·023	233·471	20·784	39·280	136·777	20·000	222·205
June 1	65·218	26·011	335·291	328·470	28·963	193·733	231·829	20·652	42·567	135·135	20·119	225·441
July 1	63·630	26·128	338·808	326·882	27·899	197·321	230·241	20·527	45·735	133·547	20·237	228·570
Aug. 1	61·988	26·246	342·446	325·240	27·830	201·026	228·599	20·400	49·007	131·905	20·360	231·815
Sept. 1	60·347	26·362	346·091	323·599	27·757	204·728	226·958	20·275	52·247	130·264	20·497	235·075
Oct. 1	58·758	26·471	349·622	322·010	27·682	208·309	225·369	20·156	55·380	128·675	20·612	238·239
Nov. 1	57·116	26·583	353·276	320·368	27·603	212·007	223·727	20·036	58·606	127·033	20·743	241·521
Dec. 1	55·528	26·688	356·815	318·780	27·524	215·582	222·139	19·923	61·716	125·445	20·872	244·709
		1892			1897			1902			1907	
Jan. 1	53·885	26·794	0·478	317·138	27·439	219·273	220·497	19·809	64·918	123·803	21·007	248·015
Feb. 1	52·244	26·898	4·145	315·497	27·351	222·960	218·856	19·698	68·107	122·162	21·143	251·333
Mar. 1	50·708	26·993	7·579	314·014	27·269	226·290	217·373	19·602	70·979	120·679	21·268	254·341
Apr. 1	49·067	27·091	11·253	312·373	27·176	229·970	215·732	19·497	74·146	119·038	21·408	257·682
May 1	47·478	27·185	14·812	310·784	27·081	233·529	214·143	19·400	77·200	117·449	21·544	260·926
June 1	45·836	27·277	18·494	309·142	26·984	237·204	212·501	19·302	80·345	115·807	21·687	264·290
July 1	44·248	27·365	22·061	307·554	26·885	240·756	210·913	19·212	83·379	114·219	21·826	267·557
Aug. 1	42·606	27·453	25·748	305·912	26·782	244·421	209·271	19·123	86·502	112·577	21·969	270·942
Sept. 1	40·965	27·537	29·440	304·271	26·675	248·084	207·630	19·037	89·616	110·936	22·115	274·340
Oct. 1	39·376	27·616	33·016	302·682	26·569	251·623	206·041	18·958	92·619	109·347	22·256	277·638
Nov. 1	37·734	27·695	36·714	301·040	26·458	255·276	204·399	18·881	95·714	107·705	22·402	281·058
Dec. 1	36·146	27·768	40·295	299·452	26·348	258·807	202·811	18·811	98·699	106·117	22·544	284·376
		1893			1898			1903			1908	
Jan. 1	34·504	27·841	43·998	297·810	26·231	262·450	201·169	18·743	101·774	104·475	22·691	287·816
Feb. 1	32·863	27·909	47·703	296·169	26·113	266·087	199·528	18·679	104·842	102·834	22·838	291·266
Mar. 1	31·380	27·970	51·054	294·686	25·004	269·370	198·045	18·626	107·607	101·298	22·976	294·503
Apr. 1	29·739	28·034	54·763	293·045	25·881	272·997	196·404	18·572	110·660	99·657	23·123	297·971
May 1	28·150	28·093	58·355	291·456	25·762	276·503	194·815	18·524	113·609	98·068	23·265	301·338
June 1	26·508	28·149	62·070	289·814	25·634	280·120	193·173	18·479	116·650	96·426	23·413	304·826
July 1	24·920	28·202	65·666	288·226	25·511	283·614	191·585	18·441	119·587	94·838	23·554	308·210
Aug. 1	23·278	28·252	69·383	286·584	25·381	287·218	189·943	18·406	122·617	93·196	23·700	311·715
Sept. 1	21·637	28·299	73·103	284·943	25·238	290·818	188·302	18·377	125·644	91·555	23·845	315·231
Oct. 1	20·048	28·342	76·703	283·354	25·119	294·294	186·713	18·353	128·569	89·966	23·985	318·640
Nov. 1	18·406	28·382	80·426	281·712	24·984	297·879	185·071	18·334	131·589	88·324	24·128	322·172
Dec. 1	16·818	28·418	84·030	280·124	24·851	301·342	183·483	18·320	134·509	86·736	24·267	325·597
		1894			1899			1904			1909	
Jan. 1	15·176	28·452	87·754	278·482	24·713	304·915	181·841	18·312	137·525	85·094	24·409	329·145
Feb. 1	13·535	28·482	91·479	276·841	24·574	308·478	180·200	18·308	140·540	83·453	24·549	332·700
Mar. 1	12·052	28·507	94·846	275·358	24·448	311·692	178·664	18·310	143·361	81·970	24·676	335·918
Apr. 1	10·411	28·531	98·573	273·717	24·306	315·241	177·023	18·317	146·376	80·329	24·814	339·488
May 1	8·822	28·551	102·181	272·128	24·168	318·668	175·434	18·329	149·296	78·740	24·946	342·949
June 1	7·180	28·568	105·915	270·486	24·025	322·203	173·792	18·346	152·315	77·098	25·082	346·532
July 1	5·592	28·582	109·520	268·898	23·885	325·614	172·204	18·369	155·239	75·510	25·212	350·007
Aug. 1	3·950	28·591	113·248	267·256	23·740	329·131	170·562	18·397	158·264	73·868	25·345	353·603
Sept. 1	2·309	28·598	116·979	265·615	23·594	332·639	168·921	18·430	161·296	72·227	25·475	357·206
Oct. 1	0·720	28·602	120·589	264·026	23·452	336·026	167·332	18·466	164·229	70·638	25·599	0·699
Nov. 1	359·078	28·601	124·320	262·384	23·306	339·517	165·690	18·510	167·269	68·996	25·726	4·314
Dec. 1	357·490	28·598	127·929	260·796	23·164	342·886	164·102	18·556	170·215	67·408	25·848	7·819

TABLE 6.—*Values of N, I, and P for Greenwich midnight, beginning each month, from 1850 to 1949*—Continued.

Month.	N	I	P	N	I	P	N	I	P	N	I	P
		1910			1915			1920			1925	
Jan. 1	65°765	25°970	11°446	329°071	27°988	229°741	232°377	20°695	78°837	135°630	20°083	261°833
Feb. 1	64·124	26·092	15·077	327·430	27·922	233·449	230·736	20·565	82·115	133·989	20·204	265·064
Mar. 1	62·641	26·199	18·363	325·947	27·860	236·797	229·200	20·446	85·171	132·506	20·315	267·992
Apr. 1	61·000	26·316	22·004	324·306	27·788	240·500	227·559	20·318	88·425	130·865	20·441	271·245
May 1	59·411	26·427	25·534	322·717	27·716	244·082	225·970	20·201	91·562	129·276	20·564	274·406
June 1	57·769	26·539	29·186	321·075	27·638	247·781	224·328	20·080	94·793	127·634	20·695	277·684
July 1	56·181	26·645	32·723	319·487	27·559	251·357	222·740	19·966	97·907	126·046	20·822	280·868
Aug. 1	54·539	26·753	36·384	317·845	27·476	255·049	221·098	19·851	101·112	124·404	20·957	284·169
Sept. 1	52·898	26·865	40·049	316·204	27·389	258·739	219·457	19·739	104·307	122·763	21·093	287·483
Oct. 1	51·309	26·956	43·600	314·615	27·302	262·307	217·868	19·634	107·387	121·174	21·226	290·701
Nov. 1	49·667	27·056	47·273	312·973	27·210	265·990	216·226	19·528	110·558	119·532	21·365	294·039
Dec. 1	48·079	27·149	50·831	311·385	27·118	269·550	214·638	19·430	113·616	117·944	21·502	297·280
		1911			1916			1921			1926	
Jan. 1	46·437	27·244	54·512	309·743	27·021	273·224	212·996	19·332	116·765	116·302	21·644	300·641
Feb. 1	44·796	27·335	58·195	308·102	26·919	276·897	211·355	19·237	119·902	114·661	21·787	304·012
Mar. 1	43·313	27·415	61·526	306·566	26·823	280·328	209·872	19·155	122·727	113·178	21·916	307·068
Apr. 1	41·672	27·501	65·216	304·925	26·718	283·991	208·231	19·068	125·843	111·537	22·062	310·461
May 1	40·083	27·581	68·790	303·336	26·613	287·532	206·642	18·988	128·850	109·948	22·203	313·756
June 1	38·441	27·661	72·487	301·694	26·502	291·187	205·000	18·909	131·948	108·306	22·348	317·171
July 1	36·853	27·736	76·066	300·106	26·393	294·719	203·412	18·837	134·937	106·718	22·490	320·486
Aug. 1	35·211	27·810	79·768	298·464	26·277	298·365	201·770	18·767	138·016	105·076	22·637	323·922
Sept. 1	33·570	27·881	83·473	296·823	26·160	302·005	200·129	18·702	141·087	103·435	22·784	327·368
Oct. 1	31·981	27·946	87·060	295·234	26·044	305·523	198·540	18·643	144·052	101·846	22·927	330·712
Nov. 1	30·339	28·011	90·770	293·592	25·923	309·153	196·898	18·588	147·107	100·204	23·074	334·179
Dec. 1	28·751	28·070	94·362	292·004	25·803	312·661	195·310	18·538	150·058	98·616	23·217	337·541
		1912			1917			1922			1927	
Jan. 1	27·109	28·128	98·074	290·362	25·677	316·280	193·668	18·492	153·099	96·974	23·363	341·028
Feb. 1	25·468	28·184	101·790	288·721	25·549	319·892	192·027	18·451	156·136	95·333	23·510	344·521
Mar. 1	23·932	28·232	105·268	287·238	25·432	323·151	190·544	18·418	158·875	93·850	23·641	347·686
Apr. 1	22·291	28·281	108·986	285·597	25·301	326·751	188·903	18·386	161·903	92·209	23·787	351·196
May 1	20·702	28·325	112·586	284·008	25·173	330·230	187·314	18·361	164·829	90·620	23·927	354·602
June 1	19·060	28·367	116·308	282·366	25·038	333·819	185·672	18·340	167·850	88·978	24·072	358·131
July 1	17·472	28·404	119·911	280·778	24·906	337·284	184·084	18·325	170·771	87·390	24·210	1·553
Aug. 1	15·830	28·439	123·635	279·136	24·769	340·858	182·442	18·315	173·787	85·748	24·353	5·096
Sept. 1	14·189	28·471	127·361	277·495	24·631	344·426	180·801	18·309	176·803	84·107	24·494	8·649
Oct. 1	12·600	28·498	130·966	275·906	24·495	347·871	179·212	18·309	179·721	82·518	24·629	12·094
Nov. 1	10·958	28·524	134·695	274·264	24·354	351·424	177·570	18·315	182·737	80·876	24·768	15·662
Dec. 1	9.370	28·545	138·302	272·676	24·215	354·853	175·982	18·324	185·655	79·288	24·902	19·121
		1913			1918			1923			1928	
Jan. 1	7·728	28·563	142·032	271·034	24·073	358·389	174·340	18·340	188·674	77·646	25·037	22·701
Feb. 1	6·087	28·578	145·760	269·393	23·929	1·917	172·699	18·361	190·695	76·005	25·172	26·290
Mar. 1	4·604	28·588	149·130	267·910	23·798	5·097	171·216	18·385	194·426	74·469	25·296	29·653
Apr. 1	2·963	28·596	152·858	266·269	23·652	8·608	169·575	18·415	197·453	72·828	25·427	33·253
May 1	1·374	28·600	156·468	264·680	23·510	11·999	167·986	18·450	200·387	71·239	25·552	36·744
June 1	359·732	28·602	160·199	263·038	23·364	15·493	166·344	18·491	203·425	69·597	25·680	40·356
July 1	358·144	28·599	163·809	261·450	23·223	18·866	164·756	18·534	206·368	68·009	25·802	43·859
Aug. 1	356·502	28·594	167·539	259·808	23·075	22·341	163·114	18·587	209·417	66·367	25·926	47·483
Sept. 1	354·861	28·584	171·268	258·167	22·929	25·807	161·473	18·643	212·473	64·726	26·047	51·113
Oct. 1	353·272	28·572	174·877	256·578	22·785	29·152	159·884	18·701	215·437	63·137	26·163	54·631
Nov. 1	351·630	28·557	178·607	254·936	22·638	32·599	158·242	18·772	218·508	61·495	26·280	58·272
Dec. 1	350·042	28·537	182·215	453·348	22·496	35·924	156·654	18·834	221·488	59·907	26·393	61·800
		1914			1919			1924			1929	
Jan. 1	348·400	28·514	185·944	251·706	22·350	39·350	155·011	18·908	224·575	58·264	26·505	65·450
Feb. 1	346·759	28·488	189·670	250·065	22·204	42·765	153·370	18·987	227·673	56·623	26·616	69·104
Mar. 1	345·276	28·461	193·036	248·582	22·072	45·841	151·834	19·065	230·580	55·140	26·714	72·409
Apr. 1	343·635	28·428	196·761	246·941	21·927	49·235	150·193	19·152	233·696	53·499	26·819	76·072
May 1	342·046	28·393	200·364	245·352	21·788	52·508	148·604	19·239	236·714	51·910	26·919	79·622
June 1	340·404	28·354	204·088	243·710	21·645	55·881	146·962	19·334	239·860	50·268	27·020	83·294
July 1	338·816	28·312	207·689	242·122	21·507	59·133	145·374	19·429	242·907	48·680	27·114	86·851
Aug. 1	337·174	28·265	211·408	240·480	21·366	62·482	143·732	19·531	246·067	47·038	27·210	90·529
Sept. 1	335·533	28·216	215·127	238·839	21·228	65·820	142·091	19·636	249·238	45·397	27·302	94·212
Oct. 1	333·944	28·164	218·723	237·250	21·094	69·038	140·502	19·742	252·318	43·808	27·388	97·779
Nov. 1	332·302	28·108	222·438	235·608	20·958	72·353	138·860	19·854	255·513	42·166	27·475	101·469
Dec. 1	330·714	28·051	226·031	234·020	20·828	75·548	137·272	19·965	258·616	40·578	27·556	105·042

TABLE 6.—*Values of N, I, and P for Greenwich midnight, beginning each month, from 1850 to 1949*—Continued.

Month.	*N*	*I*	*P*	*N*	*I*	*P*	*N*	*I*	*P*	*N*	*I*	*P*
		1930			1935			1940			1945	
Jan. 1	38°·936	27°·637	108°·738	302°·242	26°·539	327°·333	205°·548	18°·935	168°·280	108°·801	22°·305	353°·506
Feb. 1	37·295	27·715	112·436	300·601	26·428	330·985	203·907	18·859	171·372	107·160	22·451	356·928
Mar. 1	35·812	27·783	115·780	299·118	26·324	334·280	202·371	18·793	174·256	105·677	22·583	0·029
Apr. 1	34·171	27·855	119·483	297·477	26·207	337·922	200·730	18·725	177·329	104·036	22·731	3·470
May 1	32·582	27·922	123·069	295·888	26·093	341·441	199·141	18·665	180·296	102·447	22·873	6·811
June 1	30·940	28·003	126·778	294·246	25·971	345·074	197·499	18·607	183·355	100·805	23·020	10·275
July 1	29·352	28·048	130·368	292·658	25·853	348·584	195·911	18·556	186·308	99·217	23·162	13·634
Aug. 1	27·710	28·107	134·081	291·016	25·727	352·204	194·269	18·509	189·352	97·575	23·309	17·115
Sept. 1	26·069	28·164	137·796	289·375	25·600	355·820	192·628	18·465	192·392	95·934	23·456	20·606
Oct. 1	24·480	28·216	141·392	287·786	25·476	359·313	191·039	18·429	195·327	94·345	23·598	23·993
Nov. 1	22·838	28·265	145·111	286·144	25·346	2·916	189·397	18·396	198·356	92·703	23·743	27·502
Dec. 1	21·250	28·310	148·710	284·556	25·217	6·396	187·809	18·369	201·284	91·115	23·884	30·905
		1931			1936			1941			1946	
Jan. 1	19·608	28·353	152·432	282·914	25·083	9·987	186·167	18·346	204·306	89·473	24·028	34·431
Feb. 1	17·967	28·393	156·154	281·273	24·947	13·570	184·526	18·328	207·324	87·832	24·171	37·965
Mar. 1	16·484	28·425	159·519	279.737	24·819	16·916	183·043	18·317	210·050	86·349	24·300	41·164
Apr. 1	14·843	28·459	163·243	278·096	24·681	20·486	181·402	18·311	213·066	84·708	24·442	44·714
May 1	13·254	28·487	166·849	276·507	24·546	23·934	179·813	18·308	215·983	83·119	24·578	48·155
June 1	11·612	28·514	170·576	274·865	24·405	27·489	178·171	18·312	218·998	81·477	24·717	51·721
July 1	10·024	28·537	174·184	273·277	24·268	30·922	176·583	18·320	221·917	79·889	24·850	55·177
Aug. 1	8·382	28·556	177·912	271·635	24·125	34·462	174·941	18·333	224·934	78·247	24·987	58·755
Sept. 1	6·741	28·572	181·642	269·994	23·981	37·992	173·300	18·353	227·955	76·606	25·122	62·340
Oct. 1	5·152	28·584	185·250	268·405	23·842	41·402	171·711	18·377	230·879	75·017	25·252	65·817
Nov. 1	3·510	28·594	188·980	266·763	23·696	44·917	170·069	18·406	233·906	73·375	25·384	68·416
Dec. 1	1·922	28·599	192·674	265·175	23·555	48·310	168·481	18·440	236·839	71·787	25·511	72·904
		1932			1937			1942			1947	
Jan. 1	0·280	28·602	196·319	263·533	23·409	51·807	166·839	18·479	239·873	70·145	25·638	76·515
Feb. 1	358·639	28·600	200·050	261·892	23·261	55·294	165·198	18·523	242·914	68·504	25·765	80·132
Mar. 1	357·103	28·596	203·539	260·409	23·130	58·437	163·715	18·568	245·666	67·021	25·876	83·404
Apr. 1	355·462	28·589	207·268	258·768	22·983	61·906	162·074	18·621	248·719	65·380	25·999	87·031
May 1	353·873	28·577	210·877	257·179	22·839	65·254	160·485	18·679	251·680	63·791	26·117	90·547
June 1	352·231	28·562	214·607	255·537	22·692	68·704	158·843	18·742	254·749	62·149	26·234	94·186
July 1	350·643	28·545	218·216	253·949	22·550	72·033	157·255	18·808	257·725	60·561	26·347	97·712
Aug. 1	349·001	28·523	221·943	252·307	22·403	75·463	155·613	18·881	260·809	58·919	26.461	101·360
Sept. 1	347·360	28·498	225·671	250·666	22·257	78·882	153·972	18·958	263·904	57·278	26·570	105·013
Oct. 1	345·771	28·470	229·277	249·077	22·116	82·180	152·383	19·036	266·907	55·689	26·678	108·552
Nov. 1	344·129	28·439	233·003	247·435	21·970	85·579	150·741	19·122	270·021	54·047	26·784	112·215
Dec. 1	342·541	28·404	236·607	245·847	21·832	88·855	149·153	19·208	273·043	52·459	26·885	115·762
		1933			1938			1943			1948	
Jan. 1	340·899	28·366	240·330	244·205	21·688	92·231	147·510	19·302	276·178	50·816	26·986	119·433
Feb. 1	339·258	28·324	244·052	242·564	21·546	95·594	145·869	19·399	279·322	49·175	27·085	123·107
Mar. 1	337·775	28·283	247·412	241·081	21·418	98·623	144·386	19·490	282·173	47·639	27·175	126·548
Apr. 1	336·134	28·234	251·131	239·440	21·278	101·965	142·745	19·594	285·339	45·998	27·268	130·228
May 1	334·545	28·184	254·728	237·851	21·144	105·187	141·156	19·698	288·415	44·409	27·356	133·795
June 1	332·903	28·129	258·444	236·209	21·008	108·506	139·514	19·808	291·605	42·767	27·444	137·494
July 1	331·315	28·072	262·037	234·621	20·877	111·705	137·926	19·920	294·703	41·179	27·526	141·055
Aug. 1	329·673	28·012	265·748	232·979	20·744	115·000	136·284	20·036	297·916	39·537	27·608	144·750
Sept. 1	328·032	27·946	269·458	231·338	20·613	118·283	134·643	20·155	301·141	37·896	27·687	148·447
Oct. 1	326·443	27·881	273·045	229·749	20·488	121·447	133·054	20·274	304·275	36·307	27·761	152·028
Nov. 1	324·801	27·810	276·750	228·107	20·361	124·706	131·412	20·399	307·524	34·665	27·834	155·731
Dec. 1	323·213	27·739	280·333	226·519	20·241	127·847	129·824	20·522	310·681	33·077	27·901	159·316
		1934			1939			1944			1949	
Jan. 1	321·570	27·661	284·032	224·876	20·120	131·082	128·182	20·651	313·946	31·435	27·968	163·025
Feb. 1	319·929	27·581	287·728	223·235	20·001	134·303	126·541	20·783	317·241	29·794	28·032	166·737
Mar. 1	318·446	27·507	291·065	221·752	19·896	137·203	125·005	20·908	320·325	28·311	28·086	170·088
Apr. 1	316·805	27·421	294·755	220·111	19·783	140·402	123·364	21·043	323·634	26·670	28·143	173·801
May 1	315·216	27·336	298·323	218·522	19·676	143·487	121·775	21·176	326·849	25·081	28·196	177·397
June 1	313·574	27·244	302·007	216·880	19·569	146·662	120·133	21·315	330·182	23·439	28·247	181·115
July 1	311·986	27·153	305·570	215·292	19·470	149·725	118·545	21·450	333·419	21·851	28·293	184·714
Aug. 1	310·344	27·057	309·246	213·650	19·370	152·877	116·903	21·592	336·775	20·209	28·338	188·433
Sept. 1	308·703	26·957	312·919	212·009	19·274	156·019	115·262	21·734	340·143	18·568	28·378	192·157
Oct. 1	307·114	26·858	316·470	210·420	19·185	159·049	113·673	21·873	343·413	16·979	28·414	195·760
Nov. 1	305·472	26·753	320·136	208·778	19·095	162·179	112·031	22·018	346·803	15·337	28·449	199·486
Dec. 1	303·884	26·649	323·678	207·190	19·015	165·180	110·443	22·158	350·094	13·749	28·479	203·091

TABLE 7.—*Values of I, ν, ξ, ν', and $2\nu''$, corresponding to each half degree of N.*

N	I	ν	ξ	ν'	$2\nu''$	N	N	I	ν	ξ	ν'	$2\nu''$	N
°	°	°	°	°	°	°	°	°	°	°	°	°	°
0.0	28.602	0.000	0.000	0.000	0.000	360.0	30.0	28.024	5.478	4.939	3.903	8.249	330.0
0.5	28.602	0.094	0.084	0.067	0.143	359.5	30.5	28.005	5.564	5.017	3.964	8.377	329.5
1.0	28.601	0.188	0.169	0.134	0.285	359.0	31.0	27.985	5.651	5.095	4.025	8.504	329.0
1.5	28.600	0.281	0.253	0.201	0.427	358.5	31.5	27.965	5.736	5.173	4.085	8.631	328.5
2.0	28.599	0.375	0.337	0.268	0.569	358.0	32.0	27.945	5.822	5.251	4.146	8.757	328.0
2.5	28.598	0.468	0.421	0.335	0.711	357.5	32.5	27.925	5.907	5.328	4.206	8.883	327.5
3.0	28.596	0.562	0.506	0.402	0.853	357.0	33.0	27.904	5.992	5.406	4.266	9.008	327.0
3.5	28.594	0.656	0.590	0.469	0.995	356.5	33.5	27.884	6.077	5.482	4.326	9.133	326.5
4.0	28.591	0.749	0.674	0.536	1.137	356.0	34.0	27.862	6.162	5.559	4.386	9.257	326.0
4.5	28.589	0.843	0.758	0.603	1.279	355.5	34.5	27.841	6.246	5.636	4.445	9.381	325.5
5.0	28.585	0.936	0.842	0.670	1.421	355.0	35.0	27.819	6.330	5.712	4.504	9.504	325.0
5.5	28.582	1.030	0.926	0.737	1.563	354.5	35.5	27.797	6.414	5.788	4.563	9.626	324.5
6.0	28.578	1.123	1.010	0.804	1.705	354.0	36.0	27.775	6.497	5.864	4.621	9.748	324.0
6.5	28.574	1.217	1.094	0.871	1.847	353.5	36.5	27.752	6.580	5.939	4.679	9.869	323.5
7.0	28.570	1.310	1.178	0.938	1.989	353.0	37.0	27.729	6.663	6.015	4.737	9.989	323.0
7.5	28.565	1.403	1.262	1.004	2.130	352.5	37.5	27.706	6.745	6.090	4.797	10.109	322.5
8.0	28.560	1.496	1.346	1.070	2.271	352.0	38.0	27.682	6.828	6.164	4.855	10.228	322.0
8.5	28.555	1.590	1.430	1.137	2.413	351.5	38.5	27.658	6.909	6.239	4.912	10.347	321.5
9.0	28.549	1.683	1.514	1.203	2.554	351.0	39.0	27.634	6.991	6.313	4.969	10.465	321.0
9.5	28.543	1.776	1.598	1.270	2.695	350.5	39.5	27.610	7.072	6.387	5.026	10.583	320.5
10.0	28.537	1.869	1.681	1.337	2.836	350.0	40.0	27.585	7.153	6.461	5.083	10.700	320.0
10.5	28.530	1.962	1.765	1.403	2.977	349.5	40.5	27.560	7.234	6.534	5.139	10.816	319.5
11.0	28.523	2.054	1.848	1.469	3.117	349.0	41.0	27.535	7.314	6.608	5.195	10.932	319.0
11.5	28.516	2.147	1.932	1.535	3.257	348.5	41.5	27.510	7.394	6.680	5.251	11.047	318.5
12.0	28.508	2.240	2.015	1.601	3.397	348.0	42.0	27.484	7.473	6.753	5.307	11.161	318.0
12.5	28.500	2.332	2.099	1.667	3.536	347.5	42.5	27.458	7.553	6.825	5.362	11.275	317.5
13.0	28.492	2.424	2.182	1.733	3.676	347.0	43.0	27.432	7.631	6.897	5.416	11.388	317.0
13.5	28.483	2.517	2.265	1.799	3.816	346.5	43.5	27.405	7.710	6.969	5.471	11.500	316.5
14.0	28.475	2.609	2.348	1.864	3.955	346.0	44.0	27.378	7.788	7.040	5.526	11.612	316.0
14.5	28.465	2.701	2.431	1.930	4.094	345.5	44.5	27.351	7.866	7.111	5.580	11.723	315.5
15.0	28.456	2.793	2.514	1.996	4.233	345.0	45.0	27.324	7.943	7.182	5.634	11.833	315.0
15.5	28.446	2.885	2.596	2.061	4.372	344.5	45.5	27.296	8.020	7.253	5.687	11.942	314.5
16.0	28.436	2.977	2.679	2.127	4.510	334.0	46.0	27.268	8.097	7.323	5.740	12.051	314.0
16.5	28.425	3.068	2.762	2.192	4.648	343.5	46.5	27.240	8.173	7.392	5.793	12.159	313.5
17.0	28.414	3.160	2.844	2.257	4.786	343.0	47.0	27.212	8.249	7.462	5.846	12.266	313.0
17.5	28.403	3.251	2.927	2.322	4.923	342.5	47.5	27.183	8.324	7.531	5.898	12.372	312.5
18.0	28.392	3.342	3.009	2.387	5.060	342.0	48.0	27.154	8.399	7.600	5.950	12.477	312.0
18.5	28.380	3.433	3.091	2.452	5.197	341.5	48.5	27.125	8.474	7.668	6.002	12.582	311.5
19.0	28.368	3.524	3.173	2.517	5.334	341.0	49.0	27.095	8.548	7.736	6.053	12.686	311.0
19.5	28.356	3.615	3.255	2.581	5.471	340.5	49.5	27.066	8.622	7.804	6.104	12.789	310.5
20.0	28.343	3.705	3.337	2.646	5.607	340.0	50.0	27.036	8.695	7.871	6.154	12.892	310.0
20.5	28.330	3.796	3.418	2.710	5.743	339.5	50.5	27.006	8.768	7.938	6.205	12.994	309.5
21.0	28.317	3.886	3.500	2.774	5.878	339.0	51.0	26.975	8.841	8.005	6.255	13.095	309.0
21.5	28.303	3.976	3.581	2.838	6.013	338.5	51.5	26.944	8.913	8.071	6.305	13.195	308.5
22.0	28.289	4.066	3.662	2.902	6.148	338.0	52.0	26.913	8.985	8.137	6.354	13.294	308.0
22.5	28.275	4.156	3.743	2.966	6.282	337.5	52.5	26.882	9.056	8.203	6.403	13.392	307.5
23.0	28.260	4.245	3.824	3.030	6.416	337.0	53.0	26.851	9.127	8.268	6.452	13.489	307.0
23.5	28.245	4.335	3.905	3.093	6.550	336.5	53.5	26.819	9.197	8.333	6.500	13.586	306.5
24.0	28.230	4.424	3.985	3.156	6.683	336.0	54.0	26.787	9.267	8.397	6.547	13.682	306.0
24.5	28.215	4.513	4.066	3.219	6.816	335.5	54.5	26.755	9.336	8.461	6.594	13.777	305.5
25.0	28.199	4.602	4.146	3.282	6.948	335.0	55.0	26.723	9.405	8.525	6.641	13.871	305.0
25.5	28.183	4.690	4.226	3.345	7.080	334.5	55.5	26.690	9.474	8.588	6.688	13.964	304.5
26.0	28.166	4.779	4.306	3.408	7.212	334.0	56.0	26.657	9.542	8.651	6.735	14.057	304.0
26.5	28.149	4.867	4.386	3.471	7.343	333.5	56.5	26.624	9.609	8.713	6.781	14.148	303.5
27.0	28.132	4.955	4.466	3.534	7.474	333.0	57.0	26.591	9.676	8.775	6.827	14.238	303.0
27.5	28.115	5.043	4.545	3.596	7.604	332.5	57.5	26.557	9.743	8.836	6.872	14.327	302.5
28.0	28.097	5.130	4.624	3.658	7.734	332.0	58.0	26.523	9.809	8.897	6.917	14.415	302.0
28.5	28.079	5.218	4.703	3.720	7.863	331.5	58.5	26.489	9.874	8.958	6.961	14.502	301.5
29.0	28.061	5.305	4.782	3.781	7.992	331.0	59.0	26.455	9.939	9.018	7.005	14.588	301.0
29.5	28.043	5.391	4.861	3.842	8.121	330.5	59.5	26.421	10.004	9.078	7.048	14.673	300.5
30.0	28.024	5.478	4.939	3.903	8.249	330.0	60.0	26.386	10.068	9.137	7.091	14.757	300.0

See Tables 1 and 5 for explanation of symbols. I is always positive; ν, ξ, ν', and $2\nu''$ are positive when N is between 0° and 180°, and negative when N is between 180° and 360°. N is the longitude of the moon's ascending node.

$u(J_1) = -\nu$; $u(K_1) = -\nu'$; $u(K_2) = -2\nu''$; $u(L_2) = 2\xi - 2\nu - R$; see Table 8 for R; $u(M_1) = \xi - \nu + Q$; see Table 9 for Q; $u(M_2) = 2\xi - 2\nu$; $u(M_3) = 3\xi - 3\nu$; $u(M_4) = 4\xi - 4\nu$; $u(M_6) = 6\xi - 6\nu$; $u(M_8) = 8\xi - 8\nu$; $u(N_2) = 2\xi - 2\nu = u(M_2)$; $u(2N) = 2\xi - 2\nu = u(M_2)$; $u(O_1) = 2\xi - \nu$; $u(OO) = -2\xi - \nu$; $u(P_1) = 0$; $u(Q_1) = 2\xi - \nu = u(O_1)$; $u(R_2) = 0$; $u(S_{1,2,3,4}) = 0$; $u(T_2) = 0$; $u(\lambda_2) = u(\mu_2) = u(\nu_2) = 2\xi - 2\nu = u(M_2)$; $u(MK) = 2\xi - 2\nu - \nu' = u(M_2) + u(K_1)$; $u(2MK) = 4\xi - 4\nu + \nu' = u(M_4) - u(K_1)$; $u(MN) = 4\xi - 4\nu = u(M_4)$; $u(MS) = 2\xi - 2\nu = u(M_2)$; $u(2MS) = 4\xi - 4\nu = u(M_4)$; $u(2SM) = -2\xi + 2\nu = -u(M_2)$; $u(Mf) = -2\xi$; $u(MSf) = -2\xi + 2\nu = -u(M_2)$; $u(Mm) = 0$; $u(Sa) = 0$; $u(Ssa) = 0$.

TABLE 7.—*Values of I, ν, ξ, ν', and $2\nu''$, corresponding to each half degree of N*—Continued.

N	I	ν	ξ	ν'	$2\nu''$	N	N	I	ν	ξ	ν'	$2\nu''$	N
°	°	°	°	°	°	°	°	°	°	°	°	°	°
60.0	26.386	10.068	9.137	7.091	14.757	300.0	90.0	23.982	12.751	11.681	8.797	17.783	270.0
60.5	26.351	10.131	9.196	7.134	14.841	299.5	90.5	23.938	12.772	11.704	8.808	17.795	269.5
61.0	26.316	10.194	9.255	7.177	14.924	299.0	91.0	23.894	12.793	11.725	8.819	17.805	269.0
61.5	26.280	10.256	9.313	7.219	15.006	298.5	91.5	23.850	12.814	11.745	8.829	17.814	268.5
62.0	26.245	10.318	9.370	7.261	15.087	298.0	92.0	23.806	12.833	11.765	8.839	17.822	268.0
62.5	26.209	10.379	9.427	7.302	15.167	297.5	92.5	23.761	12.851	11.784	8.848	17.829	267.5
63.0	26.173	10.440	9.484	7.343	15.246	297.0	93.0	23.717	12.869	11.802	8.856	17.834	267.0
63.5	26.137	10.500	9.539	7.383	15.324	296.5	93.5	23.673	12.886	11.819	8.863	17.837	266.5
64.0	26.101	10.560	9.595	7.423	15.401	296.0	94.0	23.628	12.901	11.835	8.870	17.839	266.0
64.5	26.064	10.619	9.650	7.462	15.477	295.5	94.5	23.584	12.916	11.851	8.876	17.840	265.5
65.0	26.027	10.677	9.705	7.501	15.551	295.0	95.0	23.539	12.930	11.866	8.882	17.840	265.0
65.5	25.990	10.735	9.759	7.539	15.624	294.5	95.5	23.495	12.943	11.880	8.887	17.838	264.5
66.0	25.953	10.793	9.812	7.577	15.696	294.0	96.0	23.450	12.955	11.893	8.891	17.835	264.0
66.5	25.916	10.849	9.865	7.614	15.767	293.5	96.5	23.406	12.966	11.905	8.895	17.830	263.5
67.0	25.878	10.906	9.918	7.651	15.837	293.0	97.0	23.361	12.976	11.916	8.898	17.824	263.0
67.5	25.841	10.961	9.970	7.688	15.906	292.5	97.5	23.316	12.985	11.927	8.900	17.816	262.5
68.0	25.803	11.016	10.021	7.724	15.974	292.0	98.0	23.271	12.994	11.936	8.902	17.807	262.0
68.5	25.765	11.070	10.072	7.760	16.042	291.5	98.5	23.227	13.001	11.945	8.903	17.796	261.5
69.0	25.726	11.124	10.123	7.795	16.109	291.0	99.0	23.182	13.007	11.953	8.903	17.784	261.0
69.5	25.688	11.177	10.173	7.830	16.174	290.5	99.5	23.137	13.013	11.960	8.902	17.770	260.5
70.0	25.649	11.230	10.222	7.864	16.238	290.0	100.0	23.092	13.017	11.966	8.901	17.755	260.0
70.5	25.610	11.282	10.271	7.898	16.300	289.5	100.5	23.047	13.021	11.971	8.899	17.739	259.5
71.0	25.571	11.333	10.319	7.932	16.361	289.0	101.0	23.003	13.023	11.975	8.896	17.721	259.0
71.5	25.532	11.383	10.366	7.965	16.421	288.5	101.5	22.958	13.024	11.979	8.892	17.702	258.5
72.0	25.493	11.433	10.414	7.997	16.480	288.0	102.0	22.913	13.025	11.981	8.888	17.681	258.0
72.5	25.453	11.482	10.460	8.029	16.538	287.5	102.5	22.868	13.024	11.983	8.883	17.659	257.5
73.0	25.413	11.531	10.506	8.061	16.594	287.0	103.0	22.823	13.023	11.983	8.878	17.636	257.0
73.5	25.374	11.579	10.551	8.092	16.649	286.5	103.5	22.778	13.020	11.983	8.872	17.611	256.5
74.0	25.334	11.626	10.596	8.122	16.703	286.0	104.0	22.734	13.017	11.982	8.865	17.584	256.0
74.5	25.293	11.673	10.640	8.152	16.756	285.5	104.5	22.689	13.012	11.979	8.858	17.556	255.5
75.0	25.253	11.719	10.684	8.181	16.808	285.0	105.0	22.644	13.006	11.976	8.850	17.526	255.0
75.5	25.213	11.764	10.726	8.209	16.859	284.5	105.5	22.599	13.000	11.972	8.841	17.495	254.5
76.0	25.172	11.809	10.769	8.237	16.909	284.0	106.0	22.554	12.992	11.967	8.831	17.463	254.0
76.5	25.131	11.852	10.811	8.264	16.958	283.5	106.5	22.510	12.983	11.961	8.821	17.430	253.5
77.0	25.090	11.895	10.852	8.291	17.005	283.0	107.0	22.465	12.974	11.954	8.810	17.395	253.0
77.5	25.049	11.938	10.892	8.318	17.051	282.5	107.5	22.420	12.963	11.946	8.799	17.358	252.5
78.0	25.008	11.980	10.932	8.344	17.096	282.0	108.0	22.376	12.951	11.937	8.787	17.320	252.0
78.5	24.966	12.021	10.971	8.370	17.140	281.5	108.5	22.331	12.938	11.927	8.774	17.281	251.5
79.0	24.925	12.061	11.009	8.395	17.183	281.0	109.0	22.287	12.924	11.916	8.760	17.240	251.0
79.5	24.883	12.100	11.047	8.420	17.225	280.5	109.5	22.242	12.909	11.904	8.745	17.197	250.5
80.0	24.841	12.139	11.084	8.444	17.265	280.0	110.0	22.198	12.892	11.891	8.729	17.153	250.0
80.5	24.800	12.177	11.121	8.467	17.303	279.5	110.5	22.153	12.875	11.877	8.713	17.107	249.5
81.0	24.757	12.214	11.157	8.490	17.340	279.0	111.0	22.109	12.857	11.862	8.696	17.060	249.0
81.5	24.715	12.251	11.192	8.512	17.376	278.5	111.5	22.065	12.837	11.846	8.678	17.011	248.5
82.0	24.673	12.287	11.227	8.533	17.410	278.0	112.0	22.021	12.817	11.829	8.659	16.961	248.0
82.5	24.631	12.322	11.261	8.554	17.443	277.5	112.5	21.976	12.795	11.811	8.639	16.910	247.5
83.0	24.588	12.356	11.294	8.574	17.475	277.0	113.0	21.932	12.772	11.792	8.619	16.858	247.0
83.5	24.545	12.389	11.326	8.594	17.505	276.5	113.5	21.888	12.748	11.772	8.599	16.805	246.5
84.0	24.503	12.422	11.358	8.613	17.534	276.0	114.0	21.845	12.723	11.750	8.578	16.750	246.0
84.5	24.460	12.454	11.389	8.631	17.562	275.5	114.5	21.801	12.697	11.728	8.556	16.693	245.5
85.0	24.417	12.485	11.419	8.649	17.589	275.0	115.0	21.757	12.670	11.705	8.533	16.635	245.0
85.5	24.374	12.515	11.449	8.666	17.614	274.5	115.5	21.713	12.642	11.681	8.509	16.576	244.5
86.0	24.331	12.545	11.478	8.683	17.638	274.0	116.0	21.670	12.612	11.655	8.484	16.515	244.0
86.5	24.287	12.573	11.506	8.700	17.661	273.5	116.5	21.627	12.581	11.629	8.458	16.453	243.5
87.0	24.244	12.601	11.533	8.716	17.683	273.0	117.0	21.583	12.550	11.601	8.432	16.389	243.0
87.5	24.200	12.628	11.560	8.731	17.703	272.5	117.5	21.540	12.517	11.572	8.405	16.323	242.5
88.0	24.157	12.654	11.586	8.745	17.722	272.0	118.0	21.497	12.483	11.543	8.378	16.256	242.0
88.5	24.113	12.680	11.611	8.759	17.739	271.5	118.5	21.454	12.447	11.512	8.350	16.188	241.5
89.0	24.070	12.704	11.635	8.772	17.755	271.0	119.0	21.411	12.411	11.480	8.321	16.118	241.0
89.5	24.026	12.728	11.659	8.785	17.770	270.5	119.5	21.368	12.373	11.447	8.291	16.047	240.5
90.0	23.982	12.751	11.681	8.797	17.783	270.0	120.0	21.326	12.335	11.413	8.260	15.975	240.0

TABLE 7.—*Values of I, ν, ξ, ν', and $2\nu''$, corresponding to each half degree of N*—Continued.

N	I	ν	ξ	ν'	$2\nu''$	N	N	I	ν	ξ	ν'	$2\nu''$	N
°	°	°	°	°	°	°	°	°	°	°	°	°	°
120·0	21·326	12·335	11·413	8·260	15·975	240·0	150·0	19·162	7·854	7·324	5·096	9·392	210·0
120·5	21·283	12·295	11·378	8·229	15·902	239·5	150·5	19·135	7·745	7·223	5·024	9·251	209·5
121·0	21·241	12·254	11·342	8·197	15·827	239·0	151·0	19·108	7·635	7·120	4·951	9·109	209·0
121·5	21·199	12·211	11·304	8·165	15·751	238·5	151·5	19·082	7·523	7·017	4·878	8·966	208·5
122·0	21·157	12·168	11·266	8·132	15·673	238·0	152·0	19·056	7·411	6·913	4·805	8·823	208·0
122·5	21·115	12·123	11·226	8·098	15·594	237·5	152·5	19·030	7·298	6·809	4·730	8·679	207·5
123·0	21·073	12·078	11·186	8·063	15·513	237·0	153·0	19·005	7·184	6·703	4·655	8·534	207·0
123·5	21·032	12·031	11·144	8·028	15·431	236·5	153·5	18·981	7·069	6·596	4·579	8·389	206·5
124·0	20·990	11·983	11·101	7·992	15·347	236·0	154·0	18·956	6·953	6·488	4·502	8·243	206·0
124·5	20·949	11·933	11·057	7·955	15·262	235·5	154·5	18·933	6·836	6·380	4·424	8·096	205·5
125·0	20·908	11·883	11·012	7·917	15·176	235·0	155·0	18·909	6·718	6·270	4·346	7·948	205·0
125·5	20·867	11·831	10·965	7·878	15·089	234·5	155·5	18·886	6·599	6·160	4·268	7·800	204·5
126·0	20·826	11·778	10·918	7·839	15·001	234·0	156·0	18·863	6·480	6·049	4·189	7·651	204·0
126·5	20·786	11·724	10·869	7·799	14·911	233·5	156·5	18·841	6·359	5·937	4·110	7·502	203·5
127·0	20·746	11·669	10·820	7·758	14·820	233·0	157·0	18·819	6·238	5·824	4·030	7·352	203·0
127·5	20·705	11·612	10·769	7·716	14·728	232·5	157·5	18·798	6·116	5·710	3·950	7·201	202·5
128·0	20·666	11·555	10·717	7·674	14·635	232·0	158·0	18·777	5·993	5·596	3·869	7·050	202·0
128·5	20·626	11·496	10·664	7·631	14·540	231·5	158·5	18·756	5·869	5·480	3·787	6·898	201·5
129·0	20·586	11·436	10·610	7·587	14·444	231·0	159·0	18·736	5·744	5·364	3·705	6·745	201·0
129·5	20·547	11·374	10·554	7·542	14·347	230·5	159·5	18·716	5·618	5·247	3·623	6·592	200·5
130·0	20·508	11·312	10·498	7·496	14·248	230·0	160·0	18·697	5·492	5·130	3·541	6·438	200·0
130·5	20·469	11·248	10·440	7·449	14·148	229·5	160·5	18·678	5·365	5·011	3·458	6·284	199·5
131·0	20·430	11·184	10·382	7·401	14·048	229·0	161·0	18·660	5·237	4·892	3·375	6·130	199·0
131·5	20·392	11·118	10·322	7·353	13·946	228·5	161·5	18·642	5·109	4·773	3·291	5·975	198·5
132·0	20·353	11·050	10·261	7·304	13·842	228·0	162·0	18·624	4·980	4·652	3·207	5·819	198·0
132·5	20·315	10·982	10·199	7·255	13·737	227·5	162·5	18·607	4·850	4·531	3·122	5·663	197·5
133·0	20·278	10·912	10·135	7·205	13·631	227·0	163·0	18·591	4·719	4·409	3·037	5·506	197·0
133·5	20·240	10·841	10·071	7·154	13·524	226·5	163·5	18·575	4·588	4·287	2·952	5·349	196·5
134·0	20·203	10·769	10·005	7·102	13·416	226·0	164·0	18·559	4·456	4·164	2·866	5·192	196·0
134·5	20·166	10·696	9·939	7·050	13·306	225·5	164·5	18·544	4·323	4·040	2·780	5·034	195·5
135·0	20·129	10·622	9·871	6·998	13·195	225·0	165·0	18·529	4·190	3·916	2·694	4·875	195·0
135·5	20·092	10·546	9·802	6·945	13·083	224·5	165·5	18·515	4·056	3·791	2·607	4·716	194·5
136·0	20·056	10·469	9·732	6·891	12·970	224·0	166·0	18·501	3·922	3·665	2·520	4·557	194·0
136·5	20·020	10·391	9·660	6·837	12·856	223·5	166·5	18·487	3·787	3·539	2·433	4·397	193·5
137·0	19·984	10·312	9·588	6·782	12·741	223·0	167·0	18·475	3·651	3·413	2·345	4·238	193·0
137·5	19·949	10·232	9·515	6·727	12·624	222·5	167·5	18·462	3·515	3·286	2·257	4·078	192·5
138·0	19·913	10·150	9·440	6·671	12·506	222·0	168·0	18·450	3·379	3·158	2·169	3·918	192·0
138·5	19·878	10·068	9·364	6·614	12·387	221·5	168·5	18·439	3·242	3·030	2·081	3·757	191·5
139·0	19·844	9·984	9·287	6·556	12·268	221·0	169·0	18·428	3·104	2·902	1·992	3·596	191·0
139·5	19·809	9·899	9·209	6·498	12·148	220·5	169·5	18·417	2·966	2·773	1·903	3·435	190·5
140·0	19·775	9·813	9·130	6·439	12·027	220·0	170·0	18·407	2·828	2·644	1·814	3·273	190·0
140·5	19·742	9·725	9·050	6·379	11·904	219·5	170·5	18·398	2·689	2·514	1·725	3·110	189·5
141·0	19·708	9·637	8·969	6·318	11·780	219·0	171·0	18·388	2·550	2·384	1·635	2·948	189·0
141·5	19·675	9·547	8·887	6·256	11·655	218·5	171·5	18·380	2·410	2·254	1·546	2·786	188·5
142·0	19·642	9·457	8·803	6·193	11·529	218·0	172·0	18·372	2·270	2·123	1·456	2·623	188·0
142·5	19·610	9·365	8·719	6·129	11·402	217·5	172·5	18·364	2·130	1·992	1·365	2·460	187·5
143·0	19·577	9·272	8·633	6·064	11·274	217·0	173·0	18·357	1·989	1·860	1·275	2·297	187·0
143·5	19·545	9·177	8·546	5·999	11·145	216·5	173·5	18·350	1·848	1·729	1·185	2·133	186·5
144·0	19·514	9·082	8·458	5·933	11·016	216·0	174·0	18·344	1·707	1·597	1·094	1·970	186·0
144·5	19·483	8·986	8·370	5·866	10·886	215·5	174·5	18·338	1·566	1·464	1·003	1·806	185·5
145·0	19·452	8·888	8·280	5·798	10·755	215·0	175·0	18·333	1·424	1·332	0·912	1·642	185·0
145·5	19·421	8·790	8·189	5·730	10·623	214·5	175·5	18·328	1·282	1·199	0·821	1·478	184·5
146·0	19·391	8·690	8·097	5·661	10·490	214·0	176·0	18·324	1·140	1·067	0·730	1·314	184·0
146·5	19·361	8·589	8·004	5·592	10·356	213·5	176·5	18·320	0·998	0·934	0·639	1·150	183·5
147·0	19·332	8·487	7·910	5·522	10·221	213·0	177·0	18·317	0·856	0·801	0·548	0·986	183·0
147·5	19·302	8·384	7·814	5·452	10·084	212·5	177·5	18·315	0·713	0·667	0·457	0·822	182·5
148·0	19·273	8·280	7·718	5·382	9·947	212·0	178·0	18·312	0·571	0·534	0·366	0·658	182·0
148·5	19·245	8·175	7·621	5·311	9·809	211·5	178·5	18·311	0·428	0·401	0·274	0·493	181·5
149·0	19·217	8·069	7·523	5·240	9·671	211·0	179·0	18·309	0·286	0·267	0·183	0·329	181·0
149·5	19·189	7·962	7·424	5·168	9·532	210·5	179·5	18·309	0·143	0·133	0·091	0·165	180·5
150·0	19·162	7·854	7·324	5·096	9·392	210·0	180·0	18·308	0·000	0·000	0·000	0·000	180·0

TABLE 8.—*Values of R for completing u for* L_2.

P	I = Inclination of moon's orbit.											
	18°	19°	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°
°	°	°	°	°	°	°	°	°	°	°	°	°
0	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
5	1·76	2·00	2·27	2·57	2·90	3·27	3·67	4·14	4·65	5·22	5·88	6·57
10	3·43	3·90	4·42	5·00	5·63	6·32	7·09	7·94	8·89	9·94	11·12	12·43
15	4·94	5·61	6·39	7·15	8·03	8·99	10·04	11·20	12·47	13·86	15·40	17·10
20	6·24	7·06	7·97	8·94	10·00	11·16	12·40	13·76	15·23	16·82	18·56	20·43
25	7·28	8·21	9·22	10·32	11·49	12·76	14·12	15·57	17·14	18·81	20·60	22·50
30	8·02	9·02	10·10	11·26	12·48	13·80	15·19	16·68	18·25	19·90	21·66	23·50
35	8·48	9·51	10·60	11·77	13·00	14·31	15·68	17·13	18·64	20·23	21·89	23·61
40	8·66	9·69	10·75	11·89	13·08	14·34	15·65	17·02	18·44	19·92	21·44	23·02
45	8·56	9·54	10·57	11·64	12·77	13·95	15·17	16·43	17·73	19·08	20·46	21·87
50	8·22	9·13	10·11	11·09	12·12	13·20	14·30	15·44	16·61	17·81	19·03	20·28
55	7·66	8·49	9·36	10·26	11·18	12·14	13·12	14·13	15·16	16·20	17·27	18·34
60	6·91	7·64	8·40	9·19	10·00	10·83	11·68	12·55	13·43	14·32	15·23	16·14
65	6·00	6·62	7·27	7·94	8·62	9·32	10·03	10·75	11·49	12·23	12·98	13·73
70	4·96	5·46	5·99	6·53	7·08	7·64	8·21	8·79	9·38	9·97	10·56	11·16
75	3·80	4·19	4·59	5·00	5·36	5·84	6·26	6·70	7·14	7·58	8·02	8·47
80	2·58	2·84	3·11	3·38	3·66	3·94	4·23	4·52	4·81	5·10	5·40	5·69
85	1·30	1·43	1·57	1·70	1·84	1·98	2·13	2·27	2·42	2·56	2·71	2·86
90	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00	0·00
95	358·24	358·00	357·73	357·43	357·10	356·73	356·33	355·86	355·35	354·78	354·12	353·43
100	356·57	356·10	355·58	355·00	354·37	353·68	352·91	352·06	351·11	350·06	348·88	347·57
105	355·06	354·39	353·61	352·85	351·97	351·01	349·96	348·80	347·53	346·14	344·60	342·90
110	353·76	352·94	352·03	351·06	350·00	348·84	347·60	346·24	344·77	343·18	341·44	339·57
115	352·72	351·79	350·78	349·68	348·51	347·24	345·88	344·43	342·86	341·19	339·40	337·50
120	351·98	350·98	349·90	348·74	347·52	346·20	344·81	343·32	341·75	340·10	338·34	336·50
125	351·52	350·49	349·40	348·23	347·00	345·69	344·32	342·87	341·36	339·77	338·11	336·39
130	351·34	350·31	349·25	348·11	346·92	345·66	344·35	342·98	341·56	340·08	338·56	336·98
135	351·44	350·46	349·43	348·36	347·23	346·05	344·83	343·57	342·27	340·92	339·54	338·13
140	351·78	350·87	349·89	348·91	347·88	346·80	345·70	344·56	343·39	342·19	340·97	339·72
145	352·34	351·51	350·64	349·74	348·82	347·86	346·88	345·87	344·84	343·80	342·73	341·66
150	353·09	352·36	351·60	350·81	350·00	349·17	548·32	347·45	346·57	345·68	344·77	343·86
155	354·00	353·38	352·73	352·06	351·38	350·68	349·97	349·25	348·51	347·77	347·02	346·27
160	355·04	354·54	354·01	353·47	352·92	352·36	351·79	351·21	350·62	350·03	349·44	348·84
165	356·20	355·81	355·41	355·00	354·64	354·16	353·74	353·30	352·86	352·42	351·98	351·53
170	357·42	357·16	356·89	356·62	356·34	356·06	355·77	355·48	355·19	354·90	354·60	354·31
175	358·70	358·57	358·43	358·30	358·16	358·02	357·87	357·73	357·58	357·44	357·29	357·14
180	360·00	360·00	360·00	360·00	360·00	360·00	360·00	360·00	360·00	360·00	360·00	360·00

u for $L_2 = 2(\xi - \nu) - R$. The values of ξ and ν and to be obtained from Table 7; and the above values of R were computed from the equation $\tan R = \dfrac{\sin 2P}{\frac{1}{6}\cot^2 \frac{1}{2} I - \cos 2P}$.

The values of I and P for the first day of every month are given in Table 6.

When P lies between 180° and 360°, subtract 180° from it and enter the table with the remainder.

TABLE 9.—*Values of Q for completing u for* M_1.

P	Q	P	Q	P	Q	P	Q	P	Q	P	Q
°	°	°	°	°	°	°	°	°	°	°	°
0	0·00	60	40·89	120	139·11	180	180·00	240	220·89	300	319·11
1	0·50	61	42·05	121	140·24	181	180·50	241	222·05	301	320·24
2	1·00	62	43·24	122	141·34	182	181·00	242	223·24	302	321·34
3	1·50	63	44·46	123	142·41	183	181·50	243	224·46	303	322·41
4	2·00	64	45·71	124	143·45	184	182·00	244	225·71	304	323·45
5	2·50	65	47·00	125	144·47	185	182·50	245	227·00	305	324·47
6	3·01	66	48·32	126	145·46	186	183·01	246	228·32	306	325·46
7	3·51	67	49·67	127	146·44	187	183·51	247	229·67	307	326·44
8	4·02	68	51·06	128	147·38	188	184·02	248	231·06	308	327·38
9	4·53	69	52·48	129	148·31	189	184·53	249	232·48	309	328·31
10	5·04	70	53·95	130	149·21	190	185·04	250	233·95	310	329·21
11	5·55	71	55·45	131	150·09	191	185·55	251	235·45	311	330·09
12	6·07	72	56·98	132	150·96	192	186·07	252	236·98	312	330·96
13	6·58	73	58·56	133	151·80	193	186·58	253	238·56	313	331·80
14	7·10	74	60·17	134	152·63	194	187·10	254	240·17	314	332·63
15	7·63	75	61·81	135	153·44	195	187·63	255	241·81	315	333·44
16	8·16	76	63·50	136	154·23	196	188·16	256	243·50	316	334·23
17	8·69	77	65·22	137	155·00	197	188·69	257	245·22	317	335·00
18	9·23	78	66·97	138	155·76	198	189·23	258	246·97	318	335·76
19	9·77	79	68·76	139	156·51	199	189·77	259	248·76	319	336·51
20	10·32	80	70·58	140	157·24	200	190·32	260	250·58	320	337·24
21	10·86	81	72·42	141	157·96	201	190·86	261	252·42	321	337·96
22	11·42	82	74·30	142	158·66	202	191·42	262	254·30	322	338·66
23	11·98	83	76·20	143	159·36	203	191·98	263	256·20	323	339·36
24	12·55	84	78·13	144	160·04	204	192·55	264	258·13	324	340·04
25	13·12	85	80·08	145	160·70	205	193·12	265	260·08	325	340·70
26	13·70	86	82·04	146	161·36	206	193·70	266	262·04	326	341·36
27	14·29	87	84·02	147	162·01	207	194·29	267	264·02	327	342·01
28	14·89	88	86·00	148	162·65	208	194·89	268	266·00	328	342·65
29	15·49	89	88·00	149	163·28	209	195·49	269	268·00	329	343·28
30	16·10	90	90·00	150	163·90	210	196·10	270	270·00	330	343·90
31	16·72	91	92·00	151	164·51	211	196·72	271	272·00	331	344·51
32	17·35	92	94·00	152	165·11	212	197·35	272	274·00	332	345·11
33	17·99	93	95·98	153	165·71	213	197·99	273	275·98	333	345·71
34	18·64	94	97·96	154	166·30	214	198·64	274	277·96	334	346·30
35	19·30	95	99·92	155	166·88	215	199·30	275	279·92	335	346·88
36	19·96	96	101·87	156	167·45	216	199·96	276	281·87	336	347·45
37	20·64	97	103·80	157	168·02	217	200·64	277	283·80	337	348·02
38	21·34	98	105·70	158	168·58	218	201·34	278	285·70	338	348·58
39	22·04	99	107·58	159	169·14	219	202·04	279	287·58	339	349·14
40	22·76	100	109·42	160	169·68	220	202·76	280	289·42	340	349·68
41	23·49	101	111·24	161	170·23	221	203·49	281	291·24	341	350·23
42	24·24	102	113·03	162	170·77	222	204·24	282	293·03	342	350·77
43	25·00	103	114·78	163	171·31	223	205·00	283	294·78	343	351·31
44	25·77	104	116·50	164	171·84	224	205·77	284	296·50	344	351·84
45	26·56	105	118·19	165	172·37	225	206·56	285	298·19	345	352·37
46	27·37	106	119·83	166	172·90	226	207·37	286	299·83	346	352·90
47	28·20	107	121·44	167	173·42	227	208·20	287	301·44	347	353·42
48	29·04	108	123·02	168	173·93	228	209·04	288	303·02	348	353·93
49	29·91	109	124·55	169	174·45	229	209·91	289	304·55	349	354·45
50	30·79	110	126·05	170	174·96	230	210·79	290	306·05	350	354·96
51	31·69	111	127·52	171	175·47	231	211·69	291	307·52	351	335·47
52	32·62	112	128·94	172	175·98	232	212·62	292	308·94	352	355·98
53	33·56	113	130·33	173	176·49	233	213·56	293	310·33	353	356·49
54	34·54	114	131·68	174	176·99	234	214·54	294	311·68	354	356·99
55	35·53	115	133·00	175	177·50	235	215·53	295	313·00	355	357·50
56	36·55	116	134·29	176	178·00	236	216·55	296	314·29	356	358·00
57	37·59	117	135·54	177	178·50	237	217·59	297	315·54	357	358·50
58	38·66	118	136·76	178	179·00	238	218·66	298	316·76	358	359·00
59	39·76	119	137·95	179	179·50	239	219·76	299	317·95	359	359·50

u for $M_1 = \xi - \nu + Q$. The values of ξ and ν are to be obtained from Table 7; the above values of Q were computed from the equation $\tan Q = \frac{1}{2} \tan P$.

The value of P for the first day of every month is given in Table 6.

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years.*

Component.	1850		1851		1852		1853		1854		1855	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, [M_1]*	1·1216 0·0498	0·8916 9·9502	1·0553 0·0234	0·9476 9·9766	0·9935 9·9972	1·0066 0·0028	0·9428 9·9744	1·0607 0·0256	0·9047 9·9565	1·1053 0·0435	0·8788 9·9439	1·1380 0·0561
K_1	1·0840 0·0350	0·9225 9·9650	1·0426 0·0181	0·9591 9·9819	1·0011 0·0005	0·9989 9·9995	0·9647 9·9844	1·0366 0·0156	0·9358 9·9712	1·0686 0·0288	0·9153 9·9616	1·0925 0·0384
K_2	1·2263 0·0886	0·8155 9·9114	1·1276 0·0521	0·8869 9·9479	1·0238 0·0102	0·9767 9·9898	0·9305 9·9687	1·0747 0·0313	0·8559 9·9324	1·1684 0·0676	0·8026 9·9045	1·2459 0·0955
L_2	0·8599 9·9344	1·1629 0·0656	1·1050 0·0434	0·9050 9·9566	1·3801 0·1399	0·7246 9·8601	0·9476 9·9766	1·0553 0·0234	0·7920 9·8987	1·2626 0·1013	1·0594 0·0251	0·9439 9·9749
[L_2]*	0·9733 9·9882	1·0274 0·0118	0·9830 9·9925	1·0173 0·0075	0·9947 9·9977	1·0053 0·0023	1·0073 0·0032	0·9928 9·9968	1·0192 0·0083	0·9811 9·9917	1·0291 0·0124	0·9717 9·9876
M_1	0·9777 9·9902	1·0228 0·0098	0·5972 9·7761	1·6746 0·2239	0·5066 9·7046	1·9741 0·2954	0·6414 9·8072	1·5591 0·1929	0·8942 9·9514	1·1183 0·0486	0·5378 9·7306	1·8595 0·2694
M_2, M S	0·9733 9·9882	1·0274 0·0118	0·9830 9·9925	1·0173 0·0075	0·9947 9·9977	1·0053 0·0023	1·0073 0·0032	0·9928 9·9968	1·0192 0·0083	0·9811 9·9917	1·0291 0·0124	0·9717 9·9876
M_3	0·9602 9·9824	1·0415 0·0174	0·9746 9·9888	1·0261 0·0112	0·9921 9·9966	1·0080 0·0034	1·0110 0·0047	0·9891 9·9953	1·0289 0·0124	0·9719 9·9876	1·0440 0·0187	0·9579 9·9813
M_4, M N	0·9473 9·9765	1·0557 0·0235	0·9662 9·9851	1·0350 0·0149	0·9895 9·9954	1·0106 0·0046	1·0147 0·0063	0·9856 9·9937	1·0388 0·0165	0·9626 9·9835	1·0590 0·0249	0·9443 9·9751
M_6	0·9220 9·9647	1·0846 0·0353	0·9498 9·9776	1·0529 0·0224	0·9843 9·9931	1·0159 0·0069	1·0221 0·0095	0·9784 9·9905	1·0588 0·0248	0·9445 9·9752	1·0898 0·0374	0·9176 9·9626
M_8	0·8973 9·9530	1·1144 0·0470	0·9336 9·9702	1·0711 0·0298	0·9791 9·9908	1·0213 0·0092	1·0295 0·0126	0·9713 9·9874	1·0791 0·0331	0·9267 9·9669	1·1215 0·0498	0·8917 9·9502
N_2, 2 N	0·9733 9·9882	1·0274 0·0118	0·9830 9·9925	1·0173 0·0075	0·9947 9·9977	1·0053 0·0023	1·0073 0·0032	0·9928 9·9968	1·0192 0·0083	0·9811 9·9917	1·0291 0·0124	0·9717 9·9876
O_1, Q_1	1·1449 0·0588	0·8735 9·9412	1·0715 0·0300	0·9333 9·9700	1·0023 0·0010	0·9978 9·9990	0·9446 9·9752	1·0587 0·0248	0·9006 9·9545	1·1104 0·0455	0·8700 9·9395	1·1495 0·0605
OO	1·5841 0·1998	0·6313 9·8002	1·2732 0·1049	0·7855 9·8951	1·0175 0·0075	0·9829 9·9925	0·8305 9·9194	1·2041 0·0806	0·7031 9·8470	1·4223 0·1530	0·6218 9·7936	1·6084 0·2064
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1,0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	0·9733 9·9882	1·0274 0·0118	0·9830 9·9925	1·0173 0·0075	0·9947 9·9977	1·0053 0·0023	1·0073 0·0032	0·9928 9·9968	1·0192 0·0083	0·9811 9·9917	1·0291 0·0124	0·9717 9·9876
M K	1·0550 0·0233	0·9479 9·9767	1·0249 0·0107	0·9757 9·9893	0·9958 9·9982	1·0042 0·0018	0·9717 9·9875	1·0291 0·0125	0·9538 9·9795	1·0484 0·0205	0·9420 9·9740	1·0616 0·0260
2 M K	1·0268 0·0115	0·9739 9·9885	1·0074 0·0032	0·9926 9·9968	0·9906 9·9959	1·0095 0·0041	0·9788 9·9907	1·0217 0·0093	0·9722 9·9877	1·0286 0·0123	0·9693 9·9865	1·0316 0·0135
2 M S	0.9473 9·9765	1·0557 0·0235	0·9662 9·9851	1·0350 0·0149	0·9895 9·9954	1·0106 0·0046	1·0147 0·0063	0·9856 9·9937	1·0388 0·0165	0·9626 9·9835	1·0590 0·0249	0·9443 9·9751
Msf, 2 S M	0·9733 9·9882	1·0274 0·0118	0·9830 9·9925	1·0173 0·0075	0·9947 9·9977	1·0053 0·0023	1·0073 0·0032	0·9928 9·9968	1·0192 0·0083	0·9811 9·9917	1·0291 0·0124	0·9717 9·9876
Mf	1·3467 0·1293	0·7426 9·8707	1·1680 0·0674	0·8562 9·9326	1·0098 0·0042	0·9903 9·9958	0·8857 9·9473	1·1290 0·0527	0·7957 9·9008	1·2567 0·0992	0·7355 9·8666	1·3597 0·1334
Mm	0·9138 9·9608	1·0944 0·0392	0·9446 9·9752	1·0587 0·0248	0·9837 9·9929	1·0165 0·0071	1·0278 0·0119	0·9729 9·9881	1·0720 0·0302	0·9328 9·9698	1·1106 0·0455	0·9005 9·9545
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1856		1857		1858		1859		1860		1861	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, [M_1]	0·8636 9·9363	1·1580 0·0637	0·8583 9·9336	1·1651 0.0664	0·8625 9·9357	1·1595 0·0643	0·8764 9·9427	1·1411 0·0573	0·9010 9·9547	1·1098 0·0453	0·9376 9·9720	1·0665 0·0280
K_1	0·9030 9·9557	1·1074 0·0443	0·8987 9·9536	1·1128 0·0464	0·9021 9·9553	1·1085 0·0447	0·9134 9·9607	1·0948 0·0393	0·9330 9·9699	1·0718 0·0301	0·9609 9·9827	1·0407 0·0173
K_2	0·7707 9·8869	1·2976 0·1131	0·7593 9·8804	1·3170 0·1196	0·7683 9·8855	1·3017 0·1145	0·7977 9·9018	1·2537 0·0982	0·8484 9·9286	1·1786 0·0714	0·9207 9·9641	1·0862 0·0359
L_2	2·1324 0·3289	0·4690 9·6711	1·0393 0·0167	0·9622 9·9833	0·7796 9·8919	1·2827 0·1081	0·9992 9·9996	1·0008 0·0004	1·7598 0·2455	0·5683 9·7545	1·0822 0·0343	0·9241 9·9657
[L_2]	1·0357 0·0152	0·9655 9·9848	1·0382 0·0163	0·9632 9·9837	1·0362 0·0155	0·9650 9·9845	1·0301 0·0129	0·9708 9·9871	1·0205 0·0088	0·9799 9·9912	1·0088 0·0038	0·9913 9·9962
M_1	0·4260 9·6294	2·3475 0·3706	0·5343 9·7278	1·8716 0·2722	0·8499 9·9294	1·1766 0·0706	0·5631 9·7506	1·7759 0·2494	0·4490 9·6522	2·2274 0·3478	0·5581 9·7467	1·7919 0·2533
M_2, MS	1·0357 0·0152	0·9655 9·9848	1·0382 0·0163	0·9632 9·9837	1·0362 0·0155	0·9650 9·9845	1·0301 0·0129	0·9708 9·9871	1·0205 0·0088	0·9799 9·9912	1·0088 0·0038	0·9913 9·9962
M_3	1·0540 0·0229	0·9487 9·9771	1·0578 0·0244	0·9453 9·9756	1·0548 0·0232	0·9480 9·9768	1·0454 0·0193	0·9566 9·9807	1·0309 0·0132	0·9700 9·9868	1·0131 0·0057	0·9870 9·9943
M_4, MN	1·0727 0·0305	0·9322 9·9695	1·0779 0·0326	0·9278 9·9674	1·0738 0·0309	0·9313 9·9691	1·0611 0·0257	0·9425 9·9743	1·0415 0·0177	0·9602 9·9823	1·0176 0·0076	0·9827 9·9924
M_6	1·1110 0·0457	0·9001 9·9543	1·1190 0·0488	0·8936 9·9512	1·1127 0·0464	0·8987 9·9536	1·0930 0·0386	0·9149 9·9614	1·0629 0·0265	0·9408 9·9735	1·0265 0·0114	0·9742 9·9886
M_8	1·1507 0·0610	0·8690 9·9390	1·1618 0·0651	0·8608 9·9349	1·1530 0·0618	0·8673 9·9382	1·1258 0·0515	0·8882 9·9485	1·0847 0·0353	0·9219 9·9647	1·0355 0·0152	0·9657 9·9848
N_2, 2 N	1·0357 0·0152	0·9655 9·9848	1·0382 0·0163	0·9632 9·9837	1·0362 0·0155	0·9650 9·9845	1·0301 0·0129	0·9708 9·9871	1·0205 0·0088	0·9799 9·9912	1·0088 0·0038	0·9913 9·9962
O_1, Q_1	0·8519 9·9304	1·1739 0·0696	0·8455 9·9271	1·1827 0·0729	0·8505 9·9297	1·1758 0·0703	0·8672 9·9381	1·1532 0·0619	0·8962 9·9524	1·1158 0·0476	0·9387 9·9725	1·0654 0·0275
OO	0·5763 9·7606	1·7352 0·2394	0·5607 9·7488	1·7834 0·2512	0·5730 9·7581	1·7453 0·2419	0·6145 9·7885	1·6273 0·2115	0·6912 9·8396	1·4467 0·1604	0·8127 9·9099	1·2305 0·0901
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2 μ_2 ν_2	1·0357 0·0152	0·9655 9·9848	1·0382 0·0163	0·9632 9·9837	1·0362 0·0155	0·9650 9·9845	1·0301 0·0129	0·9708 9·9871	1·0205 0·0088	0·9799 9·9912	1·0088 0·0038	0·9913 9·9962
MK	0·9353 9·9709	1·0692 0·0292	0·9330 9·9699	1·0718 0·0301	0·9348 9·9707	1·0697 0·0293	0·9409 9·9736	1·0628 0·0264	0·9521 9·9787	1·0503 0·0213	0·9693 9·9864	1·0317 0·0136
2 MK	0·9687 9·9862	1·0323 0·0138	0·9686 9·9862	1·0324 0·0138	0·9687 9·9862	1·0323 0·0138	0·9692 9·9864	1·0318 0·0136	0·9717 9·9875	1·0291 0·0125	0·9778 9·9902	1·0227 0·0098
2 MS	1·0727 0·0305	0·9322 9·9695	1·0779 0·0326	0·9278 9·9674	1·0738 0·0309	0·9313 9·9691	1·0611 0·0257	0·9425 9·9743	1·0415 0·0177	0·9602 9·9823	1·0176 0·0076	0·9827 9·9924
Msf, 2 SM	1·0357 0·0152	0·9655 9·9848	1·0382 0·0163	0·9632 9·9837	1·0362 0·0155	0·9650 9·9845	1·0301 0·0129	0·9708 9·9871	1·0205 0·0088	0·9799 9·9912	1·0088 0·0038	0·9913 9·9962
Mf	0·7007 9·8455	1·4272 0·1545	0·6886 9·8379	1·4523 0·1621	0·6981 9·8439	1·4325 0·1561	0·7300 9·8633	1·3699 0·1367	0·7871 9·8960	1·2705 0·1040	0·8734 9·9412	1·1449 0·0588
Mm	1·1374 0·0559	0·8792 9.9441	1·1476 0·0598	0·8713 9·9402	1·1395 0·0567	0·8776 9·9433	1·1145 0·0471	0·8973 9·9529	1·0770 0·0322	0·9285 9·9678	1·0331 0·0142	0·9679 9·9858
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnigh' for leap years*—Continued.

Component.	1862		1863		1864		1865		1866		1867	
	F $\log F$	f $\log f$	F $\log F$	f $\log f$	F $\log F$	f $\log f$	F $\log F$	f $\log f$	F $\log F$	f $\log f$	F $\log F$	f $\log f$
J_1, [M_1]	0·9868 9·9942	1·0134 0·0058	1·0474 0·0201	0·9547 9·9799	1·1139 0·0468	0·8978 9·9532	1·1731 0·0693	0·8524 9·9307	1·2068 0·0816	0·8287 9·9184	1·2013 0·0796	0·8325 9·9204
K_1	0·9964 9·9984	1·0036 0·0016	1·0375 0·0160	0·9638 9·9840	1·0793 0·0331	0·9265 9·9669	1·1140 0·0469	0·8977 9·9531	1·1327 0·0541	0·8828 9·9459	1·1297 0·0530	0·8852 9·9470
K_2	1·0120 0·0052	0·9882 9·9948	1·1150 0·0473	0·8969 9·9527	1·2154 0·0847	0·8228 9·9153	1·2941 0·1120	0·7727 9·8880	1·3344 0·1253	0·7494 9·8747	1·3280 0·1232	0·7530 9·8768
L_2	0·8162 9·9118	1·2252 0·0882	0·8950 9·9518	1·1173 0·0482	1·1564 0·0631	0·8647 9·9369	1·1371 0·0558	0·8794 9·9442	0·9239 9·9656	1·0824 0·0344	0·8405 9·9245	1·1898 0·0755
[L_2]	0·9962 9·9984	1·0038 0·0016	0·9843 9·9931	1·0159 0·0069	0·9743 9·9887	1·0264 0·0113	0·9672 9·9855	1·0339 0·0145	0·9638 9·9840	1·0376 0·0160	0·9643 9·9842	1·0370 0·0158
M_1	0·9558 9·9804	1·0462 0·0196	0·7939 9·8998	1·2596 0·1002	0·5952 9·7746	1·6802 0·2254	0·6215 9·7934	1·6091 0·2066	0·8590 9·9340	1·1641 0·0660	1·2312 0·0903	0·8122 9·9097
M_2, MS	0·9962 9·9984	1·0038 0·0016	0·9843 9·9931	1·0159 0·0069	0·9743 9·9887	1·0264 0·0113	0·9672 9·9855	1·0339 0·0145	0·9638 9·9840	1·0376 0·0160	0·9643 9·9842	1·0370 0·0158
M_3	0·9944 9·9975	1·0057 0·0025	0·9766 9·9897	1·0240 0·0103	0·9617 9·9830	1·0398 0·0170	0·9513 9·9783	1·0512 0·0217	0·9462 9·9760	1·0569 0·0240	0·9470 9·9763	1·0560 0·0237
M_4, MN	0·9925 9·9967	1·0076 0·0033	0·9689 9·9863	1·0321 0·0137	0·9492 9·9774	1·0535 0·0226	0·9355 9·9711	1·0689 0·0289	0·9289 9·9680	1·0765 0·0320	0·9300 9·9685	1·0753 0·0315
M_6	0·9887 9·9951	1·0114 0·0049	0·9537 9·9794	1·0486 0·0206	0·9248 9·9661	1·0813 0·0339	0·9049 9·9566	1·1051 0·0434	0·8953 9·9520	1·1169 0·0480	0·8968 9·9527	1·1151 0·0473
M_8	0·9850 9·9934	1·0152 0·0066	0·9387 9·9725	1·0653 0·0275	0·9011 9·9548	1·1098 0·0452	0·8752 9·9421	1·1426 0·0579	0·8629 9·9360	1·1589 0·0640	0·8648 9·9369	1·1563 0·0631
N_2, 2 N	0·9962 9·9984	1·0038 0·0016	0·9843 9·9931	1·0159 0·0069	0·9743 9·9887	1·0264 0·0113	0·9672 9·9855	1·0339 0·0145	0·9638 9·9840	1·0376 0·0160	0·9643 9·9842	1·0370 0·0158
O_1, Q_1	0·9947 9·9977	1·0053 0·0023	1·0627 0·0264	0·9410 9·9736	1·1363 0·0555	0·8800 9·9445	1·2014 0·0797	0·8324 9·9203	1·2382 0·0928	0·8076 9·9072	1·2322 0·0907	0·8116 9·9093
OO	0·9917 9·9964	1·0084 0·0036	1·2389 0·0930	0·8072 9·9070	1·5457 0·1891	0·6470 9·8109	1·8536 0·2680	0·5395 9·7320	2·0436 0·3104	0·4894 9·6896	2·0118 0·3036	0·4971 9·6964
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	0·9962 9·9984	1·0038 0·0016	0·9843 9·9931	1·0159 0·0069	0·9743 9·9887	1·0264 0·0113	0·9672 9·9855	1·0339 0·0145	0·9638 9·9840	1·0376 0·0160	0·9643 9·9842	1·0370 0·0158
MK	0·9927 9·9968	1·0074 0·0032	1·0212 0·0091	0·9792 9·9909	1·0516 0·0218	0·9510 9·9782	1·0775 0·0324	0·9281 9·9676	1·0917 0·0381	0·9160 9·9619	1·0894 0·0372	0·9179 9·9628
2 MK	0·9889 9·9952	1·0112 0·0048	1·0052 0·0022	0·9948 9·9978	1·0245 0·0105	0·9761 9·9895	1·0422 0·0179	0·9595 9.9821	1·0522 0·0221	0·9504 9·9779	1·0506 0·0214	0·9519 9·9786
2 MS	0·9925 9·9967	1·0076 0·0033	0·9689 9·9863	1·0321 0·0137	0·9492 9·9774	1·0535 0·0226	0·9355 9·9711	1·0689 0·0289	0·9289 9·9680	1·0765 0·0320	0·9300 9·9685	1·0753 0·0315
MSf, 2 SM	0·9962 9·9984	1·0038 0·0016	0·9843 9·9931	1·0159 0·0069	0·9743 9·9887	1·0264 0·0113	0·9672 9·9855	1·0339 0·0145	0·9638 9·9840	1·0376 0·0160	0·9643 9·9842	1·0370 0·0158
Mf	0·9932 9·9970	1·0069 0·0030	1·1474 0·0597	0·8715 9·9403	1·3253 0·1223	0·7546 9·8777	1·4923 0·1738	0·6701 9·8262	1·5907 0·2016	0·6287 9·7984	1·5744 0·1971	0·6352 9·8029
Mm	0·9888 9·9951	1·0113 0·0049	0·9489 9·9772	1·0539 0·0228	0·9169 9·9623	1·0906 0·0377	0·8951 9·9519	1·1172 0·0481	0·8848 9·9468	1·1302 0·0532	0·8864 9·9476	1·1281 0·0524
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	868		1869		1870		1871		1872		1873	
	F log *F*	*f* log *f*	*F* log F	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*
J_1, [M_1]	1·1589 0·0640	0·8629 9·9360	1·0961 0·0398	0·9124 9·9602	1·0303 0·0130	0·9706 9·9870	0·9724 9·9878	1·0284 0·0122	0·9266 9·9669	1·0793 0·0331	0·8933 9·9510	1·1195 0·0490
K_1	1·1059 0·0437	0·9043 9·9563	1·0684 0·0287	0·9360 9·9713	1·0262 0·0112	0·9745 9·9888	0·9862 9·9940	1·0140 0·0060	0·9526 9·9789	1·0498 0·0211	0·9269 9·9670	1·0789 0·0330
K_2	1·2762 0·1059	0·7836 9·8941	1·1898 0·0755	0·8405 9·9245	1·0869 0·0362	0·9201 9·9638	0·9859 9·9938	1·0143 0·0062	0·8992 9·9539	1·1121 0·0461	0·8327 9·9205	1·2010 0·0795
L_2	0·9164 9·9621	1·0912 0·0379	1·1901 0·0756	0·8403 9·9244	1·2073 0·0818	0·8283 9 9182	0·8708 9·9399	1·1484 0·0601	0·8172 9·9123	1·2237 0·0877	1·2260 0·0885	0·8157 9·9115
[L_2]	0·9688 9·9862	1·0322 0·0138	0·9767 9·9898	1·0238 0·0102	0·9874 9·9945	1·0128 0·0055	0·9996 9·9998	1·0004 0·0002	1·0121 0·0052	0·9881 9·9948	1·0234 0·0100	0·9772 9·9900
M_1	0·8409 9·9247	1·1892 0·0753	0·5776 9·7616	1·7313 0·2384	0·5521 9·7420	1·8113 0·2580	0·7693 9·8861	1·2999 0·1139	0·8438 9·9262	1·1851 0·0738	0·4990 9·6981	2·0040 0·3019
M_2, MS	0·9688 9·9862	1·0322 0·0138	0·9767 9·9898	1·0238 0·0102	0·9874 9·9945	1·0128 0·0055	0·9996 9·9998	1·0004 0·0002	1·0121 0·0052	0·9881 9·9948	1·0234 0·0100	0·9772 9·9900
M_3	0·9535 9·9793	1·0487 0·0207	0·9653 9·9847	1·0359 0·0153	0·9811 9·9917	1·0192 0·0083	0·9994 9·9997	1·0006 0·0003	1·0182 0·0078	0·9821 9·9922	1·0352 0·0150	0·9660 9·9850
M_4, MN	0·9386 9·9725	1·0655 0·0275	0·9540 9·9795	1·0482 0·0205	0·9749 9·9890	1·0257 0·0110	0·9992 9·9996	1·0008 0·0004	1·0243 0·0104	0·9763 9·9896	1·0473 0·0201	0·9549 9·9799
M_6	0·9093 9·9587	1·0998 0·0413	0·9318 9·9693	1·0732 0·0307	0·9626 9·9834	1·0389 0·0166	0·9988 9·9995	1·0012 0·0005	1·0366 0·0156	0·9647 9·9844	1·0717 0·0301	0·9331 9·9699
M_8	0·8809 9·9449	1·1352 0·0551	0·9101 9·9591	1·0988 0·0409	0·9504 9·9779	1·0522 0·0221	0·9983 9·9993	1·0017 0·0007	1·0492 0·0208	0·9531 9·9792	1·0968 0·0401	0·9118 9·9599
N_2, 2 N	0·9688 9·9862	1·0322 0·0138	0·9767 9·9898	1·0238 0·0102	0·9874 9·9945	1·0128 0·0055	0·9996 9·9998	1·0004 0·0002	1·0121 0·0052	0·9881 9·9948	1·0234 0·0100	0·9772 9·9900
O_1, Q_1	1·1858 0·0740	0·8433 9·9260	1·1167 0·0479	0·8955 9·9521	1·0436 0·0185	0·9582 9·9815	0·9784 9·9905	1·0221 0·0095	0·9259 9·9666	1·0800 0·0334	0·8871 9·9480	1·1272 0·0520
OO	1·7768 0·2496	0·5628 9·7504	1·4596 0·1642	0·6852 9·8358	1·1658 0·0666	0·8578 9·9334	0·9373 9·9719	1·0669 0·0281	0·7750 9·8893	1·2904 0·1107	0·6667 9·8239	1·5000 0·1761
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0009	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2 ν_2	0·9688 9·9862	1·0322 0·0138	0·9767 9·9898	1·0238 0·0102	0·9874 9·9945	1·0128 0·0055	0·9996 9·9998	1·0004 0·0002	1·0121 0·0052	0·9881 9·9948	1·0234 0·0100	0·9772 9·9900
MK	1·0714 0·0299	0.9334 9·9701	1·0435 0·0185	0·9583 9·9815	1·0132 0·0057	0·9870 9·9943	0·9858 9·9938	1·0144 0·0062	0·9641 9·9841	1·0373 0·0159	0·9485 9·9770	1·0543 0·0230
2 MK	1·0379 0·0162	0·9635 9·9838	1·0192 0·0083	0·9811 9·9917	1·0004 0·0002	0·9996 9·9998	0·9854 9·9936	1·0148 0·0064	0·9757 9·9893	1·0249 0·0107	0·9707 9·9871	1·0302 0·0129
2 MS	0·9386 9·9725	1·0655 0·0275	0·9540 9·9795	1·0482 0·0205	0·9749 9·9890	1·0257 0·0110	0·9992 9·9996	1·0008 0·0004	1·0243 0·0104	0·9763 9·9896	1·0473 0·0201	0·9549 9·9799
MSf, 2 SM	0·9688 9·9862	1·0322 0·0138	0·9767 9·9898	1·0238 0·0102	0·9874 9·9945	1·0128 0·0055	0·9996 9·9998	1·0004 0·0002	1·0121 0·0052	0·9881 9·9948	1·0234 0·0100	0·9772 9·9900
M f	1·4515 0·1618	0·6889 9.8382	1·2766 0·1061	0·7833 9·8939	1·1030 0·0426	0·9066 9·9574	0·9576 9·9812	1·0443 0·0188	0·8471 9·9279	1·1806 0·0721	0·7690 9·8860	1·3003 0·1140
M m	0·8999 9·9542	1·1112 0·0458	0·9246 9·9659	1·0816 0·0341	0·9589 9·9818	1·0428 0·0182	1·0004 0·0002	0·9996 9·9998	1·0452 0·0192	0·9568 9·9808	1·0880 0·0366	0·9191 9·9634
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1874		1875		1876		1877		1878		1879	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, [M_1]	0·8717 9·9404	1·1472 0·0596	0·8604 9·9347	1·1622 0·0653	0·8588 9·9339	1·1644 0·0661	0·8667 9·9378	1·1538 0·0622	0·8846 9·9468	1·1305 0·0532	0·9137 9·9608	1·0945 0·0392
K_1	0·9096 9·9589	1·0994 0·0411	0·9004 9·9544	1·1106 0·0456	0·8991 9·9538	1·1122 0·0462	0·9056 9·9569	1·1043 0·0431	0·9200 9·9638	1·0869 0·0362	0·9428 9·9744	1·0607 0·0256
K_2	0·7878 9·8964	1·2694 0·1036	0·7638 9·8830	1·3092 0·1170	0·7604 9·8810	1·3151 0·1190	0·7772 9·8905	1·2867 0·1095	0·8148 9·9110	1·2273 0·0890	0·8738 9·9414	1·1444 0·0586
L_2	1·8337 0·2633	0·5453 9·7367	0·9196 9·9636	1·0874 0·0364	0·7874 9·8962	1·2700 0·1038	1·1657 0·0666	0·8578 9·9334	1·8420 0·2653	0·5429 9·7347	0·9649 9·9845	1·0364 0·0155
[L_2]	1·0321 0·0137	0·9689 9·9863	1·0372 0·0159	0·9641 9·9841	1·0380 0·0162	0·9634 9·9838	1·0343 0·0146	0·9668 9·9854	1·0267 0·0115	0·9740 9·9885	1·0162 0·0070	0·9841 9·9930
M_1	0·4374 9·6408	2·2864 0·3592	0·6038 9·7809	1·6562 0·2191	0·8150 9·9111	1·2270 0·0889	0·5005 9·6994	1·9981 0·3006	0·4408 9·6442	2·2688 0·3558	0·6079 9·7838	1·6450 0·2162
M_2, MS	1·0321 0·0137	0·9689 9·9863	1·0372 0·0159	0·9641 9·9841	1·0380 0·0162	0·9634 9·9838	1·0343 0·0146	0·9668 9·9854	1·0267 0·0115	0·9740 9·9885	1·0162 0·0070	0·9841 9·9930
M_3	1·0486 0·0206	0·9537 9·9794	1·0563 0·0238	0·9467 9·9762	1·0575 0·0243	0·9456 9·9757	1·0519 0·0220	0·9506 9·9780	1·0404 0·0172	0·9612 9·9828	1·0244 0·0105	0·9762 9·9895
M_4, MN	1·0653 0·0275	0·9387 9·9725	1·0758 0·0317	0·9296 9·9683	1·0774 0·0324	0·9282 9·9676	1·0698 0·0293	0·9348 9·9707	1·0542 0·0229	0·9486 9·9771	1·0326 0·0139	0·9684 9·9861
M_6	1·0995 0·0412	0·9095 9·9588	1·1158 0·0476	0·8962 9·9524	1·1183 0·0485	0·8943 9·9515	1·1065 0·0440	0·9038 9·9561	1·0824 0·0344	0·9239 9·9656	1·0493 0·0209	0·9530 9·9791
M_8	1·1348 0·0549	0·8812 9·9451	1·1573 0·0634	0·8641 9·9366	1·1607 0·0647	0·8615 9·9353	1·1445 0·0586	0·8738 9·9414	1·1113 0·0458	0·8998 9·9542	1·0663 0·0279	0·9378 9·9721
N_2, 2 N	1·0321 0·0137	0·9689 9·9863	1·0372 0·0159	0·9641 9·9841	1·0380 0·0162	0·9634 9·9838	1·0343 0·0146	0·9668 9·9854	1·0267 0·0115	0·9740 9·9885	1·0162 0·0070	0·9841 9·9930
O_1, Q_1	0·8615 9·9353	1·1607 0·0647	0·8480 9·9284	1·1792 0·0716	0·8461 9·9274	1·1819 0·0726	0·8556 9·9323	1·1688 0·0677	0·8769 9·9430	1·1404 0·0570	0·9110 9·9595	1·0977 0·0405
OO	0·6003 9·7784	1·6658 0·2216	0·5669 9·7535	1·7639 0·2465	0·5622 9·7499	1·7787 0·2501	0·5854 9·7675	1·7082 0·2325	0·6396 9·8059	1·5634 0·1941	0·7322 9·8646	1·3658 0·1354
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	1·0321 0·0137	0·9689 9·9863	1·0372 0·0159	0·9641 9·9841	1·0380 0·0162	0·9634 9·9838	1·0343 0·0146	0·9668 9·9854	1·0267 0·0115	0·9740 9·9885	1·0162 0·0070	0·9841 9·9930
MK	0·9388 9·9726	1·0651 0·0274	0·9339 9·9703	1·0708 0·0297	0·9332 9·9700	1·0716 0·0300	0·9366 9·9716	1·0677 0·0284	0·9446 9·9753	1·0586 0·0248	0·9580 9·9814	1·0438 0·0186
2 MK	0·9690 9·9863	1·0320 0·0137	0·9686 9·9862	1·0324 0·0138	0·9686 9·9862	1·0324 0·0138	0·9687 9·9862	1·0323 0·0138	0·9699 9·9867	1·0311 0·0133	0·9735 9·9883	1·0272 0·0117
2 MS	1·0653 0·0275	0·9387 9·9725	1·0758 0·0317	0·9296 9·9683	1·0774 0·0324	0·9282 9·9676	1·0698 0·0293	0·9348 9·9707	1·0542 0·0229	0·9486 9·9771	1·0326 0·0139	0·9684 9·9861
MSf, 2 SM	1·0321 0·0137	0·9689 9·9863	1·0372 0·0159	0·9641 9·9841	1·0380 0·0162	0·9634 9·9838	1·0343 0·0146	0·9668 9·9854	1·0267 0·0115	0·9740 9·9885	1·0162 0·0070	0·9841 9·9930
Mf	0·7192 9·8568	1·3905 0·1432	0·6934 9·8410	1·4422 0·1590	0·6897 9·8387	1·4499 0·1613	0·7077 9·8499	1·4130 0·1501	0·7489 9·8744	1·3352 0·1256	0·8167 9·9121	1·2244 0·0879
Mm	1·1226 0·0502	0·8908 9·9498	1·1435 0·0582	0·8745 9·9418	1·1466 0·0594	0·8721 9·9406	1·1316 0·0537	0·8837 9·9463	1·1012 0·0418	0·9081 9·9582	1·0605 0·0255	0·9430 9·9745
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1880		1881		1882		1883		1884		1885	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, [M_1]	0·9551 9·9801	1·0470 0·0199	1·0092 0·0040	0·9909 9·9960	1·0731 0·0306	0·9319 9·9694	1·1386 0·0564	0·8783 9·9436	1·1902 0·0756	0·8402 9·9244	1·2095 0·0826	0·8268 9·9174
K_1	0·9738 9·9884	1·0269 0·0116	1·0119 0·0052	0·9882 9·9948	1·0540 0·0229	0·9488 9·9772	1·0940 0·0390	0·9140 9·9610	1·1236 0·0506	0·8900 9·9494	1·1342 0·0547	0·8817 9·9453
K_2	0·9540 9·9795	1·0483 0·0205	1·0512 0·0217	0·9512 9·9783	1·1553 0·0627	0·8656 9·9373	1·2494 0·0967	0·8004 9·9033	1·3149 0·1189	0·7605 9·8811	1·3376 0·1263	0·7476 9·8737
L_2	0·8024 9·9044	1·2463 0·0956	0·9803 9·9914	1.0201 0·0086	1·2726 0·1047	0·7858 9·8953	1·0594 0·0250	0·9440 9·9750	0·8678 9·9384	1·1523 0·0616	0·8538 9·9314	1·1712 0.0686
[L_2]	1·0039 0·0017	0·9961 9·9983	0·9914 9·9963	1·0086 0·0037	0·9801 9·9913	1·0203 0·0087	0·9712 9·9873	1·0297 0·0127	0·9654 9·9847	1·0358 0·0153	0·9635 9·9839	1·0378 0·0161
M_1	0·9564 9·9807	1·0456 0.0193	0·6543 9·8158	1·5284 0·1842	0·5482 9·7389	1·8242 0·2611	0·6541 9·8156	1·5289 0·1844	1·0314 0·0134	0·9696 9·9866	1.1408 0·0572	0·8766 9·9428
M_2, MS	1·0039 0·0017	0·9961 9·9983	0·9914 9·9963	1·0086 0·0037	0·9801 9·9913	1·0203 0·0087	0·9712 9·9873	1·0297 0·0127	0·9654 9·9847	1·0358 0·0153	0·9635 9·9839	1·0378 0·0161
M_3	1·0059 0·0026	0·9941 9·9974	0·9872 9·9944	1·0130 0·0056	0·9703 9·9869	1·0306 0·0131	0·9571 9·9810	1·0448 0·0190	0·9487 9·9771	1·0541 0·0229	0·9458 9·9758	1·0573 0·0242
M_4, MN	1·0079 0·0034	0·9922 9·9966	0·9830 9·9925	1·0173 0·0075	0·9606 9·9826	1·0410 0·0174	0·9432 9·9746	1·0602 0·0254	0·9321 9·9695	1·0729 0·0305	0·9284 9·9677	1·0771 0·0323
M_6	1·0118 0·0051	0·9883 9·9949	0·9746 9·9888	1·0261 0·0112	0·9415 9·9738	1·0621 0·0262	0·9160 9·9619	1·0917 0·0381	0·8999 9·9542	1·1112 0·0458	0·8946 9·9516	1·1179 0·0484
M_8	1·0158 0·0068	0·9845 9·9932	0·9662 9·9851	1·0350 0·0149	0·9228 9·9651	1·0836 0·0349	0·8896 9·9492	1·1241 0·0508	0·8688 9·9389	1·1510 0·0611	0·8619 9·9355	1·1602 0·0645
N_2, 2 N	1·0039 0·0017	0·9961 9·9983	0·9914 9·9963	1·0086 0·0037	0·9801 9·9913	1·0203 0·0087	0·9712 9·9873	1·0297 0·0127	0·9654 9·9847	1·0358 0·0153	0·9635 9·9839	1·0378 0·0161
O_1, Q_1	0·9587 9·9817	1·0430 0·0183	1·0199 0·0086	0·9804 9·9914	1·0912 0·0379	0·9164 9·9621	1·1635 0·0658	0·8595 9·9342	1·2201 0·0864	0·8196 9·9136	1·2412 0·0938	0·8057 9·9062
OO	0·8744 9·9417	1·1437 0·0583	1·0794 0·0332	0·9264 9·9668	1·3526 0·1312	0·7393 9·8688	1·6700 0·2227	0·5988 9·7773	1·9486 0·2897	0·5132 9·7103	2·0597 0·3138	0·4855 9·6862
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	1·0039 0·0017	0·9961 9·9983	0·9914 9·9963	1·0086 0·0037	0·9801 9·9913	1·0203 0·0087	0·9712 9·9873	1·0297 0·0127	0·9654 9·9847	1·0358 0·0153	0·9635 9·9839	1·0378 0·0161
MK	0·9776 9·9902	1·0229 0·0098	1·0033 0·0014	0·9967 9·9986	1·0331 0·0141	0·9680 9·9859	1·0625 0·0263	0·9412 9·9737	1·0848 0·0353	0·9219 9·9647	1·0929 0·0386	0·9150 9·9614
2 MK	0·9814 9·9918	1·0189 0·0082	0·9947 9·9977	1·0053 0·0023	1·0125 0·0054	0·9876 9·9946	1·0319 0·0136	0·9691 9·9864	1·0473 0·0201	0·9549 9·9799	1·0530 0·0224	0·9496 9·9776
2 MS	1·0079 0·0034	0·9922 9·9966	0·9830 9·9925	1·0173 0·0075	0·9606 9·9826	1·0410 0·0174	0·9432 9·9746	1·0602 0·0254	0·9321 9·9695	1·0729 0·0305	0·9284 9·9677	1·0771 0·0323
MSf, 2 SM	1·0039 0·0017	0·9961 9·9983	0·9914 9·9963	1·0086 0·0037	0·9801 9·9913	1·0203 0·0087	0·9712 9·9873	1·0297 0·0127	0·9654 9·9847	1·0358 0·0153	0·9635 9·9839	1·0378 0·0161
Mf	0·9156 9·9617	1·0922 0·0383	1·0493 0·0209	0·9530 9·9791	1·2149 0·0845	0·8231 9·9155	1·3939 0·1442	0·7174 9·8558	1·5419 0·1880	0·6486 9·8120	1·5989 0·2038	0·6255 9·7962
Mm	1·0157 0·0068	0·9845 9·9932	0·9725 9·9879	1·0283 0·0121	0·9354 9·9710	1·0691 0·0290	0·9072 9·9577	1·1023 0·0423	0·8897 9·9493	1·1239 0·0507	0·8840 9·9465	1·1312 0·0535
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1886		1887		1888		1889		1890		1891	
	F log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*
J_1, [M_1]	1·1885 0·0750	0·8414 9·9250	1·1360 0·0554	0·8803 9·9446	1·0702 0·0295	0·9344 9·9705	1·0066 0·0028	0·9935 9·9972	0·9531 9·9792	1·0492 0·0208	0·9123 9·9601	1·0962 0·0399
K_1	1·1226 0·0502	0·8908 9·9498	1·0925 0·0384	0·9153 9·9616	1·0522 0·0221	0·9504 9·9779	1·0101 0·0044	0·9900 9·9956	0·9723 9·9878	1·0285 0·0122	0·9417 9·9739	1·0620 0·0261
K_2	1·3129 0·1182	0·7617 9·8818	1·2460 0·0955	0·8026 9·9045	1·1508 0·0610	0·8689 9·9390	1·0467 0·0198	0·9554 9·9802	0·9502 9·9778	1·0524 0·0222	0·8710 9·9400	1·1481 0·0600
L_2	0·9943 9·9975	1·0058 0·0025	1·2143 0·0843	0·8235 9·9157	1·0581 0·0245	0·9451 9·9755	0·8301 9·9191	1·2047 0·0809	0·8671 9·9380	1·1533 0·0620	1·4101 0·1492	0·7092 9·8508
[L_2]	0·9656 9·9848	1·0356 0·0152	0·9715 9·9874	1·0294 0·0126	0·9806 9·9915	1·0198 0·0085	0·9920 9·9965	1·0081 0·0035	1·0045 0·0019	0·9956 9·9981	1·0166 0·0072	0·9836 9·9928
M_1	0·7330 9·8651	1·3643 0·1349	0·5810 9·7642	1·7211 0·2358	0·6278 9·7978	1·5929 0·2022	0·9302 9·9686	1·0750 0·0314	0·7560 9·8785	1·3228 0·1215	0·4783 9·6797	2·0907 0·3203
M_2, MS	0·9656 9·9848	1·0356 0·0152	0·9715 9·9874	1·0294 0·0126	0·9806 9·9915	1·0198 0·0085	0·9920 9·9965	1·0081 0·0035	1·0045 0·0019	0·9956 9·9981	1·0166 0·0072	0·9836 9·9928
M_3	0·9489 9·9772	1·0539 0·0228	0·9575 9·9811	1·0444 0·0189	0·9710 9·9872	1·0299 0·0128	0·9880 9·9948	1·0122 0·0052	1·0067 0·0029	0·9933 9·9971	1·0251 0·0108	0·9755 9·9892
M_4, MN	0·9324 9·9696	1·0725 0·0304	0·9438 9·9749	1·0596 0·0251	0·9615 9·9830	1·0400 0·0170	0·9840 9·9930	1·0162 0·0070	1·0089 0·0039	0·9912 9·9961	1·0336 0·0143	0·9675 9·9857
M_6	0·9003 9·9544	1·1107 0·0456	0·9168 9·9623	1·0907 0·0377	0·9428 9·9744	1·0606 0·0256	0·9761 9·9895	1·0245 0·0105	1·0134 0·0058	0·9868 9·9942	1·0508 0·0215	0·9517 9·9785
M_8	0·8694 9·9392	1·1503 0·0608	0·8907 9·9497	1·1227 0·0503	0·9245 9·9659	1·0816 0·0341	0·9683 9·9860	1·0328 0·0140	1·0179 0·0077	0·9824 9·9923	1·0683 0·0287	0·9361 9·9713
N_2, 2 N	0·9656 9·9848	1·0356 0·0152	0·9715 9·9874	1·0294 0·0126	0·9806 9·9915	1·0198 0·0085	0·9920 9·9965	1·0081 0·0035	1·0045 0·0019	0·9956 9·9981	1·0166 0·0072	0·9836 9·9928
O_1, Q_1	1·2182 0·0857	0·8209 9·9143	1·1607 0·0647	0·8616 9·9353	1·0880 0·0366	0·9191 9·9634	1·0170 0·0073	0·9833 9·9927	0·9564 9·9806	1·0456 0·0194	0·9093 9·9587	1·0997 0·0413
OO	1·9390 0·2876	0·5157 9·7124	1·6568 0·2192	0·6036 9·7808	1·3395 0·1270	0·7465 9·8730	1·0689 0·0289	0·9356 9·9711	0·8671 9·9381	1·1532 0·0619	0·7275 9·8618	1·3746 0·1382
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	0·9656 9·9848	1·0356 0·0152	0·9715 9·9874	1·0294 0·0126	0·9806 9·9915	1·0198 0·0085	0·9920 9·9965	1·0081 0·0035	1·0045 0·0019	0·9956 9·9981	1·0166 0·0072	0·9836 9·9928
MK	1·0840 0·0350	0·9225 9·9650	1·0614 0·0259	0·9422 9·9741	1·0317 0·0136	0·9692 9·9864	1·0020 0·0009	0·9980 9·9991	0·9766 9·9897	1·0239 0·0103	0·9573 9·9811	1·0446 0·0189
2 MK	1·0467 0·0198	0·9554 9·9802	1·0311 0·0133	0·9698 9·9867	1·0117 0·0050	0·9884 9·9950	0·9940 9·9974	1·0061 0·0026	0·9810 9·9917	1·0194 0·0083	0·9733 9·9882	1·0275 0·0118
2 MS	0·9324 9·9696	1·0725 0·0304	0·9438 9·9749	1·0596 0·0251	0·9615 9·9830	1·0400 0·0170	0·9840 9·9930	1·0162 0·0070	1·0089 0·0039	0·9912 9·9961	1·0336 0·0143	0·9675 9·9857
MSf, 2 SM	0·9656 9·9848	1·0356 0·0152	0·9715 9·9874	1·0294 0·0126	0·9806 9·9915	1·0198 0·0085	0·9920 9·9965	1·0081 0·0035	1·0045 0·0019	0·9956 9·9981	1·0166 0·0072	0·9836 9·9928
Mf	1·5369 0·1866	0·6506 9·8134	1·3867 0·1420	0·7212 9·8580	1·2072 0·0818	0·8283 9·9182	1·0426 0·0181	0·9591 9·9819	0·9107 9·9594	1·0981 0·0406	0·8133 9·9103	1·2295 0·0897
Mm	0·8903 9·9495	1·1233 0·0505	0·9082 9·9582	1·1011 0·0418	0·9368 9·9717	1·0674 0·0283	0·9744 9·9887	1·0263 0·0113	1·0176 0·0076	0·9827 9·9924	1·0623 0·0262	0·9414 9·9738
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1892		1893		1894		1895		1896		1897	
	F log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*
J_1, [M_1]	0·8836 9·9463	1·1317 0·0537	0·8661 9·9376	1·1546 0·0624	0·8587 9·9338	1·1646 0·0662	0·8607 9·9348	1·1619 0·0652	0·8724 9·9407	1·1463 0·0593	0·8945 9·9516	1·1179 0·0484
K_1	0·9192 9·9634	1·0878 0·0366	0·9051 9·9567	1·1049 0·0433	0·8990 9·9538	1·1124 0·0462	0·9006 9·9546	1·1103 0·0454	0·9102 9·9591	1·0987 0·0409	0·9278 9·9675	1·0778 0·0325
K_2	0·8128 9·9100	1·2303 0·0900	0·7760 9·8899	1·2886 0·1101	0·7601 9·8809	1·3156 0·1191	0·7644 9·8833	1·3082 0·1167	0·7892 9·8972	1·2670 0·1028	0·8352 9·9218	1·1974 0·0782
L_2	1·4651 0·1659	0·6826 9·8341	0·8437 9·9262	1·1853 0·0738	0·8202 9·9139	1·2192 0·0861	1·4211 0·1526	0·7037 9·8474	1·6465 0·2166	0·6073 9·7834	0·8766 9·9428	1·1408 0·0572
[L_2]	1·0271 0·0116	0·9736 9·9884	1·0346 0·0148	0·9666 9·9852	1·0380 0·0162	0·9634 9·9838	1·0371 0·0158	0·9643 9·9842	1·0318 0·0136	0·9692 9·9864	1·0229 0·0098	0·9776 9·9902
M_1	0·4633 9·6659	2·1582 0·3341	0·6975 9·8435	1·4337 0·1565	0·7305 9·8636	1·3689 0·1364	0·4596 9·6624	2·1758 0·3376	0·4465 9·6498	2·2398 0·3502	0·6799 9·8324	1·4708 0·1676
M_2, MS	1·0271 0·0116	0·9736 9·9884	1·0346 0·0148	0·9666 9·9852	1·0380 0·0162	0·9634 9·9838	1·0371 0·0158	0·9643 9·9842	1·0318 0·0136	0·9692 9·9864	1·0229 0·0098	0·9776 9·9902
M_3	1·0410 0·0174	0·9607 9·9826	1·0523 0·0221	0·9503 9·9779	1·0576 0·0243	0·9455 9·9757	1·0562 0·0237	0·9468 9·9763	1·0481 0·0204	0·9541 9·9796	1·0346 0·0148	0·9666 9·9852
M_4, MN	1·0550 0·0232	0·9479 9·9768	1·0703 0·0295	0·9343 9·9705	1·0775 0·0324	0·9281 9·9676	1·0755 0·0316	0·9298 9·9684	1·0646 0·0272	0·9393 9·9728	1·0464 0·0197	0·9557 9·9803
M_6	1·0836 0·0349	0·9229 9·9651	1·1073 0·0443	0·9031 9·9557	1·1185 0·0486	0·8941 9·9514	1·1154 0·0474	0·8965 9·9526	1·0985 0·0408	0·9103 9·9592	1·0703 0·0295	0·9343 9·9705
M_8	1·1130 0·0465	0·8985 9·9535	1·1456 0·0590	0·8729 9·9410	1·1610 0·0648	0·8613 9·9352	1·1568 0·0632	0·8645 9·9368	1·1334 0·0544	0·8823 9·9456	1·0949 0·0394	0·9134 9·9606
N_2, 2 N	1·0271 0·0116	0·9736 9·9884	1·0346 0·0148	0·9666 9.9852	1·0380 0·0162	0·9634 9·9838	1·0371 0·0158	0·9643 9·9842	1·0318 0·0136	0·9692 9·9864	1·0229 0·0098	0·9776 9·9902
O_1, Q_1	0·8758 9·9424	1·1419 0·0576	0·8549 9·9319	1·1697 0·0681	0·8459 9·9273	1·1821 0·0727	0·8484 9·9286	1·1787 0·0714	0·8624 9.9357	1·1596 0·0643	0·8886 9·9487	1·1254 0·0513
OO	0·6367 9·8039	1·5707 0·1961	0·5837 9·7662	1·7131 0·2338	0·5618 9·7496	1·7800 0·2504	0·5677 9·7541	1·7614 0·2459	0·6024 9·7799	1·6600 0·2201	0·6706 9·8264	1·4914 0·1736
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1-2-3-4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000. 0·0000
λ_2, μ_2, ν_2	1·0271 0·0116	0·9736 9·9884	1·0346 0·0148	0·9666 9·9852	1·0380 0·0162	0·9634 9·9838	1·0371 0·0158	0·9643 9·9842	1·0318 0·0136	0·9692 9·9864	1·0229 0·0098	0·9776 9·9902
MK	0·9442 9·9750	1·0591 0·0250	0·9364 9·9714	1·0679 0·0286	0·9332 9·9700	1·0716 0·0300	0·9340 9·9704	1·0706 0·0296	0·9391 9·9727	1·0648 0·0273	0·9491 9·9773	1·0536 0·0227
2 MK	0·9698 9·9867	1·0312 0·0133	0·9687 9·9862	1·0323 0·0138	0·9687 9·9862	1·0324 0·0138	0·9687 9·9862	1·0324 0·0138	0·9690 9·9863	1·0320 0·0137	0·9709 9·9872	1·0300 0·0128
2 MS	1·0550 0·0232	0·9479 9·9768	1·0703 0·0295	0·9343 9·9705	1·0775 0·0324	0·9281 9·9676	1·0755 0·0316	0·9298 9·9684	1·0646 0·0272	0·9393 9·9728	1·0464 0·0197	0·9557 9·9803
MSf, 2 SM	1·0271 0·0116	0·9736 9·9884	1·0346 0·0148	0·9666 9·9852	1·0380 0·0162	0·9634 9.9838	1·0371 0·0158	0·9643 9·9842	1·0318 0·0136	0·9692 9·9864	1·0229 0·0098	0·9776 9·9902
Mf	0·7467 9·8732	1·3392 0·1268	0·7064 9·8491	1·4156 0·1509	0·6894 9·8385	1·4506 0·1615	0·6940 9·8414	1·4409 0·1586	0·7208 9·8578	1·3874 0·1422	0·7719 9·8876	1·2955 0·1124
Mm	1·1027 0·0424	0·9069 9·9576	1·1326 0·0541	0·8829 9·9459	1·1469 0·0595	0·8719 9·9405	1·1430 0·0580	0·8749 9·9420	1·1214 0·0498	0·8918 9·9502	1·0862 0·0359	0·9206 9·9641
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000.0000	1·0000 0·0000

Table 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1898		1899		1900		1901		1902		1903	
	F log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*
J_1, [M_1]	0·9283 9·9677	1·0773 0·0323	0·9746 9·9888	1·0261 0·0112	1·0329 0·0141	0·9681 9·9859	1·0988 0·0409	0·9101 9·9591	1·1611 0·0649	0·8612 9·9351	1·2022 0·0800	0·8318 9·9200
K_1	0·9538 9·9795	1·0484 0·0205	0·9878 9·9947	1·0124 0·0053	1·0279 0·0120	0·9728 9·9880	1·0701 0·0294	0·9345 9·9706	1·1072 0·0442	0·9032 9·9558	1·1302 0·0532	0·8848 9·9468
K_2	0·9025 9·9555	1·1080 0·0445	0·9899 9·9956	1·0102 0·0044	1·0913 0·0379	0·9164 9·9621	1·1938 0·0769	0·8377 9·9231	1·2790 0·1069	0·7819 9·8931	1·3292 0·1236	0·7524 9.8764
L_2	0.8138 9·9105	1·2288 0·0895	1·1150 0·0473	0·8969 9·9527	1·3279 0·1232	0·7531 9·8768	0·9704 9·9870	1·0305 0·0130	0·8383 9·9234	1·1929 0·0766	0·8951 9·9518	1·1172 0·0482
[L_2]	1·0116 0·0050	0·9886 9·9950	0·9991 9·9996	1·0009 0·0004	0·9869 9·9943	1·0133 0·0057	0·9764 9·9896	1·0242 0·0104	0·9686 9·9861	1·0325 0·0139	0·9642 9·9842	1·0371 0·0158
M_1	0·8576 9·9333	1.1660 0·0667	0·5615 9·7494	1·7809 0·2506	0·5259 9·7209	1·9016 0·2791	0·7149 9·8542	1·3988 0·1458	1·1662 0·0668	0·8575 9·9332	0·9354 9·9710	1·0690 0·0290
M_2, MS	1·0116 0·0050	0·9886 9·9950	0·9991 9·9996	1·0009 0·0004	0·9869 9·9943	1·0133 0·0057	0·9764 9·9896	1·0242 0·0104	0·9686 9·9861	1·0325 0·0139	0·9642 9·9842	1·0371 0·0158
M_3	1·0173 0·0075	0·9829 9·9925	0·9986 9·9994	1·0015 0·0006	0·9804 9·9913	1·0200 0·0086	0·9648 9·9844	1·0365 0·0156	0·9532 9·9792	1·0491 0·0208	0·9468 9·9763	1·0562 0.0237
M_4, MN	1·0232 0·0100	0·9773 9·9900	0·9981 9·9992	1·0019 0·0008	0·9739 9·9885	1·0268 0·0115	0·9533 9·9792	1·0490 0·0208	0·9381 9·9722	1·0660 0·0278	0·9298 9·9684	1·0755 0·0316
M_6	1·0350 0·0149	0·9662 9·9851	0·9972 9·9988	1·0028 0·0012	0·9611 9·9828	1·0404 0·0172	0·9307 9·9688	1·0744 0·0312	0·9086 9·9584	1·1006 0·0416	0·8965 9·9526	1·1154 0·0474
M_8	1·0469 0·0199	0·9552 9·9801	0·9962 9·9984	1·0038 0·0016	0·9485 9·9770	1·0543 0·0230	0·9087 9·9584	1·1004 0·0416	0·8800 9.9445	1·1364 0·0555	0·8645 9·9368	1·1568 0·0632
N_2, 2 N	1·0116 0·0050	0·9886 9·9950	0·9991 9·9996	1·0009 0·0004	0·9869 9·9943	1·0133 0·0057	0·9764 9·9896	1·0242 0·0104	0·9686 9·9861	1·0325 0·0139	0·9642 9·9842	1·0371 0·0158
O_1, Q_1	0·9279 9·9675	1·0778 0·0325	0·9808 9·9916	1·0195 0·0084	1·0465 0·0198	0·9555 9·9802	1·1197 0·0491	0·8931 9·9509	1·1882 0·0749	0·8416 9·9251	1·2332 0·0910	0·8109 9·9090
OO	0·7807 9·8925	1·2809 0·1075	0·9454 9·9756	1·0577 0·0244	1·1769 0·0707	0·8497 9·9293	1·4725 0·1681	0·6791 9·8319	1·7884 0·2525	0·5592 9·7475	2·0172 0·3048	0·4957 9·6952
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	1·0116 0·0050	0·9886 9·9950	0·9991 9·9996	1·0009 0·0004	0·9869 9·9943	1·0133 0·0057	0·9764 9·9896	1·0242 0·0104	0·9686 9.9861	1·0325 0·0139	0·9642 9·9842	1·0371 0·0158
MK	0·9648 9·9844	1·0365 0·0156	0·9868 9·9942	1·0133 0·0058	1·0144 0·0062	0·9858 9·9938	1·0448 0·0190	0·9571 9·9810	1·0723 0·0303	0·9326 9·9697	1·0898 0·0374	0·9176 9·9626
2 MK	0·9760 9·9894	1·0246 0·0106	0·9859 9·9938	1·0143 0·0062	1·0011 0·0005	0·9989 9·9995	1·0201 0·0086	0·9803 9·9914	1·0386 0·0164	0·9629 9·9836	1·0508 0·0215	0·9516 9·9785
2 MS	1·0232 0·0100	0·9773 9·9900	0·9981 9·9992	1·0019 0·0008	0·9739 9·9885	1·0268 0·0115	0·9533 9·9792	1·0490 0·0208	0·9381 9·9722	1·0660 0·0278	0·9298 9·9684	1·0755 0·0316
MSf, 2 SM	1·0116 0·0050	0·9886 9·9950	0·9991 9·9996	1·0009 0·0004	0·9869 9·9943	1·0133 0·0057	0·9764 9·9896	1·0242 0·0104	0·9686 9·9861	1·0325 0·0139	0·9642 9·9842	1·0371 0·0158
Mf	0·8511 9·9300	1·1750 0·0700	0·9630 9·9836	1·0384 0·0164	1·1098 0·0452	0·9011 9·9548	1·2840 0·1086	0·7788 9.8914	1·4578 0·1637	0·6860 9·8363	1·5772 0·1979	0·6340 9·8021
Mm	1·0433 0·0184	0·9585 9·9816	0·9986 9·9994	1·0014 0·0006	0·9573 9·9811	1·0446 0·0189	0·9234 9·9654	1·0830 0·0346	0·8991 9·9538	1·1122 0·0462	0·8861 9·9475	1·1285 0·0525
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1904		1905		1906		1907		1908		1909	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, [M_1]	1·2061	0·8291	1·1711	0·8539	1·1113	0·8999	1·0449	0·9570	0·9846	1·0156	0·9358	1·0686
	0·0814	9·9186	0·0686	9·9314	0·0458	9·9542	0·0191	9·9809	9·9933	0·0067	9·9712	0·0288
K_1	1·1323	0·8831	1·1128	0·8986	1·0777	0·9279	1·0358	0·9654	0·9949	1·0052	0·9595	1·0422
	0·0540	9·9460	0·0464	9·9536	0·0325	9·9675	0·0153	9·9847	9·9978	0·0023	9·9821	0·0179
K_2	1·3336	0·7499	1·2916	0·7742	1·2117	0·8253	1·1109	0·9002	1·0080	0·9920	0·9172	1·0902
	0·1250	9·8750	0·1111	9·8889	0·0834	9·9166	0·0457	9·9543	0·0035	9·9965	9·9625	0·0375
L_2	1·0807	0·9254	1·1650	0·8584	0·9514	1·0511	0·8192	1·2207	0·9416	1·0620	1·5314	0·6530
	0·0337	9·9663	0·0663	9·9337	9·9783	0·0217	9·9134	0·0866	9·9739	0·0261	0·1851	9·8149
[L_2]	0·9639	1·0375	0·9674	1·0336	0·9746	1·0260	0·9848	1·0155	0·9967	1·0033	1·0093	0·9908
	9·9840	0·0160	9·9856	0·0144	9·9888	0·0112	9·9933	0·0067	9·9986	0·0014	0·0040	9·9960
M_1	0·6636	1·5070	0·6086	1·6432	0·7465	1·3396	1·0574	0·9457	0·6762	1·4789	0·4735	2·1120
	9·8219	0·1781	9·7843	0·2157	9·8730	0·1270	0·0242	9·9758	9·8300	0·1700	9·6753	0·3247
M_2, MS	0·9639	1·0375	0·9674	1·0336	0·9746	1·0260	0·9848	1·0155	0·9967	1·0033	1·0093	0·9908
	9·9840	0·0160	9·9856	0·0144	9·9888	0·0112	9·9933	0·0067	9·9986	0·0014	0·0040	9·9960
M_3	0·9463	1·0567	0·9515	1·0509	0·9623	1·0392	0·9773	1·0233	0·9951	1·0049	1·0140	0·9862
	9·9760	0·0240	9·9784	0·0216	9·9833	0·0167	9·9900	0·0100	9·9979	0·0021	0·0060	9·9940
M_4, MN	0·9291	1·0764	0·9360	1·0684	0·9499	1·0527	0·9698	1·0312	0·9934	1·0066	1·0186	0·9817
	9·9680	0·0320	9·9713	0·0287	9·9777	0·0223	9·9867	0·0133	9·9971	0·0029	0·0080	9·9920
M_6	0·8955	1·1167	0·9055	1·1044	0·9259	1·0801	0·9550	1·0472	0·9902	1·0099	1·0281	0·9727
	9·9521	0·0479	9·9569	0·0431	9·9666	0·0334	9·9800	0·0200	9·9957	0·0043	0·0120	9·9880
M_8	0·8631	1·1586	0·8760	1·1415	0·9024	1·1082	0·9404	1·0634	0·9869	1·0133	1·0376	0·9637
	9·9361	0·0639	9·9425	0·0575	9·9554	0·0446	9·9733	0·0267	9·9943	0·0057	0·0160	9·9840
N_2, 2 N	0·9639	1·0375	0·9674	1·0336	0·9746	1·0260	0·9848	1·0155	0·9967	1·0033	1·0093	0·9908
	9·9840	0·0160	9·9856	0·0144	9·9888	0·0112	9·9933	0·0067	9·9986	0·0014	0·0040	9·9960
O_1, Q_1	1·2375	0·8081	1·1992	0·8339	1·1335	0·8823	1·0599	0·9435	0·9922	1·0078	0·9366	1·0677
	0·0925	9·9075	0·0789	9·9211	0·0544	9·9456	0·0253	9·9747	9·9966	0·0034	9·9716	0·0284
OO	2·0396	0·4903	1·8424	0·5428	1·5329	0·6524	1·2279	0·8144	0·9833	1·0170	0·8066	1·2398
	0·3095	9·6905	0·2654	9·7346	0·1855	9·8145	0·0892	9·9108	9·9927	0·0073	9·9066	0·0934
P_1, R_2, T_2	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
$S_{1,2,3,4}$	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
λ_2, μ_2, ν_2	0·9639	1·0375	0·9674	1·0336	0·9746	1·0260	0·9848	1·0155	0·9967	1·0033	1·0093	0·9908
	9·9840	0·0160	9·9856	0·0144	9·9888	0·0112	9·9933	0·0067	9·9986	0·0014	0·0040	9·9960
MK	1·0914	0·9162	1·0766	0·9288	1·0504	0·9520	1·0201	0·9803	0·9916	1·0085	0·9684	1·0326
	0·0380	9·9620	0·0321	9·9679	0·0214	9·9786	0·0086	9·9914	9·9963	0·0037	9·9861	0·0139
2 MK	1·0520	0·9506	1·0416	0·9601	1·0238	0·9768	1·0045	0·9955	0·9883	1·0118	0·9774	1·0231
	0·0220	9·9780	0·0177	9·9823	0·0102	9·9898	0·0020	9·9980	9·9949	0·0051	9·9901	0·0099
2 MS	0·9291	1·0764	0·9360	1·0684	0·9499	1·0527	0·9698	1·0312	0·9934	1·0066	1·0186	0·9817
	9·9680	0·0320	9·9713	0·0287	9·9777	0·0223	9·9867	0·0133	9·9971	0·0029	0·0080	9·9920
MSf, 2 SM	0·9639	1·0375	0·9674	1·0336	0·9746	1·0260	0·9848	1·0155	0·9967	1·0033	1·0093	0·9908
	9·9840	0·0160	9·9856	0·0144	9·9888	0·0112	9·9933	0·0067	9·9986	0·0014	0·0040	9·9960
Mf	1·5887	0·6295	1·4864	0·6728	1·3181	0·7586	1·1408	0·8766	0·9878	1·0124	0·8692	1·1506
	0·2010	9·7990	0·1721	9·8279	0·1200	9·8800	0·0572	9·9428	9·9947	0·0053	9·9391	0·0609
Mm	0·8850	1·1299	0·8958	1·1163	0·9180	1·0893	0·9503	1·0523	0·9905	1·0096	1·0350	0·9662
	9·9470	0·0530	9·9522	0·0478	9.9628	0·0372	9·9779	0·0221	9·9958	0·0042	0·0149	9·9851
Sa, Ssa	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1910		1911		1912		1913		1914		1915	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, $[M_1]$	0·8998 9·9542	1·1113 0·0458	0·8757 9·9423	1·1420 0·0577	0·8621 9·9356	1·1599 0·0644	0·8583 9·9336	1·1650 0·0664	0·8640 9·9365	1·1574 0·0635	0·8796 9·9443	1·1369 0·0557
K_1	0·9320 9·9694	1·0729 0·0306	0·9128 9·9604	1·0955 0·0396	0·9018 9·9551	1·1089 0·0449	0·8987 9·9536	1·1127 0·0464	0·9034 9·9559	1·1070 0·0414	0·9160 9·9619	1·0917 0·0381
K_2	0·8460 9·9274	1·1820 0·0726	0·7962 9·9010	1·2560 0·0990	0·7675 9·8851	1·3030 0·1149	0·7594 9·8805	1·3168 0·1195	0·7715 9·8873	1·2962 0·1127	0·8043 9·9054	1·2433 0·0946
L_2	1·1998 0·0791	0·8335 9·9209	0·8024 9·9044	1·2463 0·0956	0·8811 9·9450	1·1349 0·0550	1·7834 0·2512	0·5607 9·7488	1·3712 0·1371	0·7293 9·8629	0·8193 9·9134	1·2206 0·0866
$[L_2]$	1·0210 0·0090	0·9795 9·9910	1·0304 0·0130	0·9705 9·9870	1·0364 0·0155	0·9649 9·9845	1·0382 0·0163	0·9632 9·9837	1·0355 0·0152	0·9657 9·9848	1·0288 0·0123	0·9720 9·9877
M_1	0·5070 9·7050	1·9723 0·2950	0·8010 9·9036	1·2484 0·0964	0·6424 9·8078	1·5567 0·1922	0·4354 9·6388	2·2969 0·3612	0·4661 9·6685	2·1455 0·3315	0·7636 9·8829	1·3096 0·1171
M_2, MS	1·0210 0·0090	0·9795 9·9910	1·0304 0·0130	0·9705 9·9870	1·0364 0·0155	0·9649 9·9845	1·0382 0·0163	0·9632 9·9837	1·0355 0·0152	0·9657 9·9848	1·0288 0·0123	0·9720 9·9877
M_3	1·0316 0·0135	0·9693 9·9864	1·0459 0·0195	0·9561 9·9805	1·0551 0·0233	0·9477 9·9767	1·0578 0·0244	0·9454 9·9756	1·0538 0·0227	0·9490 9·9773	1·0435 0·0185	0·9583 9·9815
M_4, MN	1·0424 0·0180	0·9594 9·9820	1·0617 0·0260	0·9419 9·9740	1·0741 0·0311	0·9310 9·9689	1·0778 0·0325	0·9278 9·9675	1·0723 0·0303	0·9326 9·9697	1·0584 0·0246	0·9448 9·9754
M_6	1·0642 0·0270	0·9397 9·9730	1·0940 0·0390	0·9141 9·9610	1·1132 0·0466	0·8983 9·9534	1·1189 0·0488	0·8937 9·9512	1·1104 0·0455	0·9006 9·9545	1·0888 0·0370	0·9184 9·9630
M_8	1·0865 0·0360	0·9204 9·9640	1·1272 0·0520	0·8872 9·9480	1·1538 0·0621	0·8667 9·9379	1·1617 0·0651	0·8608 9·9349	1·1498 0·0606	0·8697 9·9394	1·1202 0·0493	0·8927 9·9507
N_2, 2 N	1·0210 0·0090	0·9795 9·9910	1·0304 0·0130	0·9705 9·9870	1·0364 0·0155	0·9649 9·9845	1·0382 0·0163	0·9632 9·9837	1·0355 0·0152	0·9657 9·9848	1·0288 0·0123	0·9720 9·9877
O_1, Q_1	0·8948 9·9517	1·1175 0·0483	0·8663 9·9377	1·1543 0·0623	0·8501 9·9295	1·1763 0·0705	0·8455 9·9271	1·1827 0·0729	0·8524 9·9306	1·1732 0·0694	0·8710 9·9400	1·1482 0·0600
OO	0·6874 9·8372	1·4548 0·1628	0·6124 9·7870	1·6330 0·2130	0·5719 9·7574	1·7484 0·2426	0·5609 9·7488	1·7830 0·2512	0·5775 9·7615	1·7317 0·2385	0·6242 9·7953	1·6020 0·2047
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	1·0210 0·0090	0·9795 9·9910	1·0304 0·0130	0·9705 9·9870	1·0364 0·0155	0·9649 9·9845	1·0382 0·0163	0·9632 9·9837	1·0355 0·0152	0·9657 9·9848	1·0288 0·0123	0·9720 9·9877
MK	0·9516 9·9784	1·0509 0·0216	0·9406 9·9734	1·0632 0·0266	0·9347 9·9706	1·0699 0·0294	0·9330 9·9699	1·0718 0·0301	0·9354 9·9710	1·0690 0·0290	0·9423 9·9742	1·0612 0·0258
2 MK	0·9715 9·9874	1·0293 0·0126	0·9692 9·9864	1·0318 0·0136	0·9687 9·9862	1·0323 0·0138	0·9686 9·9862	1·0324 0·0138	0·9687 9·9862	1·0323 0·0138	0·9695 9·9865	1·0315 0·0135
2 MS	1·0424 0·0180	0·9594 9·9820	1·0617 0·0260	0·9419 9·9740	1·0741 0·0311	0·9310 9·9689	1·0778 0·0325	0·9278 9·9675	1·0723 0·0303	0·9326 9·9697	1·0584 0·0246	0·9448 9·9754
MSf, 2 SM	1·0210 0·0090	0·9795 9·9910	1·0304 0·0130	0·9705 9·9870	1·0364 0·0155	0·9649 9·9845	1·0382 0·0163	0·9632 9·9837	1·0355 0·0152	0·9657 9·9848	1·0288 0·0123	0·9720 9·9877
Mf	0·7843 9·8945	1·2750 0·1055	0·7284 9·8623	1·3730 0·1377	0·6973 9·8434	1·4341 0·1566	0·6886 9·8380	1·4521 0·1620	0·7016 9·8461	1·4254 0·1539	0·7373 9·8677	1·3562 0·1323
Mm	1·0787 0·0329	0·9270 9·9671	1·1157 0·0476	0·8963 9·9524	1·1402 0·0570	0·8770 9·9430	1·1476 0·0598	0·8714 9·9402	1·1366 0·0556	0·8798 9·9444	1·1092 0·0450	0·9015 9·9550
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1916		1917		1918		1819		1920		1921	
	F log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*
J_1, [M_1]	0·9060 9·9571	1·1038 0·0429	0·9446 9·9752	1·0587 0·0248	0·9958 9·9982	1·0042 0·0018	1·0578 0·0244	0·9453 9·9756	1·1242 0·0508	0·8895 9·9492	1·1806 0·0721	0·8470 9·9279
K_1	0·9368 9·9717	1·0674 0·0283	0·9660 9·9850	1·0352 0·0150	1·0027 0·0012	0·9973 9·9988	1·0443 0·0188	0·9576 9·9812	1·0855 0·0356	0·9212 9·9644	1·1182 0·0485	0·8943 9·9515
K_2	0·8585 9·9337	1·1649 0·0663	0·9340 9·9704	1·0707 0·0296	1·0278 0·0119	0·9729 9·9881	1·1316 0·0537	0·8838 9·9463	1·2299 0·0899	0·8131 9·9101	1·3034 0·1151	0·7672 9·8849
L_2	0·8529 9·9309	1·1725 0·0691	1·3141 0·1186	0·7610 9·8814	1·2821 0·1079	0·7800 9·8921	0·8944 9·9515	1·1181 0·0485	0·8351 9·9218	1·1975 0·0782	0·9669 9·9854	1·0342 0·0146
[L_2]	1·0188 0·0081	0·9816 9·9919	1·0068 0·0029	0·9933 9·9971	0·9942 9·9975	1·0058 0·0025	0·9826 9·9924	1·0177 0·0076	0·9730 9·9881	1·0278 0·0119	0·9664 9·9852	1·0347 0·0148
M_1	0·7294 9·8630	1·3710 0·1370	0·5018 9·7005	1·9929 0·2995	0·5237 9·7191	1·9094 0·2809	0·8047 9·9056	1·2427 0·0944	1·1157 0·0476	0·8963 9·9524	0·7644 9·8833	1·3083 0·1167
M_2, MS	1·0188 0·0081	0·9816 9·9919	1·0068 0·0029	0·9933 9·9971	0·9942 9·9975	1·0058 0·0025	0·9826 9·9924	1·0177 0·0076	0·9730 9·9881	1·0278 0·0119	0·9664 9·9852	1·0347 0·0148
M_3	1·0283 0·0121	0·9725 9·9879	1·0102 0·0044	0·9899 9·9956	0·9914 9·9962	1·0087 0·0038	0·9740 9·9885	1·0267 0·0115	0·9597 9·9822	1·0420 0·0179	0·9501 9·9778	1·0525 0·0222
M_4, MN	1·0379 0·0162	0·9635 9·9838	1·0136 0·0059	0·9866 9·9941	0·9886 9·9950	1·0116 0·0050	0·9654 9·9847	1·0358 0·0153	0·9466 9·6762	1·0564 0·0238	0·9340 9·9703	1·0707 0·0297
M_6	1·0574 0·0242	0·9457 9·9758	1·0205 0·0088	0·9799 9·9912	0·9829 9·9925	1·0174 0·0075	0·9486 9·9771	1·0542 0·0229	0·9210 9·9643	1·0858 0·0357	0·9026 9·9555	1·1079 0·0445
M_8	1·0773 0·0323	0·9283 9·9677	1·0274 0·0118	0·9733 9·9882	0·9772 9·9900	1·0233 0·0100	0·9321 9·9694	1·0729 0·0306	0·8961 9·9524	1·1159 0·0476	0·8723 9·9407	1·1464 0·0593
N_2, 2 N	1·0188 1·0081	0·9816 9·9919	1·0068 0·0029	0·9933 9·9971	0·9942 9·9975	1·0058 0·0025	0·9826 9·9924	1·0177 0·0076	0·9730 9·9881	1·0278 0·0119	0·9664 9·9852	1·0347 0·0148
O_1, Q_1	0·9020 9·9552	1·1086 0·0448	0·9466 9·9762	1·0564 0·0238	1·0048 0·0021	0·9952 9·9979	1·0743 0·0311	0·9308 9·9689	1·1477 0·0598	0·8713 9·9402	1·2096 0·0826	0·8267 9·9174
OO	0·7072 9·8495	1·4141 0·1505	0·8369 9·9227	1·1948 0·0773	1·0263 0·0113	0·9744 9·9887	1·2843 0·1086	0·7787 9·8914	1·5969 0·2033	0·6262 9·7967	1·8950 0·2776	0·5277 9·7224
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	1·0188 0·0081	0·9816 9·9919	1·0068 0·0029	0·9933 9·9971	0·9942 9·9975	1·0058 0·0025	0·9826 9·9924	1·0177 0·0076	0·9730 9·9881	1·0278 0·0119	0·9664 9·9852	1·0347 0·0148
MK	0·9544 9·9797	1·0478 0·0203	0·9726 9·9879	1·0282 0·0121	0·9969 9·9987	1·0031 0·0013	1·0261 0·0112	0·9746 9·9888	1·0562 0·0237	0·9468 9·9763	1·0807 0·0337	0·9253 9·9663
2 MK	0·9723 9·9878	1·0284 0·0122	0·9792 9·9909	1·0213 0·0091	0·9912 9·9962	1·0089 0·0038	1·0082 0·0035	0·9919 9·9965	1·0276 0·0118	0·9732 9·9882	1·0444 0·0189	0·9575 9·9811
2 MS	1·0379 0·0162	0·9635 9·9838	1·0136 0·0059	0·9866 9·9941	0·9886 9·9950	1·0116 0·0050	0·9654 9·9847	1·0358 0·0153	0·9466 9·9762	1·0564 0·0238	0·9340 9·9703	1·0707 0·0297
MSf, 2 SM	1·0188 0·0081	0·9816 9·9919	1·0068 0·0029	0·9933 9·9971	0·9942 9·9975	1·0058 0·0025	0·9826 9·9924	1·0177 0·0076	0·9730 9·9881	1·0278 0·0119	0·9664 9·9852	1·0347 0·0148
Mf	0·7987 9·9024	1·2521 0·0976	0·8901 9·9494	1·1235 0·0506	1·0155 0·0067	0·9847 9·9933	1·1746 0·0699	0·8514 9·9301	1·3538 0·1316	0·7387 9·8684	1·5140 0·1801	0·6605 9·8199
Mm	1·0703 0·0295	0·9343 9·9705	1·0260 0·0111	0·9747 9·9889	0·9820 9·9921	1·0183 0·0079	0·9432 9·9746	1·0602 0·0254	0·9127 9·9603	1·0956 0·0397	0·8927 9·9507	1·1202 0·0493
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1922		1923		1924		1925		1926		1927	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, [M_1]	1·2087 0·0823	0·8273 9·9177	1·1967 0·0780	0·8357 9·9220	1·1498 0·0606	0·8697 9·9394	1·0854 0·0356	0·9213 9·9644	1·0204 0·0088	0·9800 9·9912	0·9643 9·9842	1·0370 0·0158
K_1	1·1338 0·0545	0·8820 9·9455	1·1272 0·0520	0·8872 9·9480	1·1006 0·0416	0·9086 9·9584	1·0618 0·0260	0·9418 9·9740	1·0195 0·0084	0·9808 9·9916	0·9804 9·9914	1·0200 0·0086
K_2	1·3366 0·1260	0·7482 9·8740	1·3226 0·1214	0·7561 9·8786	1·2644 0·1019	0·7909 9·8981	1·1740 0·0697	0·8518 9·9303	1·0703 0·0295	0·9343 9·9705	0·9710 9·9872	1·0299 0·0128
L_2	1·1489 0·0603	0·8704 9·9397	1·0734 0·0308	0·9316 9·9692	0·8828 9·9459	1·1328 0·0541	0·8340 9·9212	1·1990 0·0788	1·0387 0·0165	0·9627 9·9835	1·4953 0·1747	0·6688 9·8253
[L_2]	0·9636 9·9839	1·0378 0·0161	0·9648 9·9844	1·0365 0·0156	0·9698 9·9867	1·0311 0·0133	0·9783 9·9905	1·0222 0·0095	0·9892 9·9953	1·0109 0·0047	1·0016 0·0007	0·9984 9·9993
M_1	0·6260 9·7966	1·5974 0·2034	0·6653 9·8230	1·5030 0·1770	0·9244 9·9659	1·0818 0·0341	1·0488 0·0207	0·9535 9·9793	0·6177 9·7908	1·6190 0·2092	0·4847 9·6855	2·0630 0·3145
M_2, MS	0·9636 9·9839	1·0378 0·0161	0·9648 9·9844	1·0365 0·0156	0·9698 9·9867	1·0311 0·0133	0·9783 9·9905	1·0222 0·0095	0·9892 9·9953	1·0109 0·0047	1·0016 0·0007	0·9984 9·9993
M_3	0·9459 9·9758	1·0572 0·0242	0·9476 9·9766	1·0553 0·0234	0·9551 9·9800	1·0470 0·0200	0·9676 9·9857	1·0335 0·0143	0·9839 9·9930	1·0164 0·0070	1·0024 0·0010	0·9976 9·9990
M_4, MN	0·9285 9·9678	1·0770 0·0322	0·9308 9·9689	1·0743 0·0311	0·9406 9·9734	1·0632 0·0266	0·9570 9·9809	1·0449 0·0191	0·9786 9·9906	1·0219 0·0094	1·0032 0·0014	0·9968 9·9986
M_6	0·8947 9·9517	1·1176 0·0483	0·8980 9·9533	1·1136 0·0467	0·9122 9·9601	1·0962 0·0399	0·9362 9·9714	1·0681 0·0286	0·9681 9·9859	1·0330 0·0141	1·0048 0·0021	0·9952 9·9979
M_8	0·8622 9·9356	1·1598 0·0644	0·8664 9·9377	1·1542 0·0623	0·8847 9·9468	1·1303 0·0532	0·9159 9·9618	1·0918 0·0382	0·9576 9·9812	1·0442 0·0188	1·0064 0·0028	0·9937 9·9972
N_2, 2 N	0·9636 9·9839	1·0378 0·0161	0·9648 9·9844	1·0365 0·0156	0·9698 9·9867	1·0311 0·0133	0·9783 9·9905	1·0222 0·0095	0·9892 9·9953	1·0109 0·0047	1·0016 0·0007	0·9984 9·9993
O_1, Q_1	1·2403 0·0935	0·8063 9·9065	1·2272 0·0889	0·8149 9·9111	1·1758 0·0704	0·8504 9·9296	1·1049 0·0433	0·9050 9·9567	1·0325 0·0139	0·9685 9·9861	0·9692 9·9864	1·0318 0·0136
OO	2·0547 0·3128	0·4867 9·6872	1·9853 0·2978	0·5037 9·7022	1·7284 0·2376	0·5786 9·7624	1·4095 0·1491	0·7095 9·8509	1·1248 0·0510	0·8891 9·9490	0·9074 9·9578	1·1020 0·0422
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ, μ_2, ν_2	0·9636 9·9839	1·0378 0·0161	0·9648 9·9844	1·0365 0·0156	0·9698 9·9867	1·0311 0·0133	0·9783 9·9905	1·0222 0·0095	0·9892 9·9953	1·0109 0·0047	1·0016 0·0007	0·9984 9·9993
MK	1·0925 0·0384	0·9153 9·9616	1·0875 0·0364	0·9196 9·9636	1·0674 0·0283	0·9368 9·9717	1·0387 0·0165	0·9627 9·9835	1·0086 0·0037	0·9915 9·9963	0·9820 9·9921	1·0184 0·0079
2 MK	1·0527 0·0223	0·9499 9·9777	1·0492 0·0208	0·9531 9·9792	1·0352 0·0150	0·9660 9·9850	1·0162 0·0070	0·9841 9·9930	0·9977 9·9990	1·0023 0·0010	0·9835 9·9928	1·0168 0·0072
2 MS	0·9285 9·9678	1·0770 0·0322	0·9308 9·9689	1·0743 0·0311	0·9406 9·9734	1·0632 0·0266	0·9570 9·9809	1·0449 0·0191	0·9786 9·9906	1·0219 0·0094	1·0032 0·0014	0·9968 9·9986
MSf, 2 SM	0·9636 9·9839	1·0378 0·0161	0·9648 9·9844	1·0365 0·0156	0·9698 9·9867	1·0311 0·0133	0·9783 9·9905	1·0222 0·0095	0·9892 9·9953	1·0109 0·0047	1·0016 0·0007	0·9984 9·9993
Mf	1·5964 0·2031	0·6264 9·7969	1·5608 0·1933	0·6407 9·8066	1·4256 0·1540	0·7014 9·8460	1·2480 0·0962	0·8013 9·9038	1·0776 0·0325	0·9280 9·9675	0·9378 9·9721	1·0664 0·0279
Mm	0·8843 9·9466	1·1309 0·0534	0·8878 9·9483	1·1264 0·0517	0·9031 9·9557	1·1073 0·0443	0·9294 9·9682	1·0759 0·0318	0·9651 9·9846	1·0361 0·0154	1·0074 0·0032	0·9926 9·9968
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1928		1929		1930		1931		1932		1933	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, [M_1]	0·9204 9·9640	1·0864 0·0360	0·8891 9·9490	1·1247 0·0510	0·8692 9·9391	1·1505 0·0609	0·8595 9·9342	1·1634 0·0658	0·8594 9·9342	1·1636 0·0658	0·8688 9·9389	1·1510 0·0611
K_1	0·9479 9·9768	1·0549 0·0232	0·9236 9·9655	1·0827 0·0345	0·9076 9·9579	1·1018 0·0421	0·8997 9·9541	1·1115 0.0459	0·8996 9·9540	1·1116 0·0460	0·9073 9·9578	1·1022 0·0422
K_2	0·8872 9·9480	1·1271 0·0520	0·8241 9·9160	1·2135 0·0840	0·7826 9·8935	1·2778 0·1065	0·7619 9·8819	1·3125 0·1181	0·7617 9·8818	1·3129 0·1182	0·7818 9·8931	1·2791 0·1069
L_2	1·0256 0·0110	0·9750 9·9890	0·7874 9·8962	1·2700 0·1038	0·9781 9·9904	1·0224 0·0096	2·1230 0·3270	0·4710 9·6730	1·1452 0·0589	0·8732 9·9411	0·7874 9·8962	1·2700 0·1038
[L_2]	1·0140 0·0060	0·9862 9·9940	1·0250 0·0107	0·9756 9·9893	1·0332 0·0142	0·9679 9·9858	1·0376 0·0160	0·9637 9·9840	1·0377 0·0161	0·9637 9·9839	1·0334 0·0142	0·9677 9·9858
M_1	0·5750 9.7597	1·7392 0·2403	0·8787 9·9438	1·1380 0·0562	0·7625 9·8823	1·3115 0·1117	0·4250 9·6284	2·3529 0·3716	0·5020 9·7007	1·9921 0·2993	0·8354 9·9219	1·1970 0·0781
M_2, MS	1·0140 0·0060	0·9862 9·9940	1·0250 0·0107	0·9756 9·9893	1·0332 0·0142	0·9679 9·9858	1·0376 0·0160	0·9637 9·9840	1·0377 0·0161	0·9637 9·9839	1·0334 0·0142	0·9677 9·9858
M_3	1·0210 0·0090	0·9794 9·9910	1·0376 0·0160	0·9637 9·9840	1·0502 0·0213	0·9522 9·9787	1·0570 0·0241	0·9461 9·9759	1·0570 0·0241	0·9460 9·9759	1·0505 0·0214	0·9519 9·9786
M_4, MN	1·0282 0·0121	0·9726 9·9879	1·0506 0·0214	0·9519 9·9786	1·0675 0·0284	0·9368 9·9716	1·0767 0·0321	0·9288 9·9679	1·0768 0·0321	0·9287 9·9679	1·0678 0·0285	0·9365 9·9715
M_6	1·0425 0·0181	0·9592 9·9819	1·0768 0·0321	0·9287 9·9679	1·1029 0·0425	0·9067 9·9575	1·1772 0·0481	0·8951 9·9519	1·1173 0·0482	0·8950 9·9518	1·1034 0·0428	0·9063 9·9572
M_8	1·0571 0·0241	0·9460 9·9759	1·1037 0·0428	0·9061 9·9572	1·1395 0·0567	0·8776 9·9433	1·1592 0·0642	0·8627 9·9358	1·1594 0·0642	0·8625 9·9358	1·1402 0·0570	0·8770 9·9430
N_2, 2 N	1·0140 0·0060	0·9862 9·9940	1·0250 0·0107	0·9756 9·9893	1·0332 0·0142	0·9679 9·9858	1·0376 0·0160	0·9637 9·9840	1·0377 0·0161	0·9637 9·9839	1·0334 0·0142	0·9677 9·9858
O_1, Q_1	0·9188 9·9632	1·0883 0·0368	0·8822 9·9456	1·1335 0·0544	0·8586 9·9338	1·1647 0·0662	0·8470 9·9279	1·1807 0·0721	0·8468 9·9278	1·1809 0·0722	0·8582 9·9336	1·1653 0·0664
OO	0·7545 9·8777	1·3254 0·1223	0·6536 9·8153	1·5300 0·1847	0·5930 9·7730	1·6865 0·2270	0·5643 9·7515	1·7721 0·2485	0·5640 9·7513	1·7732 0·2487	0·5919 9·7722	1·6896 0·2278
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1,\,2,\,3,\,4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	1·0140 0·0060	0·9862 9.9940	1·0250 0·0107	0·9756 9·9893	1·0332 0·0142	0·9679 9·9858	1·0376 0·0160	0·9637 9·9840	1·0377 0·0161	0·9637 9·9839	1·0334 0·0142	0·9677 9·9858
MK	0·9612 9·9828	1·0404 0·0172	0·9467 9·9762	1·0564 0·0238	0·9377 9·9721	1·0664 0·0279	0·9335 9·9701	1·0712 0·0299	0·9335 9·9701	1·0713 0·0299	0·9376 9·9720	1·0666 0·0280
2 MK	0·9746 9·9888	1·0260 0·0112	0·9703 9·9869	1·0306 0·0131	0·9689 9·9863	1·0321 0·0137	0·9686 9·9862	1·0324 0·0138	0·9686 9·9862	1·0324 0·0138	0·9689 9·9863	1·0321 0·0137
2 MS	1·0282 0·0121	0·9726 9·9879	1·0506 0·0214	0·9519 9·9786	1·0675 0·0284	0·9368 9·9716	1·0767 0·0321	0·9288 9.9679	1·0768 0·0321	0·9287 9·9679	1·0678 0·0285	0·9365 9·9715
MSf, 2 SM	1·0140 0·0060	0·9862 9·9940	1·0250 0·0107	0·9756 9·9893	1·0332 0·0142	0·9679 9·9858	1·0376 0·0160	0·9637 9·9840	1·0377 0·0161	0·9637 9·9839	1·0334 0·0142	0·9677 9.9858
Mf	0·8326 9·9204	1·2010 0·0796	0·7594 9·8804	1·3169 0·1196	0·7135 9·8534	1·4015 0·1466	0·6913 9·8397	1·4465 0·1603	0·6911 9·8395	1·4470 0·1605	0·7127 9·8529	1·4032 0·1471
Mm	1·0523 0·0221	0·9503 9·9779	1·0942 0·0391	0·9139 9·9609	1·1270 0·0519	0·8873 9·9481	1·1452 0·0589	0·8732 9·9411	1·1455 0·0590	0·8730 9·9410	1·1277 0·0522	0·8868 9·9478
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1934		1935		1636		1937		1938		1930	
	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f	F log F	f log f
J_1, [M_1]	0·8884 9·9486	1·1256 0·0514	0·9194 9·9635	1·0877 0·0365	0·9629 9·9836	1·0385 0·0164	1·0188 0·0081	0·9815 9·9919	1·0837 0·0349	0·9228 9·9651	1·1482 0·0600	0·8710 9·9400
K_1	0·9231 9·9652	1·0834 0·0348	0·9471 9·9764	1·0558 0·0236	0·9794 9·9910	1·0210 0·0090	1·0185 0·0080	0·9819 9·9920	1·0607 0·0256	0·9428 9·9744	1·0997 0·0413	0·9094 9·9587
K_2	0·8227 9·9152	1·2155 0·0848	0·8852 9·9470	1·1297 0·0530	0·9684 9·9861	1·0326 0·0139	1·0677 0·0284	0·9366 9·9716	1·1714 0·0687	0·8537 9·9313	1·2622 0·1011	0·7923 9·8989
L_2	0·9274 9·9673	1·0783 0·0327	1·5720 0·1965	0·6361 9·8035	1·1645 0·0661	0·8588 9·9339	0·8401 9·9243	1·1903 0·0757	0·8602 9·9346	1·1625 0·0654	1·0681 0·0286	0·9363 9·9714
[L_2]	1·0252 0·0108	0·9754 9·9892	1·0143 0·0062	0·9859 9·9938	1·0019 0·0008	0·9981 9·9992	0·9895 9·9954	1·0106 0·0046	0·9785 9·9906	1·0220 0·0094	0·9700 9·9868	1 0309 0·0132
M_1	0·6196 9·7921	1·6139 0·2079	0·4655 9·6679	2·1482 0·3321	0·5407 9·7329	1·8495 0·2671	0·9109 9·9595	1·0978 0·0405	0·9269 9·9670	1·0789 0·0330	0·6518 9·8141	1·5342 0·1859
M_2, MS	1·0252 0·0108	0·9754 9.9892	1·0143 0·0062	0·9859 9·9938	1·0019 0·0008	0·9981 9·9992	0·9895 9·9954	1·0106 0·0046	0·9785 9·9906	1·0220 0·0094	0·9700 9·9868	1·0309 0·0132
M_3	1·0381 0·0162	0·9633 9·9838	1·0216 0·0093	0·9789 9·9907	1·0028 0·0012	0·9972 9·9988	0·9843 9·9931	1·0159 0·0069	0·9680 9·9859	1·0331 0·0141	0·9554 9·9802	1·0467 0·0198
M_4, MN	1·0511 0·0216	0·9514 9·9784	1·0288 0·0123	0·9720 9·9877	1·0038 0·0017	0·9962 9·9983	0·9792 9·9909	1·0213 0·0091	0·9575 9·9811	1·0444 0·0189	0·9409 9·9736	1·0628 0·0264
M_6	1·0776 0·0325	0·9280 9·9675	1·0435 0·0185	0·9583 9·9815	1·0057 0·0025	0·9943 9·9975	0·9689 9·9863	1·0321 0·0137	0·9369 9·9717	1·0673 0·0283	0·9127 9·9603	1·0956 0·0397
M_8	1·1048 0·0433	0·9051 9·9567	1·0585 0·0247	0·9448 9·9753	1·0077 0·0033	0·9924 9·9967	0·9588 9·9817	1·0430 0·0183	0·9168 9·9623	1·0907 0·0377	0·8854 9·9471	1·1295 0·0529
N_2, 2 N	1·0252 0·0108	0·9754 9·9892	1·0143 0·0062	0·9859 9·9938	1·0019 0·0008	0·9981 9·9992	0·9895 9·9954	1·0106 0·0046	0·9785 9·9906	1·0220 0·0094	0·9700 9·9868	1·0309 0·0132
O_1, Q_1	0·8814 9·9452	1·1345 0·0548	0·9176 9·9627	1·0898 0·0373	0·9676 9·9857	1·0335 0·0143	1·0307 0·0132	0·9702 9·9868	1·1030 0·0426	0·9066 9·9574	1·1740 0·0697	0·8518 9·9303
OO	0·6515 9·8139	1·5348 0·1861	0·7510 9·8757	1·3315 0·1243	0·9024 9·9554	1·1081 0·0446	1·1184 0·0486	0·8942 9·9514	1·4014 0·1466	0·7136 9·8534	1·7198 0·2355	0·5814 9·7645
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	1·0252 0·0108	0·9754 9·9892	1·0143 0·0062	0·9859 9·9938	1·0019 0·0008	0·9981 9·9992	0·9895 9·9954	1·0106 0·0046	0·9785 9·9906	1·0220 0·0094	0·9700 9·9868	1·0309 0·0132
MK	0·9463 9·9760	1·0567 0·0240	0·9607 9·9826	1·0409 0·0174	0·9813 9·9918	1·0191 0·0082	1·0078 0·0034	0·9922 9·9966	1·0379 0·0162	0·9635 9·9838	1·0667 0·0280	0·9375 9·9720
2 MK	0·9702 9·9869	1·0307 0·0131	0·9744 9·9888	1·0262 0·0112	0·9832 9·9926	1·0171 0·0074	0·9973 9·9988	1·0027 0·0012	1·0156 0·0067	0·9846 9·9933	1·0347 0·0148	0·9665 9·9852
2 MS	1·0511 0·0216	0·9514 9·9784	1·0288 0·0123	0·9720 9·9877	1·0038 0·0017	0·9962 9·9983	0·9792 9·9909	1·0213 0·0091	0·9575 9·9811	1·0444 0·0189	0·9409 9·9736	1·0628 0·0264
MSf, 2 SM	1·0252 0·0108	0·9754 9·9892	1·0143 0·0062	0·9859 9·9938	1·0019 0·0008	0·9981 9·9992	0·9895 9·9954	1·0106 0·0046	0·9785 9·9906	1·0220 0·0094	0·9700 9·9868	1·0309 0·0132
Mf	0·7578 9·8796	1·3196 0·1204	0·8302 9·9192	1·2046 0·0808	0·9344 9·9706	1·0702 0·0294	1·0736 0·0309	0·9314 9·9691	1·2433 0·0946	0·8043 9·9054	1·4210 0·1526	0·7038 9·8474
Mm	1·0952 0·0395	0·9131 9·9605	1·0535 0·0226	0·9492 9·9774	1·0086 0·0037	0·9914 9·9963	0·9662 9·9850	1·0350 0·0150	0·9302 9·9686	1·0750 0·0314	0·9037 9·9560	1·1066 0·0440
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1940		1941		1942		1943		1944		1945	
	F log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*
J_1, [M_1]	1·1958 0·0776	0·8363 9·9224	1·2089 0·0824	0·8272 9·9176	1·1818 0·0726	0·8462 9·9274	1·1260 0·0515	0·8881 9·9485	1·0597 0·0252	0·9437 9·9748	0·9973 9·9988	1·0028 0·0012
K_1	1·1267 0·0518	0·8876 9·9482	1·1339 0·0546	0·8819 9·9454	1·1189 0·0488	0·8937 9·9512	1·0866 0·0361	0·9203 9·9639	1·0454 0·0193	0·9565 9·9807	1·0037 0·0016	0·9963 9·9984
K_2	1·3216 0·1211	0·7567 9·8789	1·3369 0·1261	0·7480 9·8739	1·3048 0·1156	0·7664 9·8844	1·2323 0·0907	0·8115 9·9093	1·1344 0·0548	0·8815 9·9452	1·0305 0·0130	0·9704 9·9870
L_2	1·1635 0·0658	0·8595 9·9342	0·9794 9·9909	1·0211 0·0091	0·8474 9·9281	1·1801 0.0719	0·8738 9·9414	1·1444 0·0586	1·1415 0·0575	0·8760 9·9425	1·3364 0·1259	0·7483 9·8741
[L_2]	0·9649 9·9845	1·0364 0·0155	0·9636 9·9839	1·0378 0·0161	0·9663 9·9851	1·0349 0·0149	0·9727 9·9880	1·0280 0·0120	0·9823 9·9922	1·0181 0·0078	0·9939 9·9974	1·0061 0·0026
M_1	0·6160 9·7896	1·6234 0·2104	0·7616 9·8817	1·3131 0·1183	1·1380 0·0561	0·8787 9·9439	0·9294 9·9682	1·0760 0·0318	0·5834 9·7660	1·7142 0·2340	0·5144 9·7113	1·9442 0·2887
M_2, MS	0·9649 9·9845	1·0364 0·0155	0·9636 9·9839	1·0378 0·0161	0·9663 9·9851	1·0349 0·0149	0·9727 9·9880	1·0280 0·0120	0·9823 9·9922	1·0181 0·0078	0·9939 9·9974	1·0061 0·0026
M_3	0·9477 9·9767	1·0551 0·0233	0·9459 9·9758	1·0572 0·0242	0·9499 9·9777	1·0527 0·0223	0·9593 9·9820	1·0424 0·0180	0·9735 9·9883	1·0272 0·0117	0·9909 9·9960	1·0092 0·0040
M_4, MN	0·9310 9·9689	1·0741 0·0311	0·9285 9·9678	1·0770 0·0322	0·9338 9·9702	1·0709 0·0298	0·9462 9·9760	1·0569 0·0240	0·9649 9·9845	1·0364 0·0155	0·9879 9·9947	1·0122 0·0053
M_6	0·8983 9·9534	1·1132 0·0466	0·8947 9·9517	1·1177 0·0483	0·9023 9·9554	1·1083 0·0446	0·9204 9·9640	1·0865 0·0360	0·9477 9·9767	1·0551 0·0233	0·9819 9·9921	1·0184 0·0079
M_8	0·8667 9·9379	1·1538 0·0621	0·8621 9·9356	1·1600 0·0644	0·8719 9·9405	1·1469 0·0595	0·8953 9·9520	1·1170 0·0480	0·9309 9·9689	1·0742 0·0311	0·9760 9·9894	1·0246 0·0106
N_2, 2 N	0·9649 9·9845	1·0364 0·0155	0·9636 9·9839	1·0378 0·0161	0·9663 9·9851	1·0349 0·0149	0·9727 9·9880	1·0280 0·0120	0·9823 9·9922	1·0181 0·0078	0·9939 9·9974	1·0061 0·0026
O_1, Q_1	1·2262 0·0886	0·8155 9·9114	1·2406 0·0936	0·8061 9·9064	1·2109 0·0831	0·8258 9·9169	1·1497 0·0606	0·8698 9·9394	1·0763 0·0319	0·9291 9·9681	1·0065 0·0028	0·9935 9·9972
OO	1·9802 0·2967	0·5050 9·7033	2·0561 0·3130	0·4864 9·6870	1·9016 0·2791	0·5259 9·7209	1·6059 0·2057	0·6227 9·7943	1·2923 0·1114	0·7738 9·8886	1·0321 0·0137	0·9689 9·9863
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000
λ_2, μ_2, ν_2	0·9649 9·9845	1·0364 0·0155	0·9636 9·9839	1·0378 0·0161	0·9663 9·9851	1·0349 0·0149	0·9727 9·9880	1·0280 0·0120	0·9823 9·9922	1·0181 0·0078	0·9939 9·9974	1·0061 0·0026
MK	1·0871 0·0363	0·9199 9·9637	1·0926 0·0385	0·9153 9·9615	1·0812 0·0339	0·9249 9·9661	1·0570 0·0241	0·9461 9·9759	1·0269 0·0115	0·9738 9·9885	0·9976 9·9990	1·0024 0·0010
2 MK	1·0489 0·0207	0·9534 9·9793	1·0528 0·0224	0·9498 9·9776	1·0448 0·0190	0·9571 9·9810	1·0281 0·0120	0·9726 9·9880	1·0087 0·0038	0·9914 9·9962	0·9916 9·9963	1·0085 0·0037
2 MS	0·9310 9·9689	1·0741 0·0311	0·9285 9·9678	1·0770 0·0322	0·9338 9·9702	1·0709 0·0298	0·9462 9·9760	1·0569 0·0240	0·9649 9·9845	1·0364 0·0155	0·9879 9·9947	1·0122 0·0053
MSf, 2 SM	0·9649 9·9845	1·0364 0·0155	0·9636 9·9839	1·0378 0·0161	0·9663 9·9851	1·0349 0·0149	0·9727 9·9880	1·0280 0·0120	0·9823 9·9922	1·0181 0·0078	0·9939 9·9974	1·0061 0·0026
Mf	1·5582 0·1926	0·6417 9·8074	1·5971 0·2033	0·6262 9·7967	1·5174 0·1811	0·6590 9·8189	1·3588 0·1331	0·7360 9·8669	1·1794 0·0717	0·8479 9·9283	1·0192 0·0083	0·9811 9·9917
Mm	0·8380 9·9484	1·1261 0·0516	0·8842 9·9466	1·1310 0·0534	0·8923 9·9505	1·1206 0·0495	0·9120 9·9600	1·0965 0·0400	0·9422 9·9742	1·0613 0·0258	0·9810 9·9916	1·0194 0·0084
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000

TABLE 10.—*Factors F and f for reduction and prediction of tides; computed for the middle of each year, or for July 2, at Greenwich mean noon for common years, and at preceding midnight for leap years*—Continued.

Component.	1946		1947		1948		1949		1950		Mean value of *f* from the years 1850–1942.
	F log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	*F* log *F*	*f* log *f*	
J_1, [M_1]	0·9458 9·9758	1·0573 0·0242	0·9069 9·9576	1·1027 0·0424	0·8801 9·9446	1·1362 0·0554	0·8643 9·9367	1·1570 0·0633	0·8584 9·9337	1·1650 0·0663	1·00490
K_1	0·9669 9·9854	1·0342 0·0146	0·9375 9·9720	1·0666 0·0280	0·9164 9·9621	1·0912 0·0379	0·9036 9·9560	1·1067 0·0440	0·8987 9·9536	1·1127 0·0464	1·00609
K_2	0·9363 9·9714	1·0681 0·0286	0·8602 9·9346	1·1624 0·0654	0·8055 9·9061	1·2415 0·0939	0·7721 9·8877	1·2951 0·1123	0·7594 9·8805	1·3168 0·1195	1·02421
L_2	0·9165 9·9621	1·0911 0·0379	0·7970 9·9015	1·2547 0·0985	1·1182 0·0485	0·8943 9·9515	2·0764 0·3173	0·4816 9·6827	0·9888 9·9951	1·0113 0·0049	0·97803
[L_2]	1·0065 0·0028	0·9936 9·9972	1·0185 0·0080	0·9819 9·9920	1·0285 0·0122	0·9723 9·9878	1·0354 0·0151	0·9658 9·9849	1·0382 0·0163	0·9632 9·9837	1·00033
M_1	0·6755 9·8296	1·4804 0·1704	0·8786 9·9438	1·1382 0·0562	0·5190 9·7151	1·9269 0·2849	0·4275 9·6310	2·3390 0·3690	0·5572 9·7460	1·7947 0·2540	1·55050
M_2, MS	1·0065 0·0028	0·9936 9·9972	1·0185 0·0080	0·9819 9·9920	1·0285 0·0122	0·9723 9·9878	1·0354 0·0151	0·9658 9·9849	1·0382 0·0163	0·9632 9·9837	1·00033
M_3	1·0096 0·0042	0·9904 9·9958	1·0278 0·0119	0·9729 9·9881	1·0431 0·0183	0·9587 9·9817	1·0535 0·0226	0·9492 9·9774	1·0578 0·0244	0·9454 9·9756	1·00074
M_4, MN	1·0129 0·0056	0·9872 9·9944	1·0373 0·0159	0·9641 9·9841	1·0579 0·0244	0·9453 9·9756	1·0721 0·0302	0·9328 9·9698	1·0778 0·0325	0·9278 9·9675	1·00137
M_6	1·0195 0·0084	0·9809 9·9916	1·0565 0·0238	0·9466 9·9762	1·0881 0·0367	0·9191 9·9633	1·1100 0·0453	0·9009 9·9547	1·1189 0·0488	0·8937 9·9512	1·00307
M_8	1·0260 0·0112	0·9746 9·9888	1·0760 0·0318	0·9294 9·9682	1·1191 0·0489	0·8936 9·9511	1·1493 0·0604	0·8701 9·9396	1·1617 0·0651	0·8608 9·9349	1·00552
N_2, 2 N	1·0065 0·0028	0·9936 9·9972	1·0185 0·0080	0·9819 9·9920	1·0285 0·0122	0·9723 9·9878	1·0354 0·0151	0·9658 9·9849	1·0382 0·0163	0·9632 9·9837	1·00033
O_1, Q_1	0·9480 9·9768	1·0548 0·0232	0·9031 9·9557	1·1073 0·0443	0·8716 9·9403	1·1473 0·0597	0·8527 9·9308	1·1727 0·0692	0·8456 9·9272	1·1826 0·0728	1·00905
OO	0·8411 9·9249	1·1889 0·0751	0·7100 9·8513	1·4084 0·1487	0·6259 9·7965	1·5976 0·2035	0·5784 9·7622	1·7290 0·2378	0·5609 9·7489	1·7828 0·2511	1.10310
P_1, R_2, T_2	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·00000
$S_{1, 2, 3, 4}$	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0060	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·00000
λ_2, μ_2, ν_2	1·0065 0·0028	0·9936 9·9972	1·0185 0·0080	0·9819 9·9920	1·0285 0·0122	0·9723 9·9878	1·0354 0·0151	0·9658 9·9849	1·0382 0·0163	0·9632 9·9837	1·00033
MK	0·9731 9·9882	1·0276 0·0118	0·9548 9·9799	1·0473 0·0201	0·9426 9·9743	1·0609 0·0257	0·9356 9·9711	1·0688 0·0289	0·9330 9·9699	1·0718 0·0301	1·00428
2 MK	0·9794 9·9910	1·0210 0·0090	0·9725 9·9879	1·0283 0·0121	0·9695 9·9865	1·0315 0·0135	0·9687 9·9862	1·0323 0·0138	0·9686 9·9862	1·0324 0·0138	1·00315
2 MS	1·0129 0·0056	0·9872 9·9944	1·0373 0·0159	0·9641 9·9841	1·0579 0·0244	0·9453 9·9756	1·0721 0·0302	0·9328 9·9698	1·0778 0·0325	0·9278 9·9675	1·00137
MSf, 2 SM	1·0065 0·0028	0·9936 9·9972	1·0185 0·0080	0·9819 9·9920	1·0285 0·0122	0·9723 9·9878	1·0354 0·0151	0·9658 9·9849	1·0382 0·0163	0·9632 9·9837	1·00033
Mf	0·8930 9·9508	1·1198 0·0492	0·8008 9·9035	1·2488 0·0965	0·7386 9·8684	1·3539 0·1316	0·7023 9·8465	1·4240 0·1535	0·6887 9·8380	1·4520 0·1620	1·04317
Mm	1·0248 0·0106	0·9758 9·9894	1·0692 0·0290	0·9353 9·9710	1·1083 0·0447	0·9023 9·9553	1·1360 0·0554	0·8802 9·9446	1·1475 0·0598	0·8714 9·9,02	0·99992
Sa, Ssa	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·0000 0·0000	1·00000

TABLE 11.—*Values of log R′ for obtaining the factor F of* L_2 *from that of* M_2.

P	I = inclination of moon's orbit.											
	18°	19°	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°
°												
0	0·0778	0·0889	0·1014	0·1154	0·1312	0·1491	0·1696	0·1935	0·2216	0·2555	0·2976	0·3523
5	0·0764	0·0873	0·0994	0·1131	0·1284	0·1459	0·1658	0·1888	0·2158	0·2482	0·2881	0·3393
10	0·0722	0·0824	0·0938	0·1064	0·1206	0·1365	0·1547	0·1754	0·1995	0·2279	0·2622	0·3047
15	0·0656	0·0747	0·0847	0·0959	0·1083	0·1221	0·1377	0·1552	0·1753	0·1984	0·2255	0·2579
20	0·0569	0·0646	0·0731	0·0824	0·0926	0·1040	0·1165	0·1305	0·1462	0·1639	0·1840	0·2072
25	0·0467	0·0528	0·0595	0·0668	0·0748	0·0835	0·0930	0·1035	0·1150	0·1277	0·1418	0·1575
30	0·0354	0·0400	0·0448	0·0501	0·0558	0·0620	0·0687	0·0759	0·0837	0·0922	0·1014	0·1114
35	0·0236	0·0265	0·0296	0·0330	0·0366	0·0404	0·0445	0·0489	0·0536	0·0586	0·0640	0·0697
40	0·0117	0·0130	0·0146	0·0161	0·0178	0·0196	0·0215	0·0235	0·0256	0·0278	0·0301	0·0326
45	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
50	9·9889	9·9877	9·9864	9·9850	9·9836	9·9820	9·9805	9·9788	9·9771	9·9754	9·9736	9·9717
55	9·9787	9·9764	9·9739	9·9714	9·9687	9·9659	9·9631	9·9601	9·9570	9·9539	9·9506	9·9473
60	9·9696	9·9663	9·9628	9·9593	9·9556	9·9518	9·9479	9·9439	9·9398	9·9355	9·9312	9·9267
65	9·9616	9·9575	9·9533	9·9490	9·9445	9·9398	9·9351	9·9302	9·9252	9·9201	9·9149	9·9097
70	9·9549	9·9503	9·9454	9·9404	9·9353	9·9300	9·9246	9·9190	9·9134	9·9077	9·9018	9·8959
75	9·9497	9·9445	9·9392	9·9337	9·9281	9·9223	9·9164	9·9104	9·9043	9·8981	9·8918	9·8854
80	9·9459	9·9404	9·9347	9·9289	9·9229	9·9168	9·9106	9·9042	9·8978	9·8913	9·8846	9·8780
85	9·9436	9·9379	9·9320	9·9260	9·9198	9·9135	9·9071	9·9006	9·8939	9·8872	9·8804	9·8735
90	9·9429	9·9371	9·9312	9·9250	9·9188	9·9124	9·9059	9·8993	9·8926	9·8858	9·8790	9·8720
95	9·9436	9·9379	9·9320	9·9260	9·9198	9·9135	9·9071	9·9006	9·8939	9·8872	9·8804	9·8735
100	9·9459	9·9404	9·9347	9·9289	9·9229	9·9168	9·9106	9·9042	9·8978	9·8913	9·8846	9·8780
105	9·9497	9·9445	9·9392	9·9337	9·9281	9·9223	9·9164	9·9104	9·9043	9·8981	9·8918	9·8854
110	9·9549	9·9503	9·9454	9·9404	9·9353	9·9300	9·9246	9·9190	9·9134	9·9077	9·9018	6·8959
115	9·9616	9·9575	9·9533	9·9490	9·9445	9·9398	9·9351	9·9302	9·9252	9·9201	9·9149	9·9097
120	9·9696	9·9663	9·9628	9·9593	9·9556	9·9518	9·9479	9·9439	9·9398	9·9355	9·9312	9·9267
125	9·9787	9·9764	9·9739	9·9714	9·9687	9·9659	9·9631	9·9601	9·9570	9·9539	9·9506	9·9473
130	9·9889	9·9877	9·9864	9·9850	9·9836	9·9820	9·9805	9·9788	9·9771	9·9754	9·9736	9·9717
135	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
140	0·0117	0·0130	0·0146	0·0161	0·0178	0·0196	0·0215	0·0235	0·0256	0·0278	0·0301	0·0326
145	0·0236	0·0265	0·0296	0·0330	0·0366	0·0404	0·0445	0·0489	0·0536	0·0586	0·0640	0·0697
150	0·0354	0·0400	0·0448	0·0501	0·0558	0·0620	0·0687	0·0759	0·0837	0·0922	0·1014	0·1114
155	0·0467	0·0528	0·0595	0·0668	0·0748	0·0835	0·0930	0·1035	0·1150	0·1277	0·1418	0·1575
160	0·0569	0·0646	0·0731	0·0824	0·0926	0·1040	0·1165	0·1305	0·1462	0·1639	0·1840	0·2072
165	0·0656	0·0747	0·0847	0·0959	0·1083	0·1221	0·1377	0·1552	0·1753	0·1984	0·2255	0·2579
170	0·0722	0·0824	0·0938	0·1064	0·1206	0·1365	0·1547	0·1754	0·1995	0·2279	0·2622	0·3047
175	0·0764	0·0873	0·0994	0·1131	0·1284	0·1459	0·1658	0·1888	0·2158	0·2482	0·2881	0·3393
180	0·0778	0·0889	0·1014	0·1154	0·1312	0·1491	0·1696	0·1935	0·2216	0·2555	0·2976	0·3523

Log F (L_2) = log F (M_2) + log R', where $R' = \left(\frac{1}{1 - 12 \tan^2 \frac{1}{2} I \cos 2P}\right)^{\frac{1}{2}}$. The values of I and P for the first day of every month are given in Table 6.

When P lies between 180° and 360° subtract 180° from it, and enter the table with the remainder.

TABLE 12.—*Values of log Q′ for obtaining the factor F of* M_1 *from that of* O_1.

P	Log Q'	P	Log Q'	P	Log Q'	P	Log Q'	P	Log Q'	P	Log Q'
°		°		°		°		°		°	
0	9·6990	60	9·8785	120	9·8785	180	9·6990	240	9·8785	300	9·8785
1	9·6990	61	9·8841	121	9·8729	181	9·6990	241	9·8841	301	9·8729
2	9·6992	62	9·8898	122	9·8673	182	9·6992	242	9·8898	302	9·8673
3	9·6994	63	9·8955	123	9·8618	183	9·6994	243	9·8955	303	9·8618
4	9·6998	64	9·9012	124	9·8563	184	9·6998	244	9·9012	304	9·8563
5	9·7002	65	9·9068	125	9·8509	185	9·7002	245	9·9068	305	9·8509
6	9·7008	66	9·9125	126	9·8456	186	9·7008	246	9·9125	306	9·8456
7	9·7014	67	9·9181	127	9·8403	187	9·7014	247	9·9181	307	9·8403
8	9·7022	68	9·9237	128	9·8351	188	9·7022	248	9·9237	308	9·8351
9	9·7030	69	9·9292	129	9·8300	189	9·7030	249	9·9292	309	9·8300
10	9·7039	70	9·9347	130	9·8249	190	9·7039	250	9·9347	310	9·8249
11	9·7050	71	9·9400	131	9·8200	191	9·7050	251	9·9400	311	9·8200
12	9·7061	72	9·9453	132	9·8151	192	9·7061	252	9·9453	312	9·8151
13	9·7074	73	9·9504	133	9·8103	193	9·7074	253	9·9504	313	9·8103
14	9·7087	74	9·9554	134	9·8056	194	9·7087	254	9·9554	314	9·8056
15	9·7102	75	9·9602	135	9·8010	195	9·7102	255	9·9602	315	9·8010
16	9·7117	76	9·9649	136	9·7965	196	9·7117	256	9·9649	316	9·7965
17	9·7134	77	9·9693	137	9·7921	197	9·7134	257	9·9693	317	9·7921
18	9·7151	78	9·9735	138	6·7878	198	9·7151	258	9·9735	318	9·7878
19	9·7170	79	9·9775	139	9·7836	199	9·7170	259	9·9775	319	9·7836
20	9·7190	80	9·9812	140	9·7795	200	9·7190	260	9·9812	320	9·7795
21	9·7210	81	9·9846	141	9·7755	201	9·7210	261	9·9846	321	9·7755
22	9·7231	82	9·9877	142	9·7716	202	9·7231	262	9·9877	322	9·7716
23	9·7254	83	9·9905	143	9·7678	203	9·7254	263	9·9905	323	9·7678
24	9·7277	84	9·9930	144	9·7641	204	9·7277	264	9·9930	324	9·7641
25	9·7302	85	9·9951	145	9·7605	205	9·7302	265	9·9951	325	9·7605
26	9·7328	86	9·9968	146	9·7570	206	9·7328	266	9·9968	326	9·7570
27	9·7354	87	9·9982	147	9·7536	207	9·7354	267	9·9982	327	9·7536
28	9·7382	88	9·9992	148	9·7503	208	9·7382	268	9·9992	328	9·7503
29	9·7411	89	9·9998	149	9·7471	209	9·7411	269	9·9998	329	9·7471
30	9·7441	90	10·0000	150	9·7441	210	9·7441	270	10.0000	330	9·7441
31	9·7471	91	9·9998	151	9·7411	211	9·7471	271	9·9998	331	9·7411
32	9·7503	92	9·9992	152	9·7382	212	9·7503	272	9·9992	332	9·7382
33	9·7536	93	9·9982	153	9·7354	213	9·7536	273	9·9982	333	9·7354
34	9·7570	94	9·9968	154	9·7328	214	9·7570	274	9·9968	334	9·7328
35	9·7605	95	9·9951	155	9·7302	215	9·7605	275	9·9951	335	9·7302
36	9·7641	96	9·9930	156	9·7277	216	9·7641	276	9·9930	336	9·7277
37	9·7678	97	9·9905	157	9·7254	217	9·7678	277	9·9905	337	9·7254
38	9·7716	98	9·9877	158	9·7231	218	9·7716	278	9·9877	338	9·7231
39	9·7755	99	9·9846	159	9·7210	219	9·7755	279	9·9846	339	9·7210
40	9·7795	100	9·9812	160	9·7190	220	9·7795	280	9·9812	340	9·7190
41	9·7836	101	9·9775	161	9·7170	221	9·7836	281	9·9775	341	9·7170
42	9·7878	102	9·9735	162	9·7151	222	9·7878	282	9·9735	342	9·7151
43	9·7921	103	9·9693	163	9·7134	223	9·7921	283	9·9693	343	9·7134
44	9·7965	104	9·9649	164	9·7117	224	9·7965	284	9·9649	344	9·7117
45	9·8010	105	9·9602	165	9·7102	225	9·8010	285	9·9602	345	9·7102
46	9·8056	106	9·9554	166	9·7087	226	9·8056	286	9·9554	346	9·7087
47	9·8103	107	9·9504	167	9·7074	227	9·8103	287	9·9504	347	9·7074
48	9·8151	108	9·9453	168	9·7061	228	9·8151	288	9·9453	348	9·7061
49	9·8200	109	9·9400	169	9·7050	229	9·8200	289	9·9400	349	9·7050
50	9·8249	110	9·9347	170	9·7039	230	9·8249	290	9·9347	350	9·7039
51	9·8300	111	9·9292	171	9·7030	231	9·8300	291	9·9292	351	9·7030
52	9·8351	112	9·9237	172	9·7022	232	9·8351	292	9·9237	352	9·7022
53	9·8403	113	9·9181	173	9·7014	233	9·8403	293	9·9181	353	9·7014
54	9·8456	114	9·9125	174	9·7008	234	9·8456	294	9·9125	354	9·7008
55	9·8509	115	9·9068	175	9·7002	235	9·8509	295	9·9068	355	9·7002
56	9·8563	116	9·9012	176	9·6998	236	9·8563	296	9·9012	356	9·6998
57	9·8618	117	9·8955	177	9·6994	237	9·8618	297	9·8955	357	9·6994
58	9·8673	118	9·8898	178	9·6992	238	9·8673	298	9·8898	358	9·6992
59	9·8729	119	9·8841	179	9·6990	239	9·8729	299	9·8841	359	9·6990

$$\text{Log } F\,(M_1) = \log F\,(O_1) + \log Q', \text{ where } Q' = \frac{1}{(2{\cdot}5 + 1{\cdot}5 \cos 2P)^{\frac{1}{2}}}.$$

The value of P for the first day of every month is given in Table 6.

TABLE 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides.*

I	$F(J_1)$	Diff.	$f(J_1)$	Diff.
°				
18·3	1·21005	565	0·82640	389
·4	·20440	558	·83029	387
·5	·19882	552	·83416	385
·6	·19330	546	·83801	385
·7	·18784	539	·84186	384
·8	·18245	533	·84570	383
18·9	·17712	527	·84953	382
19·0	·17185	520	·85335	381
·1	·16665	514	·85716	379
·2	·16151	509	·86095	379
·3	·15642	503	·86474	377
·4	·15139	497	·86851	377
19·5	·14642	491	·87228	375
·6	·14151	486	·87603	375
·7	·13665	480	·87978	373
·8	·13185	475	·88351	372
19·9	·12710	470	·88723	372
20·0	·12240	464	·89095	370
·1	·11776	459	·89465	369
·2	·11317	454	·89834	368
·3	·10863	449	·90202	366
·4	·10414	444	·90568	366
20·5	·09970	439	·90934	365
·6	·09531	435	·91299	363
·7	·09096	429	·91662	362
·8	·08667	425	·92024	362
20·9	·08242	420	·92386	360
21·0	·07822	416	·92746	359
·1	·07406	411	·93105	358
·2	·06995	407	·93463	356
·3	·06588	403	·93819	356
·4	·06185	398	·94175	354
21·5	·05787	394	·94529	354
·6	·05393	389	·94883	352
·7	·05004	386	·95235	351
·8	·04618	381	·95586	350
21·9	·04237	378	·95936	348
22·0	·03859	373	·96284	348
·1	·03486	370	·96632	346
·2	·03116	365	·96978	345
·3	·02751	362	·97323	344
·4	·02389	358	·97667	343
22·5	·02031	354	·98010	341
·6	·01677	351	·98351	341
·7	·01326	347	·98692	339
·8	·00979	343	·99031	338
22·9	·00636	340	·99369	336
23·0	1·00296	337	0·99705	336
·1	0·99959	333	1·00041	334
·2	·99626	329	·00375	333
23·3	0·99297	326	1·00708	332

I	$F(J_1)$	Diff.	$f(J_1)$	Diff.
°				
23·3	0·99297	326	1·00708	332
·4	·98971	323	·01040	330
·5	·98648	319	·01370	330
·6	·98329	317	·01700	328
·7	·98012	313	·02028	327
·8	·97699	310	·02355	326
23·9	·97389	306	·02681	324
24·0	·97083	304	·03005	323
·1	·96779	300	·03328	322
·2	·96479	298	·03650	320
·3	·96181	295	·03970	320
·4	·95886	291	·04290	318
24·5	·95595	288	·04608	316
·6	·95307	286	·04924	316
·7	·95021	283	·05240	314
·8	·94738	280	·05554	313
24·9	·94458	277	·05867	312
25·0	·94181	275	·06179	310
·1	·93906	272	·06489	309
·2	·93634	269	·06798	308
·3	·93365	266	·07106	306
·4	·93099	264	·07412	305
25·5	·92835	261	·07717	304
·6	·92574	258	·08021	303
·7	·92316	256	·08324	301
·8	·92060	254	·08625	300
25·9	·91806	251	·08925	298
26·0	·91555	248	·09223	298
·1	·51307	246	·09521	296
·2	·91061	244	·09817	294
·3	·90817	241	·10111	293
·4	·90576	239	·10404	292
26·5	·90337	236	·10696	291
·6	·90101	234	·10987	289
·7	·89867	232	·11276	288
·8	·89635	230	·11564	286
26·9	·89405	227	·11850	285
27·0	·89178	225	·12135	284
·1	·88953	223	·12419	282
·2	·88730	220	·12701	281
·3	·88510	219	·12982	280
·4	·88291	216	·13262	278
27·5	·88075	214	·13540	277
·6	·87861	213	·13817	275
·7	·87648	210	·14092	274
·8	·87438	208	·14366	273
27·9	·87230	205	·14639	271
28·0	·87025	204	·14910	270
·1	·86821	202	·15180	268
·2	·86619	200	·15448	267
·3	·86419	198	·15715	266
·4	·86221	196	·15981	264
·5	·86025	194	·16245	263
28·6	0·85831		1·16508	

$$F = 1/f = \frac{\sin\omega\cos\omega\,(1 - \tfrac{3}{2}\sin^2 i)}{\sin I \cos I} = \frac{0{\cdot}72147}{\sin 2I}.$$

$F(J_1) = F$ for lunar K_1.
$f(J_1) = f$ for lunar K_1.

I is given for the first of each month in Table 6.

TABLE 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides*—Continued.

I	$F(K_1)$	Diff.	$f(K_1)$	Diff.	*I*	$F(K_1)$	Diff.	$f(K_1)$	Diff.
°					°				
18·3	1·13450	308	0·88145	240	23·3	1·00073	228	0·99927	228
·4	·13142	307	·88385	241	·4	0·99845	226	1·00155	228
·5	·12835	305	·88626	240	·5	·99619	224	·00383	227
·6	·12530	303	·88866	240	·6	·99395	223	·00610	226
·7	·12227	301	·89106	239	·7	·99172	222	·00836	226
·8	·11926	299	·89345	240	·8	·98950	221	·01062	225
18·9	·11627	298	·89585	239	23·9	·98729	219	·01287	225
19·0	·11329	296	·89824	240	24·0	·98510	217	·01512	225
·1	·11033	294	·90064	240	·1	·98293	216	·01737	224
·2	·10739	292	·90304	339	·2	·98077	215	·01961	224
·3	·10447	291	·90543	238	·3	·97862	214	·02185	224
·4	·10156	289	·90781	238	·4	·97648	213	·02409	223
19·5	·09867	288	·91019	239	24·5	·97435	210	·02632	222
·6	·09579	286	·91258	239	·6	·97225	209	·02854	222
·7	·09293	284	·91497	238	·7	·97016	208	·03076	221
·8	·09009	282	·91735	239	·8	·96808	207	·03297	221
19·9	·08727	281	·91974	238	24·9	·96601	205	·03518	220
20·0	·08446	280	·92212	239	25·0	·96396	203	·03738	220
·1	·08166	277	·92451	238	·1	·96193	202	·03958	219
·2	·07889	276	·92689	238	·2	·95991	201	·04177	219
·3	·07613	274	·92927	237	·3	·95790	200	·04396	218
·4	·07339	272	·93164	236	·4	·95590	199	·04614	218
20·5	·07067	271	·93400	237	25·5	·95391	197	·04832	217
·6	·06796	269	·93637	237	·6	·95194	195	·05049	216
·7	·06527	268	·93874	237	·7	·94999	194	·05265	216
·8	·06259	267	·94111	236	·8	·94805	193	·05481	215
20·9	·05992	265	·94347	236	25·9	·94612	193	·05696	215
21·0	·05727	263	·94583	236	26·0	·94419	191	·05911	214
·1	·05464	261	·94819	235	·1	·94228	189	·06125	213
·2	·05203	260	·95054	235	·2	·94039	188	·06338	213
·3	·04943	258	·95289	235	·3	·93851	187	·06551	213
·4	·04685	256	·95524	235	·4	·93664	185	·06764	212
21·5	·04429	255	·95759	235	26·5	·93479	184	·06976	211
·6	·04174	254	·95994	234	·6	·93295	183	·07187	211
·7	·03920	252	·96228	234	·7	·93112	182	·07398	210
·8	·03668	251	·96462	234	·8	·92930	181	·07608	209
21·9	·03417	249	·96696	233	26·9	·92749	179	·07817	209
22·0	·03168	247	·96929	233	27·0	·92570	178	·08026	208
·1	·02921	245	·97162	232	·1	·92392	176	·08234	207
·2	·02676	244	·97394	232	·2	·92216	175	·08441	206
·3	·02432	243	·97626	232	·3	·92041	174	·08647	206
·4	·02189	241	·97858	231	·4	·91867	174	·08853	206
22·5	·01948	239	·98089	231	27·5	·91693	172	·09059	205
·6	·01709	238	·08320	231	·6	·91521	170	·09264	204
·7	·01471	237	·98551	230	·7	·91351	170	·09468	204
·8	·01234	236	·98781	230	·8	·91181	168	·09672	203
22·9	·00998	234	·99011	230	27·9	·91013	167	·09875	202
23·0	·00764	232	·99241	229	28·0	·90846	166	·10077	201
·1	·00532	230	·99470	228	·1	·90680	165	·10278	200
·2	·00302	229	·99698	229	·2	·90515	163	·10478	200
23·3	1·00073	228	0·99927	228	·3	·90352	162	·10678	200
					·4	·90190	162	·10878	199
					·5	·90028	160	·11077	198
					28·6	0·89868		1·11275	

$$F = 1/f = \frac{1{\cdot}05628}{(\sin^2 2\,I + 0{\cdot}66962 \cos \nu \sin 2\,I + 0{\cdot}11210)^{\frac{1}{2}}}.$$

I is given for the first of each month in Table 6.

TABLE 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides*—Continued.

I	F (K₂)	Diff.	f (K₂)	Diff.	I	F (K₂)	Diff.	f (K₂)	Diff.
°					°				
18·3	1·33821	652	0·74732	363	23·3	1·02291	578	0·97760	556
·4	·33169	651	·75095	367	·4	·01713	576	·98316	560
·5	·32518	652	·75462	372	·5	·01137	572	·98876	563
·6	·31866	652	·75834	377	·6	·00565	569	0·99439	566
·7	·31214	652	·76211	381	·7	0·99996	567	1·00005	570
·8	·30562	652	·76592	385	·8	·99429	564	·00575	573
18·9	·29910	653	·76977	388	23·9	·98865	561	·01148	577
19·0	·29257	653	·77365	392	24·0	·98304	558	·01725	580
·1	·28604	653	·77757	397	·1	·97746	554	·02305	583
·2	·27951	653	·78154	401	·2	·97192	551	·02888	587
·3	·27298	653	·78555	405	·3	·96641	548	·03475	590
·4	·26645	652	·78960	409	·4	·96093	545	·04065	594
19·5	·25993	652	·79369	412	24·5	·95548	542	·04659	597
·6	·25341	651	·79781	417	·6	·95006	539	·05256	600
·7	·24690	651	·80198	421	·7	·94467	535	·05856	604
·8	·24039	650	·80619	425	·8	·93932	532	·06460	607
19·9	·23389	649	·81044	429	24·9	·93400	529	·07067	610
20·0	·22740	648	·81473	432	25·0	·92871	526	·07677	613
·1	·22092	647	·81905	436	·1	·92345	523	·08290	616
·2	·21445	646	·82341	440	·2	·91822	520	·08906	620
·3	·20799	645	·82781	445	·3	·91302	516	·09526	623
·4	·20154	644	·83226	449	·4	·90786	513	·10149	626
20·5	·19510	642	·83675	452	25·5	·90273	510	·10775	629
·6	·18868	641	·84127	456	·6	·89763	507	·11404	633
·7	·18227	639	·84583	460	·7	·89256	504	·12037	636
·8	·17588	638	·85043	464	·8	·88752	500	·12673	639
20·9	·16950	637	·85507	468	25·9	·88252	497	·13312	642
21·0	·16313	634	·85975	471	26·0	·87755	494	·13954	645
·1	·15679	632	·86446	475	·1	·87261	491	·14599	648
·2	·15047	630	·86921	479	·2	·86770	488	·15247	651
·3	·14417	629	·87400	483	·3	·86282	484	·15898	654
·4	·13788	627	·87883	487	·4	·85798	481	·16552	658
21·5	·13161	625	·88370	490	26·5	·85317	478	·17210	660
·6	·12536	622	·88860	494	·6	·84839	475	·17870	663
·7	·11914	620	·89354	498	·7	·84364	472	·18533	666
·8	·11294	618	·89852	502	·8	·83892	468	·19199	670
21·9	·10676	616	·90354	506	26·9	·83424	465	·19869	673
22·0	·10060	613	·90860	509	27·0	·82959	462	·20542	675
·1	·09447	610	·91369	513	·1	·82497	459	·21217	678
·2	·08837	608	·91882	516	·2	·82038	456	·21895	681
·3	·08229	606	·92398	520	·3	·81582	453	·22576	684
·4	·07623	604	·92918	524	·4	·81129	450	·23260	688
22·5	·07019	601	·93442	527	27·5	·80679	447	·23948	690
·6	·06418	598	·93969	531	·6	·80232	444	·24638	693
·7	·05820	595	·94500	534	·7	·79788	440	·25331	695
·8	·05225	592	·95034	538	·8	·79348	437	·26026	698
22·9	·04633	590	·95572	542	27·9	·78911	434	·26724	701
23·0	·04043	587	·96114	545	28·0	·78477	431	·27425	704
·1	·03456	584	·96659	549	·1	·78046	428	·28129	707
·2	·02872	581	·97208	552	·2	·77618	425	·28836	710
23·3	1·02291	578	0·97760	556	·3	·77193	422	·29546	713
					·4	·76771	420	·30259	715
					·5	·76351	417	·30974	718
					28·6	0·75934		1·31692	

$$F = 1/f = \frac{0·22915}{(\sin^4 I + 0·14527 \cos 2\nu \sin^2 I + 0·00528)^{\frac{1}{2}}}.$$

I is given for the first of each month in Table 6.

TABLE 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides*—Continued.

I	F for lunar K_2	Diff.	f for lunar K_2	Diff.
°				
18·3	1·58749	1661	0·62992	666
·4	·57088	1634	·63658	670
·5	·55454	1609	·64328	672
·6	·53845	1583	·65000	676
·7	·52262	1557	·65676	679
·8	·50705	1533	·66355	682
18·9	·49172	1509	·67037	685
19·0	·47663	1485	·67722	688
·1	·46178	1462	·68410	691
·2	·44716	1440	·69101	694
·3	·43276	1417	·69795	698
·4	·41859	1395	·70493	700
19·5	·40464	1374	·71193	703
·6	·39090	1353	·71896	706
·7	·37737	1333	·72602	710
·8	·36404	1313	·73312	712
19·9	·35091	1292	·74024	715
20·0	·33799	1274	·74739	718
·1	·32525	1255	·75457	722
·2	·31270	1236	·76179	724
·3	·30034	1218	·76903	727
·4	·28816	1200	·77630	730
20·5	·27616	1183	·78360	733
·6	·26433	1166	·79093	736
·7	·25267	1149	·79829	739
·8	·24118	1132	·80598	742
20·9	·22986	1116	·81310	745
21·0	·21870	1100	·82055	747
·1	·20770	1085	·82802	751
·2	·19685	1070	·83553	753
·3	·18615	1054	·84306	756
·4	·17561	1040	·85062	759
21·5	·16521	1025	·85821	762
·6	·15496	1012	·86583	765
·7	·14484	997	·87348	768
·8	·13487	984	·88116	770
21·9	·12503	970	·88886	773
22·0	·11533	957	·89659	776
·1	·10576	944	·90435	779
·2	·09632	931	·91214	782
·3	·08701	919	·91996	784
·4	·07782	907	·92780	787
22·5	·06875	895	·93567	790
·6	·05980	883	·94357	793
·7	·05097	871	·95150	795
·8	·04226	860	·95945	798
22·9	·03366	848	·96743	801
23·0	·02518	838	·97544	804
·1	·01680	827	·98348	806
·2	·00853	816	·99154	809
23·3	1·00037	805	0·99963	811

I	F for lunar K_2	Diff.	f for lunar K_2	Diff.
°				
23·3	1·00037	805	0·99963	811
·4	0·99232	796	1·00774	814
·5	·98436	785	·01588	817
·6	·97651	775	·02405	820
·7	·96876	766	·03225	822
·8	·96110	756	·04047	825
23·9	·95354	746	·04872	827
24·0	·94608	737	·05699	830
·1	·93871	728	·06529	833
·2	·93143	719	·07362	835
·3	·92424	710	·08197	838
·4	·91714	701	·09035	840
24·5	·91013	693	·09875	843
·6	·90320	685	·10718	845
·7	·89635	676	·11563	848
·8	·88959	668	·12411	851
24·9	·88291	660	·13262	853
25·0	·87631	652	·14115	855
·1	·86979	644	·14970	858
·2	·86335	637	·15828	861
·3	·85698	629	·16689	863
·4	·85069	622	·17552	865
25·5	·84447	614	·18417	868
·6	·83833	607	·19285	870
·7	·83226	600	·20155	873
·8	·82626	594	·21028	875
25·9	·82032	586	·21903	878
26·0	·81446	579	·22781	880
·1	·80867	573	·23661	882
·2	·80294	567	·24543	884
·3	·79727	560	·25427	887
·4	·79167	555	·26314	890
26·5	·78614	547	·27204	892
·6	·78067	541	·28096	894
·7	·77526	535	·28990	896
·8	·76991	529	·29886	899
26·9	·76462	524	·30785	901
27·0	·75938	517	·31686	903
·1	·75421	512	·32589	906
·2	·74909	506	·33495	908
·3	·74403	500	·34403	910
·4	·73903	495	·35313	912
27·5	·73408	490	·36225	915
·6	·72918	484	·37140	916
·7	·72434	479	·38056	919
·8	·71955	474	·38975	921
27·9	·71481	468	·39896	924
28·0	·71013	464	·40820	926
·1	·70549	459	·41746	927
·2	·70090	454	·42673	930
·3	·69636	449	·43603	932
·4	·69187	444	·44535	934
·5	·68743	440	·45469	937
28·6	0·68303		1·46406	

$$F = 1/f = \frac{\sin^2 \omega \, (1 - \frac{3}{2} \sin^2 i)}{\sin^2 I} = \frac{0{\cdot}15651}{\sin^2 I}.$$

I is given for the first of each month in Table 6.

Table 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides*—Continued.

I	$F(M_2)$	Diff.	$f(M_2)$	Diff.	I	$F(M_2)$	Diff.	$f(M_2)$	Diff.
°					°				
18·3	0·96349	54	1·03789	58	23·3	0·99486	72	1·00517	73
·4	·96403	55	·03731	59	·4	·99558	72	·00444	72
·5	·96458	55	·03672	59	·5	·99630	72	·00372	73
·6	·96513	55	·03613	59	·6	·99702	73	·00299	74
·7	·96568	56	·03554	60	·7	·99775	73	·00225	73
·8	·96624	56	·03494	60	·8	·99848	74	·00152	74
18·9	·96680	56	·03434	60	23·9	·99922	74	·00078	74
19·0	·96736	57	·03374	61	24·0	0·99996	74	1·00004	74
·1	·96793	57	·03313	61	·1	1·00070	75	0·99930	75
·2	·96850	57	·03252	61	·2	·00145	75	·99855	75
·3	·96907	58	·03191	61	·3	·00220	76	·99780	75
·4	·96965	58	·03130	62	·4	·00296	76	·99705	75
19·5	·97023	58	·03068	62	24·5	·00372	76	·99630	76
·6	·97081	59	·03006	62	·6	·00448	77	·99554	76
·7	·97140	59	·02944	63	·7	·00525	77	·99478	76
·8	·97199	60	·02881	62	·8	·00602	77	·99402	76
19·9	·97259	59	·02819	63	24·9	·00679	78	·99326	77
20·0	·97318	60	·02756	64	25·0	·00757	78	·99249	77
·1	·97378	61	·02692	63	·1	·00835	79	·99172	77
·2	·97439	61	·02629	64	·2	·00914	78	·99095	78
·3	·97500	61	·02565	65	·3	·00992	80	·99017	77
·4	·97561	61	·02500	64	·4	·01072	80	·98940	78
20·5	·97622	62	·02436	65	25·5	·01152	80	·98862	79
·6	·97684	62	·02371	65	·6	·01232	80	·98783	78
·7	·97746	63	·02306	65	·7	·01312	81	·98705	79
·8	·97809	62	·02241	66	·8	·01393	81	·98626	79
20·9	·97871	64	·02175	66	25·9	·01474	82	·98547	79
21·0	·97935	63	·02109	66	26·0	·01556	82	·98468	79
·1	·97998	64	·02043	67	·1	·01638	82	·98389	80
·2	·98062	64	·01976	66	·2	·01720	83	·98309	80
·3	·98126	65	·01910	67	·3	·01803	83	·98229	81
·4	·98191	65	·01843	68	·4	·01886	84	·98148	80
21·5	·98256	65	·01775	67	26·5	·01970	84	·98068	81
·6	·98321	66	·01708	68	·6	·02054	84	·97987	81
·7	·98387	66	·01640	68	·7	·02138	85	·97906	81
·8	·98453	66	·01572	69	·8	·02223	85	·97825	81
21·9	·98519	67	·01503	68	26·9	·02308	86	·97744	82
22·0	·98586	67	·01435	69	27·0	·02394	86	·97662	82
·1	·98653	67	·01366	70	·1	·02480	86	·97580	82
·2	·98720	68	·01296	69	·2	·02566	87	·97498	83
·3	·98788	68	·01227	70	·3	·02653	87	·97415	82
·4	·98856	69	·01157	70	·4	·02740	88	·97333	83
22·5	·98925	69	·01087	70	27·5	·02828	88	·97250	83
·6	·98994	69	·01017	71	·6	·02916	89	·97167	84
·7	·99063	69	·00946	71	·7	·03005	88	·97083	84
·8	·99132	70	·00875	71	·8	·03093	90	·96999	84
22·9	·99202	71	·00804	71	27·9	·03183	89	·96915	84
23·0	·99273	70	·00733	72	28·0	·03272	91	·96831	84
·1	·99343	71	·00661	72	·1	·03363	90	·96747	85
·2	·99414	72	·00589	72	·2	·03453	91	·96662	85
23·3	0·99486	72	1·00517	73	·3	·03544	91	·96577	85
					·4	·03635	92	·96492	85
					·5	·03727	92	·96407	86
					28·6	1·03819		0·96321	

$$F = 1/f = \frac{\cos^4 \frac{1}{2}\omega \cos^4 \frac{1}{2} i}{\cos^4 \frac{1}{2} I} = \frac{0{\cdot}91538}{\cos^4 \frac{1}{2} I}$$

$$F(M_2) = F(N_2) = F(2N) = F(MS) = F(2SM) = F(MSf) = F(\lambda_2) = F(\mu_2) = F(\nu_2).$$
$$f(M_2) = f(N_2) = f(2N) = f(MS) = f(2SM) = f(MSf) = f(\lambda_2) = f(\mu_2) = f(\nu_2).$$

I is given for the first of each month in Table 6.

TABLE 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides*—Continued.

I	$F(O_1)$	Diff.	$f(O_1)$	Diff.	I	$F(O_1)$	Diff.	$f(O_1)$	Diff.
°					°				
18·3	1·24178	617	0·80529	403	23·3	1·00166	368	0·99834	368
·4	·23561	610	·80932	402	·4	0·99798	364	1·00202	368
·5	·22951	603	·81334	400	·5	·99434	362	·00570	367
·6	·22348	596	·81734	400	·6	·99072	358	·00937	366
·7	·21752	590	·82134	400	·7	·98714	354	·01303	365
·8	·21162	583	·82534	399	·8	·98360	352	·01668	365
18·9	·20579	576	·82933	398	23·9	·98008	348	·02033	363
19·0	·20003	570	·83331	398	24·0	·97660	345	·02396	363
·1	·19433	564	·83729	397	·1	·97315	342	·02759	363
·2	·18869	558	·84126	397	·2	·96973	339	·03122	361
·3	·18311	552	·84523	396	·3	·96634	336	·03483	361
·4	·17759	545	·84919	395	·4	·96298	332	·03844	360
19·5	·17214	540	·85314	394	24·5	·95966	330	·04204	359
·6	·16674	533	·85708	394	·6	·95636	327	·04563	358
·7	·16141	528	·86102	394	·7	·95309	323	·04921	358
·8	·15613	523	·86496	392	·8	·94986	321	·05279	357
19·9	·15090	517	·86888	392	24·9	·94665	318	·05636	356
20·0	·14573	511	·87280	392	25·0	·94347	316	·05992	355
·1	·14062	506	·87672	390	·1	·84031	312	·06347	355
·2	·13556	501	·88062	390	·2	·93719	310	·06702	354
·3	·13055	495	·88452	390	·3	·93409	307	·07056	353
·4	·12560	491	·88842	388	·4	·93102	304	·07409	352
20·5	·12069	485	·89230	388	25·5	·92798	301	·07761	351
·6	·11584	480	·89618	388	·6	·92497	299	·08112	351
·7	·11104	475	·90006	387	·7	·92198	296	·08463	349
·8	·10629	471	·90393	386	·8	·91902	294	·08812	349
20·9	·10158	465	·90779	385	25·9	·91608	292	·09161	348
21·0	·09693	461	·91164	385	26·0	·91316	289	·09509	348
·1	·09232	457	·91549	384	·1	·01027	286	·09857	346
·2	·08775	451	·91933	383	·2	·90741	283	·10203	346
·3	·08324	447	·92316	383	·3	·90458	282	·10549	345
·4	·07877	443	·92699	382	·4	·90176	279	·10894	344
21·5	·07434	438	·93081	381	26·5	·89897	276	·11238	343
·6	·06996	434	·93462	380	·6	·89621	274	·11581	343
·7	·06562	430	·93842	380	·7	·89347	272	·11924	342
·8	·06132	425	·94222	379	·8	·89075	270	·12266	340
21·9	·05707	421	·94601	379	26·9	·88805	267	·12606	340
22·0	·05286	417	·94980	377	27·0	·88538	265	·12946	339
·1	·04869	413	·95357	377	·1	·88273	263	·13285	339
·2	·04456	409	·95734	377	·2	·88010	261	·13624	337
·3	·04047	405	·96111	375	·3	·87749	258	·13961	337
·4	·03642	401	·96486	375	·4	·87491	256	·14298	336
22·5	·03241	398	·96861	374	27·5	·87235	254	·14634	335
·6	·02843	393	·97235	374	·6	·86981	253	·14969	334
·7	·02450	390	·97609	372	·7	·86728	250	·15303	333
·8	·02060	386	·97981	372	·8	·86478	248	·15636	332
22·9	·01674	382	·98353	371	27·9	·86230	245	·15968	332
23·0	·01292	379	·98724	371	28·0	·85985	244	·16300	331
·1	·00913	375	·99095	370	·1	·85741	242	·16631	330
·2	·00538	372	·99465	369	·2	·85499	240	·16961	329
23·3	1·00166	368	0·99834	368	·3	·85259	238	·17290	328
					·4	·85021	236	·17618	327
					·5	·84785	234	·17945	326
					28·6	0·84551		1·18271	

$$F = 1/f = \frac{\sin\omega\cos^2\frac{1}{2}\omega\cos^4\frac{1}{2}i}{\sin I\cos^2\frac{1}{2}I} = \frac{0{\cdot}38005}{\sin I\cos^2\frac{1}{2}I}$$

$$F(O_1) = F(Q_1) = F(M_1) \div Q'.$$
$$f(O_1) = f(Q_1) = f(M_1) \times Q'.$$

I is given for the first of each month in Table 6.

Table 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides*—Continued.

I	*F* (OO)	Diff.	*f* (OO)	Diff.
°				
18.3	2.06270	3289	0.48482	784
.4	2.02981	3218	.49226	793
.5	1.99763	3149	.50059	802
.6	.96614	3082	.50861	810
.7	.93532	3017	.51671	818
.8	.90515	2954	.52489	827
18.9	.87561	2892	.53316	835
19.0	.84669	2832	.54151	843
.1	.81837	2774	.54994	852
.2	.79063	2717	.55846	861
.3	.76346	2662	.56707	869
.4	.73684	2607	.57576	877
19.5	.71077	2555	.58453	886
.6	.68522	2503	.59339	895
.7	.66019	2454	.60234	904
.8	.63565	2404	.61138	912
19.9	.61161	2357	.62050	921
20.0	.58804	2310	.62971	929
.1	.56494	2265	.63900	939
.2	.54229	2221	.64839	947
.3	.52008	2178	.65786	956
.4	.49830	2135	.66742	965
20.5	.47695	2095	.67707	974
.6	.45600	2055	.68681	983
.7	.43545	2015	.69664	992
.8	.41530	1977	.70656	1002
20.9	.39553	1940	.71658	1010
21.0	.37613	1904	.72668	1019
.1	.35709	1868	.73687	1028
.2	.33841	1833	.74715	1038
.3	.32008	1799	.75753	1047
.4	.30209	1766	.76800	1056
21.5	.28443	1734	.77856	1065
.6	.26709	1702	.78921	1075
.7	.25007	1672	.79996	1084
.8	.23335	1641	.81080	1093
21.9	.21694	1611	.82173	1103
22.0	.20083	1583	.83276	1112
.1	.18500	1554	.84388	1122
.2	.16946	1527	.85510	1131
.3	.15419	1500	.86641	1141
.4	.13919	1473	.87782	1150
22.5	.12446	1448	.88932	1160
.6	.10998	1422	.90092	1169
.7	.09576	1398	.91261	1179
.8	.08178	1374	.92440	1189
22.9	.06804	1350	.93629	1199
23.0	.05454	1326	.94828	1208
.1	.04128	1305	.96036	1218
.2	.02823	1281	.97254	1228
23.3	1.01542	1261	0.98482	1237

I	*F* (OO)	Diff.	*f* (OO)	Diff.
°				
23.3	1.01542	1261	0.98482	1237
.4	1.00281	1239	0.99719	1248
.5	0.99042	1218	1.00967	1258
.6	.97824	1198	.02225	1267
.7	.96626	1178	.03492	1277
.8	.95448	1159	.04769	1288
23.9	.94289	1139	.06057	1297
24.0	.93150	1121	.07354	1307
.1	.92029	1102	.08661	1318
.2	.90927	1085	.09979	1327
.3	.89842	1067	.11306	1337
.4	.88775	1049	.12643	1348
24.5	.87726	1033	.13991	1359
.6	.86693	1016	.15350	1368
.7	.85677	1000	.16718	1378
.8	.84677	984	.18096	1389
24.9	.83693	969	.19485	1399
25.0	.82724	953	.20884	1409
.1	.81771	939	.22293	1420
.2	.80832	923	.23713	1430
.3	.79909	910	.25143	1440
.4	.78999	895	.26583	1451
25.5	.78104	881	.28034	1461
.6	.77223	868	.29495	1472
.7	.76355	854	.30967	1482
.8	.75501	842	.32449	1493
25.9	.74659	829	.33942	1503
26.0	.73830	816	.35445	1514
.1	.73014	804	.36959	1525
.2	.72210	791	.38484	1535
.3	.71419	780	.40019	1546
.4	.70639	768	.41565	1556
26.5	.69871	757	.43121	1567
.6	.69114	746	.44688	1578
.7	.68368	734	.46266	1589
.8	.67634	724	.47855	1600
26.9	.66910	713	.49455	1610
27.0	.66197	703	.51065	1621
.1	.65494	693	.52686	1632
.2	.64801	682	.54318	1643
.3	.64119	673	.55961	1654
.4	.63446	663	.57615	1664
27.5	.62783	654	.59279	1676
.6	.62129	644	.60955	1686
.7	.61485	635	.62641	1698
.8	.60850	626	.64339	1708
27.9	.60224	617	66047	1720
28.0	.59607	609	.67767	1730
.1	.58998	600	.69497	1742
.2	.58398	592	.71239	1753
.3	.57806	583	.72992	1764
.4	.57223	576	.74756	1775
.5	.56647	567	.76531	1786
28.6	0.56080		1.78317	

$$F = 1/f = \frac{\sin\omega \sin^2 \frac{1}{2}\omega \cos^4 \frac{1}{2} i}{\sin I \sin^2 \frac{1}{2} I} = \frac{0.01638}{\sin I \sin^2 \frac{1}{2} I}$$

I is given for the first of each month in Table 6.

TABLE 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides*—Continued.

I	F (Mf)	Diff.	f (Mf)	Diff.
°				
18·3	1·60043	1675	0·62486	658
·4	·58368	1648	·63144	663
·5	·56720	1622	·63807	668
·6	·55098	1596	·64475	671
·7	·53502	1571	·65146	673
·8	·51931	1545	·65819	677
18·9	·50386	1521	·66496	679
19·0	·48865	1497	·67175	683
·1	·47368	1474	·67858	685
·2	·45894	1451	·68543	689
·3	·44443	1429	·69232	691
·4	·43014	1407	·69923	695
19·5	·41607	1385	·70618	697
·6	·40222	1364	·71315	701
·7	·38858	1343	·72016	704
·8	·37515	1324	·72720	707
19·9	·36191	1303	·73427	709
20·0	·34888	1284	·74136	712
·1	·33604	1265	·74848	716
·2	·32339	1246	·75564	718
·3	·31093	1228	·76282	721
·4	·29865	1210	·77003	724
20·5	·28655	1193	·77727	728
·6	·27462	1175	·78455	730
·7	·26287	1158	·79185	733
·8	·25129	1142	·79918	736
20·9	·23987	1125	·80654	738
21·0	·22862	1109	·81392	742
·1	·21753	1094	·82134	744
·2	·20659	1078	·82878	747
·3	·19581	1063	·83625	750
·4	·18518	1048	·84375	753
21·5	·17470	1034	·85128	756
·6	·16436	1020	·85884	759
·7	·15416	1005	·86643	761
·8	·14411	992	·87404	764
21·9	·13419	978	·88168	767
22·0	·12441	965	·88935	770
·1	·11476	952	·89705	773
·2	·10524	939	·90478	775
·3	·09585	926	·91253	778
·4	·08659	914	·92031	781
22·5	·07745	902	·92812	783
·6	·06843	890	·93595	786
·7	·05953	878	·94381	789
·8	·05075	867	·95170	792
22·9	·04208	856	·95962	794
23·0	·03352	844	·96756	797
·1	·02508	834	·97553	800
·2	·01674	822	·98353	803
23·3	1·00852	812	0·99156	805

I	F (Mf)	Diff.	f (Mf)	Diff.
°				
23·3	1·00852	812	0·99156	805
·4	1·00040	802	0·99961	807
·5	0·99238	792	1·00768	810
·6	·98446	782	·01578	813
·7	·97664	771	·02391	816
·8	·96893	762	·03207	818
23·9	·96131	753	·04025	821
24·0	·95378	743	·04846	823
·1	·94635	734	·05669	826
·2	·93901	725	·06495	828
·3	·93176	716	·07323	831
·4	·92460	707	·08154	834
24·5	·91753	698	·08988	836
·6	·91055	690	·09824	839
·7	·90365	682	·10663	841
·8	·89683	673	·11504	843
24·9	·89010	666	·12347	846
25·0	·88344	657	·13193	849
·1	·87687	649	·14042	851
·2	·87038	642	·14893	853
·3	·86396	634	·15746	856
·4	·85762	627	·16602	859
25·5	·85135	620	·17461	861
·6	·84515	612	·18322	863
·7	·83903	605	·19185	865
·8	·83298	598	·20050	868
25·9	·82700	591	·20918	871
26·0	·82109	584	·21789	873
·1	·81525	578	·22662	875
·2	·80947	570	·23537	877
3	·80377	565	·24414	880
·4	·79812	558	·25294	883
26 5	·79254	552	·26177	884
6	·78702	545	·27061	887
·7	·78157	540	·27984	889
·8	·77617	533	·28837	892
26·9	·77084	527	·29729	893
27·0	·76557	522	·30622	896
·1	·76035	516	31518	899
2	·75519	510	·32417	900
·3	·75009	504	·33317	903
·4	·74505	499	·34220	905
27·5	·74006	494	·35125	907
·6	·73512	488	·36032	909
·7	·73024	483	·36941	912
·8	·72541	478	·37853	914
27·9	·72063	472	·38767	916
28·0	·71591	468	·39683	918
·1	·71123	462	·40601	920
·2	·70661	458	·41521	922
·3	·70203	452	·42443	925
·4	·69751	448	·43368	927
·5	·69303	443	·44295	928
28·6	0·68860		1·45223	

$$F = 1/f = \frac{\sin^2 \omega \, \cos^4 \frac{1}{2} i}{\sin^2 I} = \frac{0{\cdot}15779}{\sin^2 I}.$$

I is given for the first of each month in Table 6.

TABLE 13.—*Factors F and f, corresponding to every tenth of a degree of I, for reduction and prediction of tides*—Continued

I	F (Mm)	Diff.	f (Mm)	Diff.
°				
18.3	0.88387	163	1.13139	208
.4	.88550	164	.12931	209
.5	.88714	165	.12722	209
.6	.88879	167	.12513	211
.7	.89046	168	.12302	212
.8	.89214	169	.12090	212
18.9	.89383	171	.11878	214
19.0	.89554	172	.11664	214
.1	.89726	174	.11450	216
.2	.89900	175	.11234	216
.3	.90075	177	.11018	218
.4	.90252	178	.10800	218
19.5	.90430	180	.10582	219
.6	.90610	181	.10363	220
.7	.90791	183	.10143	221
.8	.90974	184	.09922	222
19.9	.91158	185	.09700	223
20.0	.91343	187	.09477	224
.1	.91530	189	.09253	225
.2	.91719	191	.09028	226
.3	.91910	192	.08802	226
.4	.92102	193	.08576	228
20.5	.92295	195	.08348	228
.6	.92490	197	.08120	230
.7	.92687	198	.07890	230
.8	.92885	200	.07660	231
20.9	.93085	201	.07429	232
21.0	.93286	204	.07197	233
.1	.93490	205	.06964	234
.2	.93695	206	.06730	235
.3	.93901	209	.06495	236
.4	.94110	210	.06259	237
21.5	.94320	211	.06022	237
.6	.94531	214	.05785	238
.7	.94745	215	.05547	240
.8	.94960	217	.05307	240
21.9	.95177	219	.05067	241
22.0	.95396	221	.04826	242
.1	.95617	222	.04584	243
.2	.95839	224	.04341	243
.3	.96063	227	.04098	245
.4	.96290	228	.03853	245
22.5	.96518	230	.03608	246
.6	.96748	232	.03362	247
.7	.96980	233	.03115	248
.8	.97213	236	.02867	249
22.9	.97449	238	.02618	250
23.0	.97687	239	.02368	250
.1	.97926	242	.02118	251
.2	.98168	243	.01867	253
23.3	0.98411	246	1.01614	253

I	F (Mm)	Diff.	f (Mm)	Diff.
°				
23.3	0.98411	246	1.01614	253
.4	.98657	248	.01361	253
.5	.98905	249	.01108	255
.6	.99154	252	.00853	256
.7	.99406	254	.00597	256
.8	.99660	256	.00341	257
23.9	0.99916	258	1.00084	258
24.0	1.00174	260	0.99826	259
.1	.00434	263	.99567	259
.2	.00697	265	.99308	260
.3	.00962	267	.99048	262
.4	.01229	269	.98786	262
24.5	.01498	271	.98524	262
.6	.01769	274	.98262	264
.7	.02043	276	.97998	264
.8	.02319	278	.97734	265
24.9	.02597	281	.97469	266
25.0	.02878	283	.97203	267
.1	.03161	285	.96936	267
.2	.03446	288	.96669	268
.3	.03734	290	.96401	269
.4	.04024	293	.96132	270
25.5	.04317	295	.95862	271
.6	.04612	298	.95591	271
.7	.04910	300	.95320	272
.8	.05210	303	.95048	273
25.9	.05513	305	.94775	273
26.0	.05818	308	.94502	275
.1	.06126	311	.94227	275
.2	.06437	313	.93952	275
.3	.06750	316	.93677	277
.4	.07066	319	.93400	277
26.5	.07385	322	.93123	278
.6	.07707	324	.92845	279
.7	.08031	327	.92566	279
.8	.08358	330	.92287	280
26.9	.08688	333	.92007	281
27.0	.09021	335	.91726	282
.1	.09356	339	.91444	282
.2	.09695	342	.91162	283
.3	.10037	344	.90879	284
.4	.10381	348	.90595	284
27.5	.10729	350	.90311	285
.6	.11079	354	.90026	286
.7	.11433	357	.89740	287
.8	.11790	360	.89453	287
27.9	.12150	363	.89166	288
28.0	.12513	367	.88878	288
.1	.12880	369	.88590	289
.2	.13249	373	.88301	290
.3	.13622	377	.88011	291
.4	.13999	379	.87720	291
.5	.14378	383	.87429	292
28.6	1.14761		0.87137	

$$F = 1/f = \frac{(1-\frac{3}{2}\sin^2\omega)(1-\frac{3}{2}\sin^2 i)}{1-\frac{3}{2}\sin^2 I} = \frac{0.75316}{1-\frac{3}{2}\sin^2 I}.$$

I is given for the first of each month in Table 6.

TABLE 14.—*Factors* (F) *for reducing* Mn, K_1+O_1, *and* X *to mean values.*

	I, or inclination of orbit to equator.											
	18½°	19°	20°	21°	22°	23°	24°	25°	26°	27°	28°	28½°
						F (Mn)						
$\frac{K_1+O_1}{M_2}$ = 0·0	0·970	0·972	0·977	0·982	0·988	0·994	1·000	1·006	1·012	1·019	1·026	1·029
0·2	0·971	0·973	0·978	0·982	0·988	0·994	1·000	1·006	1·012	1·019	1·026	1·029
0·4	0·972	0·974	0·979	0·983	0·988	0·994	1·000	1·006	1·012	1·018	1.025	1·028
0·6	0·974	0·976	0·980	0·984	0·989	0·994	1·000	1·006	1·011	1·017	1·023	1·026
0·8	0·977	0·979	0·982	0·986	0·990	0·995	1·000	1·005	1·010	1·015	1·020	1·022
1·0	9·980	0·982	0·985	0·989	0·992	0·996	1·000	1·004	1·008	1·013	1·017	1·019
1·2	0·984	0·985	0·988	0·991	0·994	0·997	1·000	1·003	1·007	1·010	1·014	1·016
1·4	0·988	0·989	0·991	0·994	0·996	0·998	1·000	1·002	1·005	1·007	1·010	1·012
1·6	0·994	0.994	0·995	0·997	0·998	0·999	1·000	1·001	1·003	1·004	1·005	1·006
1·8	1·001	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	0·999
2·0	1·008	1·007	1·006	1·004	1·003	1·001	1·000	0·998	0·997	0·995	0·994	0·993
2·5	1·025	1·023	1·019	1·014	1·009	1·004	0·999	0·994	0·989	0·984	0·979	0·977
						$F(K_1+O_1)$						
	1·168	1·148	1·109	1·073	1·040	1·010	0·982	0·955	0·931	0·909	0·888	0·878
						F(X)						
X′ = 0·1					0·47	0·89	1·16	1·38	1·54	1·68	1·80	1·86
0·2	0·54	0·61	0·73	0·83	0·91	0·98	1·04	1·09	1·14	1·19	1·23	1·24
0·3	0·86	0·88	0·91	0·94	0·97	0·99	1·01	1·04	1·05	1·07	1·08	1·09
0·4	0·96	0·96	0·97	0·98	0·99	1·00	1·01	1·01	1·02	1·02	1·03	1·03
0·5	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00

$F(K_1+O_1)=\frac{2{\cdot}4066}{1{\cdot}4066\,f(K_1)+f(O_1)}$. $F(X)=\frac{X}{X'}$. $\cos(X\cdot 180^\circ)=F(K_1+O_1)\cos(X'\cdot 180^\circ)$.

This table is based upon Tables 1, 13, 21, and §§ 3, 21, 50.

TABLE 15.—*Acceleration in* HW *and* LW *of a*

[The amplitude of the principal wave is taken as unity.]

HW phase.* LW phase.*	0° 180	10° 190	20° 200	30° 210	40° 220	50° 230	60° 240	70° 250	80° 260	90° 270
Amplitude of subordinate wave.	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
0·0	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 00	0 00
0·1	0 00	0 54	1 47	2 38	3 25	4 07	4 43	5 12	5 32	5 42
0·2	0 00	1 40	3 18	4 52	6 22	7 44	8 57	9 59	10 47	11 19
0·3	0 00	2 18	4 35	6 47	8 55	10 54	12 44	14 20	15 41	16 42
0·4	0 00	2 51	5 41	8 27	11 08	13 42	16 06	18 18	20 13	21 48
0·5	0 00	3 20	6 38	9 54	13 05	16 10	19 06	21 52	24 22	26 34
0·6	0 00	3 45	7 28	11 10	14 48	18 21	21 47	25 04	28 09	30 58
0·7	0 00	4 07	8 13	12 18	16 19	20 18	24 11	27 57	31 35	35 00
0·8	0 00	4 27	8 53	13 18	17 41	22 02	26 20	30 33	34 40	38 40
0·9	0 00	4 44	9 28	14 12	18 54	23 36	28 16	32 53	37 28	41 59
1·0	0 00	5 00	10 00	15 00	20 00	25 00	30 00	35 00	40 00	45 00
HW phase. LW phase.	360° 180	350° 170	340° 160	330° 150	320° 140	310° 130	300° 120	290° 110	280° 100	270° 90

*i. e. the argument, or phase, of the subordinate component (B) at the time of HW and LW, respectively, of the principal component (A).
By § 2

$$\tan \text{acceleration in } \frac{\text{HW}}{\text{LW}} = \frac{a}{b} \frac{\frac{B\,b^2}{A\,a^2} \sin \frac{\text{HW phase}}{\text{LW phase}}}{\pm 1 + \frac{B\,b^2}{A\,a^2} \cos \frac{\text{HW phase}}{\text{LW phase}}}.$$

(When $b = a$, this formula is exact.) If t denote the time after the coospiring of A and B, § 17, then

$$\text{phase} = (b - a)\,t;$$

$$= \frac{b}{a}\,2\,n\,\pi \text{ for HW},$$

$$= \frac{b}{a}\,\overline{2\,n + 1}\,\pi \text{ for LW},$$

n being an integer denoting the number of high waters of A since the coincidence of the maxima of A and B.

TABLE 16.—*Height of* HW *and* LW *for a*

[The amplitude of the principal wave is taken as unity.]

HW phase. LW phase.	0° 180	10° 190	20° 200	30° 210	40° 220	50° 230	60° 240	70° 250	80° 260	90° 270
Amplitude of subordinate wave. 0·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
0·1	1·1000	1·0986	1·0945	1·0877	1·0785	1·0670	1·0536	1·0385	1·0221	1·0050
0·2	1·2000	1·1975	1·1899	1·1775	1·1603	1·1389	1·1135	1·0849	1·0532	1·0198
0·3	1·3000	1·2965	1·2860	1·2687	1·2448	1·2148	1·1790	1·1381	1·0928	1·0440
0·4	1·4000	1·3957	1·3827	1·3611	1·3315	1·2939	1·2490	1·1973	1·1397	1·0770
0·5	1·5000	1·4949	1·4798	1·4546	1·4198	1·3758	1·3228	1·2618	1·1931	1·1180
0·6	1·6000	1·5943	1·5772	1·5490	1·5097	1·4599	1·4000	1·3306	1·2523	1·1662
0·7	1·7000	1·6937	1·6749	1·6439	1·6007	1·5459	1·4798	1·4032	1·3165	1·2207
0·8	1·8000	1·7932	1·7730	1·7395	1·6928	1·6336	1·5620	1·4789	1·3849	1·2806
0·9	1·9000	1·8928	1·8712	1·8354	1·7858	1.7225	1·6463	1·5575	1·4569	1·3453
1·0	2·0000	1·9924	1·9696	1·9318	1·8794	1·8126	1·7320	1·6384	1·5320	1·4142
HW phase. LW phase.	360° 180	350° 170	340° 160	330° 150	320° 140	310° 130	300° 120	290° 110	280° 100	270° 90

For high waters use the tabular values as given; but for low waters alter their signs.

TABLE 20.—*Great tropic range and its duration.*

[The amplitude of the semidiurnal wave is taken as unity.]

	Elevation of great tropic HW > depression of great tropic LW.					Depression of gt. tropic LW > elevation of gt. tropic HW.				
HW phase.	0° 180	10° 170	20° 160	30° 150	40° 140	50° 130	60° 120	70° 110	80° 100	90° 90
Amplitude of diurnal wave. 1·6	3·921 7 50	4·149 7 54	4·326 7 56	4·444 7 57	4·504 7 57	4·504 7 57	4·444 7 57	4·326 7 56	4·149 7 54	3·921 7 50
1·7	4·062 7 57	4·291 7 59	4·487 8 01	4·610 8 02	4·674 8 03	4·674 8 03	4·610 8 02	4·487 8 01	4·291 7 59	4·062 7 57
1·8	4·205 8 03	4·456 8 05	4·649 8 06	4·777 8 07	4·844 8 08	4·844 8 08	4·777 8 07	4·649 8 06	4·456 8 05	4·205 8 03
1·9	4·351 8 10	4·612 8 11	4·813 8 12	4·947 8 12	5·015 8 12	5·015 8 12	4·947 8 12	4·813 8 12	4·612 8 11	4·351 8 10
2·0	4·500 8 17	4·772 8 17	4·978 8 17	5·117 8 17	5·187 8 17	5·187 8 17	5·117 8 17	4·978 8 17	4·772 8 17	4·500 8 17
2·1	4·651 8 24	4·932 8 23	5·144 8 22	5·288 8 21	5·360 8 21	5·360 8 21	5·288 8 21	5·144 8 22	4·932 8 23	4·651 8 24
2·2	4·805 8 31	5·093 8 28	5·311 8 27	5·460 8 26	5·535 8 26	5·535 8 26	5·460 8 26	5·311 8 27	5·093 8 28	4·805 8 31
2·3	4·962 8 38	5·256 8 34	5·487 8 31	5·634 8 31	5·710 8 30	5·710 8 30	5·634 8 31	5·487 8 31	5·256 8 34	4·962 8 38
2·4	5·121 8 45	5·421 8 40	5·652 8 37	5·808 8 35	5·886 8 34	5·886 8 34	5·808 8 35	5·652 8 37	5·421 8 40	5·121 8 45
2·5	5·282 8 53	5·589 8 45	5·824 8 41	5·984 8 39	6·063 8 38	6·063 8 38	5·984 8 39	5·824 8 41	5·589 8 45	5·282 8 53
2·6	5·445 9 01	5·757 8 51	5·997 8 46	6·160 8 44	6·241 8 42	6·241 8 42	6·160 8 44	5·997 8 46	5·757 8 51	5·445 9 01
2·7	5·611 9 08	5·927 8 57	6·170 8 51	6·337 8 48	6·420 8 46	6·420 8 46	6·337 8 48	6·170 8 51	5·927 8 57	5·611 9 08
2·8	5·780 9 17	6·097 9 02	6·345 8 55	6·514 8 52	6·600 8 50	6·600 8 50	6·514 8 52	6·345 8 55	6·097 9 02	5·780 9 17
2·9	5·951 9 25	6·274 9 07	6·522 8 59	6·692 8 55	6·779 8 54	6·779 8 54	6·692 8 55	6·522 8 59	6·274 9 07	5·951 9 25
3·0	6·125 9 34	6·446 9 13	6·700 9 03	6·875 8 59	6·959 8 57	6·959 8 57	6·875 8 59	6·700 9 03	6·446 9 13	6·125 9 34
4·0	8·000 12 25	8·261 10 03	8·517 9 42	8·701 9 32	8·795 9 28	8·795 9 28	8·701 9 32	8·517 9 42	8·261 10 03	8.000 12 25
5·0	10·000 12 25	10·165 10 43	10·399 10 11	10·579 9 58	10·675 9 52	10·675 9 52	10·579 9 58	10·399 10 11	10·165 10 43	10 00 12 25
10·0	20·000 12 25	20·037 11 48	20·146 11 18	20·289 11 01	20·387 10 53	20·387 10 53	20·289 11 01	20·146 11 18	20·037 11 48	20 00 12 25
HW phase.	180° 360	190° 350	200° 340	210° 330	220° 320	230° 310	240° 300	250° 290	260° 280	270° 270

The first value of each pair is the value of the great tropic range; the second, its duration in hours and minutes.
This table assumes that $d_1 = m_1$.
See §§ 25, 37, and 53.

TABLE 21.—*Effects of various tidal components upon the mean semirange of tide.* *

[The amplitude of M_2 is taken as unity.]

Amplitude of subordinate component.	Semidiurnal components.								Amplitude of subordinate component.	Diurnal components.				
	K_2	L_2	N_2	S_2	λ_2	μ_2	ν_2	$\frac{B^2}{4}$		K_1	O_1	P_1	Q_1	$\frac{B^2}{16}$
0·02	·0001	·0001	·0001	·0001	·0001	·0001	·0001	·0001	0·04	·0001	·0001	·0001	·0001	·0001
0·04	·0004	·0004	·0004	·0004	·0004	·0004	·0004	·0004	0·08	·0004	·0004	·0004	·0003	·0004
0·06	·0010	·0009	·0009	·0010	·0009	·0008	·0009	·0009	0·12	·0010	·0008	·0010	·0008	·0009
0·08	·0017	·0017	·0015	·0017	·0017	·0015	·0015	·0016	9·16	·0017	·0015	·0017	·0014	·0016
0·10	·0027	·0026	·0024	·0027	·0026	·0023	·0024	·0025	0·20	·0027	·0023	·0027	·0021	·0025
0·12	·0039	·0037	·0035	·0039				·0036	0·24	·0039	·0033	·0038	·0030	·0036
0·14	·0053	·0051	·0047	·0052				·0049	0·28	·0053	·0045	·0052	·0041	·0049
0·16	·0069		·0061	·0069				·0064	0.32	·0069	·0059	·0068		·0064
0·18	·0087		·0078	·0087				·0081	0·36	·0087	·0075	·0086		·0081
0·20	·0108		·0096	·0107				·0100	0·40	·0108	·0093	·0106		·0100
0·22	·0130		·0116	·0130				·0121	0·44	·0130	·0112	·0127		·0121
0·24	·0155		·0138	·0154				·0144	0·48	·0155	·0133	·0150		·0144
0·26			·0162	·0181				·0169	0·52	·0182	·0156	·0176		·0169
0·28			·0188	·0210				·0196	0·56	·0211	·0181			·0196
0·30			·0216	·0241				·0225	0·60	·0242	·0208			·0225
0·32				·0274				·0256	0·64	·0276	·0237			·0256
0·34				·0310				·0289	0·68	·0311	·0268			·0289
0·36				·0347				·0324	0·72	·0349	·0300			·0324
0·38				·0387				·0361	0·76	·0389	·0334			·0361
0·40				·0428				·0400	0·80	·0431	·0370			·0400
0·42				·0472				·0441	0·84	·0475	·0407			·0441
0·44				·0518				·0484	0·88	·0521	·0447			·0484
0·46				·0567				·0529	0·92	·0570	·0488			·0529
0·48				·0617				·0576	0·96	·0620	·0532			·0576
0·50				·0669				·0625	1·00	·0673	·0577			·0625
									1·04	·0728	·0624			·0676
0·60				·0964				·0900	1·08	·0785	·0673			·0729
0·70				·1312				·1225	1·12	·0845	·0723			·0784
0·80				·1715				·1600	1·16	·0906	·0776			·0841
0·90				·2169				·2025	1·20	·0969	·0831			·0900
1·00				·2678				·2500	1·24	·1035	·0887			·0961
									1·28	·1103	·0945			·1024
									1·32	·1173				·1089
									1·36	·1245				·1156
									1·40	·1320				·1225
									1·44	·1396				·1296
									1·48	·1475				·1369
									1·52	·1555				·1444
									1·56	·1637				·1521
									1·60	·1720				·1600
									1·64	·1805				·1681
									1·68	·1892				·1764

* Tabular value for component $C = \frac{C^2}{4\,M_2}\,\frac{c^2}{m_2^2}$, § 13.

TABLE 22.—*Value of* ½ Mn *when* M_2 = 1.

$\frac{K_1+O_1}{M_2}$	S_2/M_2							
	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
0·0	1·0127	1·0213	1·0357	1·0558	1·0817	1·1134	1·1508	1·1940
0·2	1·0142	1·0228	1·0372	1·0573	1·0832	1·1149	1·1523	1·1955
0·4	1·0185	1·0271	1·0415	1·0616	1·0875	1·1192	1·1566	1·1998
0·6	1·0257	1·0343	1·0487	1·0688	1·0947	1·1264	1·1638	1·2070
0·8	1·0357	1·0443	1·0587	1·0788	1·1047	1·1364	1·1738	1·2170
1·0	1·0486	1·0572	1·0716	1·0917	1·1176	1·1493	1·1867	1·2299
1·2	1·0643	1·0729	1·0873	1·1074	1·1333	1·1650	1·2024	1·2456
1·4	1·0830	1·0916	1·1060	1·1261	1·1520	1·1837	1·2211	1·2643
1·6	1·1045	1·1131	1·1275	1·1476	1·1735	1·2052	1·2426	1·2858
1·8	1·1289	1·1375	1·1519	1·1720	1·1979	1·2296	1·2670	1·3102
2·0	1·1559	1·1645	1·1789	1·1990	1·2249	1·2566	1·2940	1·3372
2·5	1·2360	1·2446	1·2590	1·2791	1·3050	1·3367	1·3741	1·4173

This table, based upon Tables 1 and 21, is computed upon the assumption that the ratios between the diurnal components, the pure lunar semidiurnals, the solar semidiurnals (including luni-solar K_2), are respectively constant for all stations; also that shallow water tides do not occur.

On account of nonpredictable inequalities, the tabular values should be multiplied by about 1·02.

TABLE 23.—*Value of* M_2 *when* ½ Mn = 1.

$\frac{K_1+O_1}{\frac{1}{2}\,Mn}$	$S_2/\frac{1}{2}Mn$							
	0.1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
0·0	0·9873	0·9785	0·9634	0·9414	0·9116	0·8725	0·8221	0·7574
0·2	0·9858	0·9770	0·9618	0·9396	0·9096	0·8702	0·8194	0·7542
0·4	0·9814	0·9725	0·9572	0·9348	0·9045	0·8648	0·8136	0·7479
0·6	0·9740	0·9650	0·9495	0·9269	0·8963	0·8562	0·8045	0·7382
0·8	0·9635	0·9543	0·9386	0·9157	0·8847	0·8442	0·7919	0·7249
1·0	0·9497	0·9403	0·9243	0·9011	0·8696	0·9286	0·7756	0·7077
1·2	0.9322	0·9225	0·9062	0·8826	0·8506	0·8090	0·7552	0·6863
1·4	0·9106	0·9006	0·8839	0·8598	0·8272	0·7849	0·7302	0·6602
1·6	0·8844	0·8740	0·8568	0·8321	0·7988	0·7557	0·7000	0·6288
1·8	0·8529	0·8421	0·8243	0·7989	0·7648	0·7207	0·6639	0·5913
2·0	0·8151	0·8038	0·7854	0·7592	0·7242	0·6789	0·6208	0·5465
2·5	0·7175	0·7049	0·6848	0·6566	0·6193	0·5710	0·5093	0·4304

This table is Table 22 reverted.

On account of nonpredictable inequalities, the tabular values should be divided by about 1·02.

TABLE 24.—*Variation in lunitidal interval and mean semirange of tide, due to the phase wave composed of* S_2 *and* μ_2

Time.			Increase in lunitidal interval due to S_2.	Increase in mean semirange of tide due to S_2 and μ_2. Length of half group.				
				0 tides.	4 tides.	6 tides.	8 tides.	10 tides.
	d.	*h.*	*m.*					
After spring tides.	0	00	0 S_2/M_2	+0·93 S_2	+0·90 S_2	+0·88 S_2	+0·84 S_2	+0·79 S_2
	0	06	− 9 "	+0·92 "	+0·90 "	+0·87 "	+0·83 "	+0·78 "
	0	12	− 18 "	+0·91 "	+0·89 "	+0·86 "	+0·82 "	+0·77 "
	0	18	− 27 "	+0·89 "	+0·87 "	+0·84 "	+0·81 "	+0·76 "
	1	00	− 35 "	+0·86 "	+0·84 "	+0·81 "	+0·78 "	+0·73 "
	1	06	− 44 "	+0·82 "	+0·80 "	+0·77 "	+0·74 "	+0·69 "
	1	12	− 52 "	+0·78 "	+0·76 "	+0·73 "	+0·70 "	+0·66 "
	1	18	− 61 "	+0·73 "	+0·71 "	+0·69 "	+0·65 "	+0·62 "
	2	00	− 69 "	+0·67 "	+0·65 "	+0·63 "	+0·59 "	+0·56 "
	2	06	− 77 "	+0·60 "	+0·58 "	+0·56 "	+0·53 "	+0·50 "
	2	12	− 84 "	+0·53 "	+0·51 "	+0·50 "	+0·47 "	+0·44 "
	2	18	− 92 "	+0·45 "	+0·44 "	+0·43 "	+0·40 "	+0·37 "
	3	00	− 98 "	+0·36 "	+0·35 "	+0·34 "	+0·32 "	+0·29 "
	3	06	−105 "	+0·27 "	+0·26 "	+0·25 "	+0·23 "	+0·21 "
	3	12	−111 "	+0·18 "	+0·18 "	+0·17 "	+0·15 "	+0·13 "
	3	18	−116 "	+0·07 "	+0·07 "	+0·07 "	+0·05 "	+0·03 "
	4	00	−120 "	−0·03 "	−0·04 "	−0·04 "	−0·06 "	−0·08 "
Before neap tides.	4	00	−108 "	+0·22 "	+0·20 "	+0·19 "	+0·19 "	+0·18 "
	3	18	−113 "	+0·12 "	+0·10 "	+0·09 "	+0·08 "	+0·07 "
	3	12	−118 "	+0·02 "	0·00 "	−0·91 "	−0·02 "	−0·03 "
	3	06	−122 "	−0·09 "	−0·10 "	−0·11 "	−0·12 "	−0·13 "
	3	00	−125 "	−0·20 "	−0·20 "	−0·22 "	−0·23 "	−0·23 "
	2	18	−127 "	−0·31 "	−0·30 "	−0·32 "	−0·32 "	−0·32 "
	2	12	−127 "	−0·42 "	−0·41 "	−0·42 "	−0·42 "	−0·40 "
	2	06	−126 "	−0·53 "	−0·52 "	−0·52 "	−0·51 "	−0·49 "
	2	00	−123 "	−0·64 "	−0·63 "	−0·61 "	−0·60 "	−0·57 "
	1	18	−117 "	−0·74 "	−0·73 "	−0·70 "	−0·68 "	−0·64 "
	1	12	−109 "	−0·84 "	−0·82 "	−0·78 "	−0·76 "	−0·70 "
	1	06	− 99 "	−0·93 "	−0·90 "	−0·86 "	−0·83 "	−0·75 "
	1	00	− 84 "	−1·01 "	−0·98 "	−0·93 "	−0·87 "	−0·80 "
	0	18	− 67 "	−1·08 "	−1·05 "	−0·99 "	−0·93 "	−0·85 "
	0	12	− 47 "	−1·13 "	−1·09 "	−1·03 "	−0·96 "	−0·88 "
	0	06	− 24 "	−1·16 "	−1·11 "	−1·05 "	−0·97 "	−0·89 "
	0	00	0 "	−1·18 "	−1·12 "	−1·06 "	−0·98 "	−0·89 "
After neap tides.	0	00	0 "	−1·18 "	−1·12 "	−1·06 "	−0·98 "	−0·89 "
	0	06	+ 24 "	−1·16 "	−1·11 "	−1·05 "	−0·97 "	−0·89 "
	0	12	+ 47 "	−1·13 "	−1·09 "	−1·03 "	−0·96 "	−0·88 "
	0	18	+ 67 "	−1·08 "	−1·05 "	−0·99 "	−0·93 "	−0·85 "
	1	00	+ 84 "	−1·01 "	−0·98 "	−0·93 "	−0·87 "	−0·80 "
	1	06	+ 99 "	−0·93 "	−0·90 "	−0·86 "	−0·83 "	−0·75 "
	1	12	+109 "	−0·84 "	−0·82 "	−0·78 "	−0·76 "	−0·70 "
	1	18	+117 "	−0·74 "	−0·73 "	−0·70 "	−0·68 "	−0·64 "
	2	00	+123 "	−0·64 "	−0·63 "	−0·61 "	−0·59 "	−0·56 "
	2	06	+126 "	−0·53 "	−0·52 "	−0·52 "	−0·50 "	−0·48 "
	2	12	+127 "	−0·42 "	−0·41 "	−0·42 "	−0·42 "	−0·40 "
	2	18	+127 "	−0·31 "	−0·30 "	−0·32 "	−0·32 "	−0·31 "
	3	00	+125 "	−0·20 "	−0·20 "	−0·21 "	−0·22 "	−0·22 "
	3	06	+122 "	−0·09 "	−0·09 "	−0·11 "	−0·11 "	−0·11 "
	3	12	+118 "	+0·02 "	+0·01 "	+0·01 "	0·00 "	0·00 "
	3	18	+113 "	+0·12 "	+0·11 "	+0·11 "	+0·10 "	+0·10 "
	4	00	+108 "	+0·22 "	+0·21 "	+0·21 "	+0·20 "	+0·19 "
Before spring tides.	4	00	+120 "	−0·03 "	−0·04 "	−0·05 "	−0·06 "	−0·07 "
	3	18	+116 "	+0·07 "	+0·07 "	+0·06 "	+0·04 "	+0·03 "
	3	12	+111 "	+0·18 "	+0·18 "	+0·17 "	+0·15 "	+0·13 "
	3	06	+105 "	+0·27 "	+0·27 "	+0·26 "	+0·24 "	+0·22 "
	3	00	+ 98 "	+0·36 "	+0·36 "	+0·35 "	+0·32 "	+0·30 "
	2	18	+ 92 "	+0·45 "	+0·45 "	+0·43 "	+0·40 "	+0·38 "
	2	12	+ 84 "	+0·53 "	+0·52 "	+0·50 "	+0·47 "	+0·45 "
	2	06	+ 77 "	+0·60 "	+0·58 "	+0·56 "	+0·53 "	+0·50 "
	2	00	+ 69 "	+0·67 "	+0·65 "	+0·63 "	+0·60 "	+0·56 "
	1	18	+ 61 "	+0·73 "	+0·71 "	+0·69 "	+0·66 "	+0·62 "
	1	12	+ 52 "	+0·78 "	+0·76 "	+0·73 "	+0·71 "	+0·66 "
	1	06	+ 44 "	+0·82 "	+0·80 "	+0·77 "	+0·74 "	+0·69 "
	1	00	+ 35 "	+0·86 "	+0·84 "	+0·81 "	+0·78 "	+0·73 "
	0	18	+ 27 "	+0·89 "	+0·87 "	+0·84 "	+0·81 "	+0·76 "
	0	12	+ 18 "	+0·91 "	+0·89 "	+0·86 "	+0·82 "	+0·77 "
	0	06	+ 9 "	+0·92 "	+0·90 "	+0·87 "	+0·83 "	+0·78 "
	0	00	0 "	+0·93 "	+0·90 "	+0·88 "	+0·84 "	+0·79 "

In clearing tides of phase or semimenstrual height inequality, apply the tabulated values as they stand to the low waters, but alter their signs for the high waters.

Spring and neap tides occur $S_2^\circ - M_2^\circ$ hours after syzygy and quadrature.

TABLE 25.—*Variation in mean semirange of tide due to the parallax wave composed of* N_2, L_2, *and* 2N.

Time.			Increase in mean semirange of tide due to N_2, L_2, and 2N. Length of half group.				Time.			Increase in mean semirange of tide due to N_2, L_2, and 2N. Length of half group.			
	d.	*h.*	0 tides.	4 tides.	8 tides.	12 tides.		*d.*	*h.*	0 tides.	4 tides.	8 tides.	12 tides.
After perigean tides.	0	00	+0·94 N_2	+0·93 N_2	+0·90 N_2	+0·85 N_2	After apogean tides.	0	00	—0·77 N_2	—0·77 N_2	—0·76 N_2	—0·73 N_2
	0	06	+0·94 "	+0·93 "	+0·90 "	+0·85 "		0	06	—0·77 "	—0·77 "	—0·76 "	—0·73 "
	0	12	+0·93 "	+0·93 "	+0·89 "	+0·84 "		0	12	—0·77 "	—0·77 "	—0·76 "	—0·73 "
	0	18	+0·92 "	+0·91 "	+0·88 "	+0·83 "		0	18	—0·76 "	—0·76 "	—0·75 "	—0·73 "
	1	00	+0·91 "	+0·91 "	+0·87 "	+0·82 "		1	00	—0·75 "	—0·75 "	—0·74 "	—0·72 "
	1	06	+0·89 "	+0·89 "	+0.85 "	+0·80 "		1	06	—0·75 "	—0·75 "	—0·74 "	—0·72 "
	1	12	+0·87 "	+0·87 "	+0·84 "	+0·79 "		1	12	—0·74 "	—0·74 "	—0·73 "	—0·71 "
	1	18	+0·85 "	+0·85 "	+0·82 "	+0·77 "		1	18	—0·73 "	—0·72 "	—0·72 "	—0·70 "
	2	00	+0·82 "	+0·82 "	+0·80 "	+0·76 "		2	00	—0·71 "	—0·70 "	—0·70 "	—0·67 "
	2	06	+0·79 "	+0·79 "	+0·77 "	+0·73 "		2	06	—0·69 "	—0·69 "	—0·68 "	—0·65 "
	2	12	+0·75 "	+0·75 "	+0·74 "	+0·70 "		2	12	—0·68 "	—0·68 "	—0·65 "	—0·64 "
	2	18	+0·72 "	+0·72 "	+0·72 "	+0·67 "		2	18	—0·66 "	—0·66 "	—0·64 "	—0·63 "
	3	00	+0·68 "	+0·68 "	+0·67 "	+0·64 "		3	00	—0·63 "	—0·63 "	—0·62 "	—0·60 "
	3	06	+0·64 "	+0·64 "	+0·63 "	+0·61 "		3	06	—0·61 "	—0·61 "	—0·60 "	—0·57 "
	3	12	+0·60 "	+0·60 "	+0·59 "	+0·57 "		3	12	—0·58 "	—0·58 "	—0·57 "	—0·54 "
Before midtime tides.	3	12	+0·61 "	+0·61 "	+0·59 "	+0·57 "	Before midtime tides.	3	12	—0·60 "	—0·60 "	—0·58 "	—0·56 "
	3	06	+0.57 "	+0·57 "	+0·55 "	+0·53 "		3	06	—0·57 "	—0·57 "	—0·55 "	—0·53 "
	3	00	+0·52 "	+0·51 "	+0·50 "	+0·49 "		3	00	—0·55 "	—0·54 "	—0·52 "	—0·50 "
	2	18	+0·48 "	+0·47 "	+0·45 "	+0·45 "		2	18	—0·51 "	—0·51 "	—0·50 "	—0·46 "
	2	12	+0·43 "	+0·43 "	+0·40 "	+0·40 "		2	12	—0·47 "	—0·47 "	—0·46 "	—0·43 "
	2	06	+0·38 "	+0·38 "	+0·36 "	+0·36 "		2	06	—0·45 "	—0·45 "	—0·44 "	—0·41 "
	2	00	+0·33 "	+0·33 "	+0·31 "	+0·31 "		2	00	—0·41 "	—0·41 "	—0·40 "	—0·37 "
	1	18	+0·28 "	+0·28 "	+0·27 "	+0·28 "		1	18	—0·37 "	—0·37 "	—0·35 "	—0·32 "
	1	12	+0·23 "	+0·23 "	+0·23 "	+0·24 "		1	12	—0·33 "	—0·33 "	—0·32 "	—0·29 "
	1	06	+0·18 "	+0·18 "	+0·19 "	+0·20 "		1	06	—0·29 "	—0·29 "	—0·27 "	—0·24 "
	1	00	+0·12 "	+0·12 "	+0·13 "	+0·14 "		1	00	—0·26 "	—0·26 "	—0·24 "	—0·21 "
	0	18	+0·08 "	+0·08 "	+0·09 "	+0·10 "		0	18	—0·21 "	—0·21 "	—0·19 "	—0·16 "
	0	12	+0·02 "	+0·02 "	+0·04 "	+0·04 "		0	12	—0·17 "	—0·17 "	—0·15 "	—0·13 "
	0	06	—0·02 "	—0·02 "	0·00 "	+0·01 "		0	06	—0·12 "	—0·12 "	—0·10 "	—0·09 "
	0	00	—0·07 "	—0·07 "	—0·06 "	—0·04 "		0	00	—0·07 "	—0·07 "	—0·06 "	—0·04 "
After midtime tides.	0	00	—0·07 "	—0·07 "	—0·06 "	—0·04 "	After midtime tides.	0	00	—0·07 "	—0·07 "	—0·06 "	—0·04 "
	0	06	—0·12 "	—0·12 "	—0·10 "	—0·09 "		0	06	—0·02 "	—0·02 "	0·00 "	+0·01 "
	0	12	—0·17 "	—0·17 "	—0·15 "	—0·13 "		0	12	+0·02 "	+0·02 "	+0·04 "	+0·04 "
	0	18	—0·21 "	—0·21 "	—0·19 "	—0·16 "		0	18	+0·08 "	+0·08 "	+0·09 "	+0·10 "
	1	00	—0·26 "	—0·26 "	—0·24 "	—0·21 "		1	00	+0·12 "	+0·12 "	+0·13 "	+0·14 "
	1	06	—0·29 "	—0·29 "	—0·27 "	—0·24 "		1	06	+0·18 "	+0·18 "	+0·19 "	+0·20 "
	1	12	—0·33 "	—0·33 "	—0·32 "	—0·29 "		1	12	+0·23 "	+0·23 "	+0·23 "	+0·24 "
	1	18	—0·37 "	—0·37 "	—0·35 "	—0·32 "		1	18	+0·28 "	+0·28 "	+0·27 "	+0·28 "
	2	00	—0·41 "	—0·41 "	—0·40 "	—0·37 "		2	00	+0·33 "	+0·33 "	+0·31 "	+0·31 "
	2	06	—0·45 "	—0·45 "	—0·44 "	—0·41 "		2	06	+0·38 "	+0·38 "	+0·36 "	+0·36 "
	2	12	—0.47 "	—0·47 "	—0·46 "	—0·43 "		2	12	+0·43 "	+0·43 "	+0·40 "	+0·40 "
	2	18	—0·51 "	—0·51 "	—0·50 "	—0·46 "		2	18	+0·48 "	+0·47 "	+0·45 "	+0·45 "
	3	00	—0·55 "	—0·54 "	—0·52 "	—0·50 "		3	00	+0·52 "	+0·51 "	+0·50 "	+0·49 "
	3	06	—0·57 "	—0·57 "	—0·55 "	—0·53 "		3	06	+0·57 "	+0·57 "	+0·55 "	+0·53 "
	3	12	—0·60 "	—0·60 "	—0·58 "	—0·56 "		3	12	+0·61 "	+0·61 "	+0·59 "	+0·57 "
Before apogean tides.	3	12	—0·58 "	—0·58 "	—0·57 "	—0·54 "	Before perigean tides.	3	12	+0·60 "	+0·60 "	+0·59 "	+0·57 "
	3	06	—0·61 "	—0·61 "	—0·60 "	—0·57 "		3	06	+0·64 "	+0·64 "	+0·63 "	+0·61 "
	3	00	—0·63 "	—0·63 "	—0·62 "	—0·60 "		3	00	+0·68 "	+0·68 "	+0·67 "	+0·64 "
	2	18	—0·66 "	—0·66 "	—0·64 "	—0·63 "		2	18	+0·72 "	+0·72 "	+0·72 "	+0·67 "
	2	12	—0·68 "	—0·68 "	—0·65 "	—0·64 "		2	12	+0·75 "	+0·75 "	+0·74 "	+0·70 "
	2	06	—0·69 "	—0·69 "	—0·67 "	—0·65 "		2	06	+0·79 "	+0·79 "	+0·77 "	+0·73 "
	2	00	—0·71 "	—0·70 "	—0·68 "	—0·67 "		2	00	+0·82 "	+0·82 "	+0·80 "	+0·76 "
	1	18	—0·73 "	—0·72 "	—0·72 "	—0·70 "		1	18	+0·85 "	+0·85 "	+0·82 "	+0·77 "
	1	12	—0·74 "	—0·74 "	—0·73 "	—0·71 "		1	12	+0·87 "	+0·87 "	+0·84 "	+0·79 "
	1	06	—0·75 "	—0·75 "	—0·74 "	—0·72 "		1	06	+0·89 "	+0·89 "	+0·85 "	+0·80 "
	1	00	—0·75 "	—0·75 "	—0·74 "	—0·72 "		1	00	+0·91 "	+0·91 "	+0·87 "	+0·82 "
	0	18	—0·76 "	—0·76 "	—0·75 "	—0·73 "		0	18	+0·92 "	+0·91 "	+0·88 "	+0·83 "
	0	12	—0·77 "	—0·77 "	—0·76 "	—0·73 "		0	12	+0·93 "	+0·93 "	+0·89 "	+0·84 "
	0	06	—0·77 "	—0·77 "	—0·76 "	—0·73 "		0	06	+0·94 "	+0·93 "	+0·90 "	+0·85 "
	0	00	—0·77 "	—0·77 "	—0·76 "	—0·73 "		0	00	+0·94 "	+0·93 "	+0·90 "	+0·85 "

In clearing tides of this inequality, apply the tabular values as they stand to the low waters, but alter their signs for the high waters.

Perigean tides, apogean tides, etc., occur $\frac{M_2^\circ - N_2^\circ}{0{\cdot}5444}$ hours after perigee, apogee, etc.

This table is based upon Tables 1, 16, and 21.

TABLE 26.—*Effect of* Q_1 *upon the amplitude of* O_1.

Time.		Resultant amp. O_1 and Q_1. $O_1 = 1$.	Time.		Resultant amp. O_1 and Q_1. $O_1 = 1$.
	d.			*d.*	
After perigee.	0	1·19	After apogee.	0	0·81
	1	1·19		1	0·81
	2	1·18		2	0·83
	3	1·16		3	0·86
Before midtime.	3	1·13	Before midtime.	3	0·89
	2	1·10		2	0·93
	1	1·06		1	0·98
	0	1·02		0	0·02
After midtime.	0	1·02	After midtime.	0	1·02
	1	0·98		1	1·06
	2	0·93		2	1·10
	3	0·89		3	1·13
Before apogee.	3	0·86	Before perigee.	3	1·16
	2	0·83		2	1·18
	1	0·81		1	1·19
	0	0·81		0	1·19

This table is based upon Tables 1 and 16.

TABLE 27.—*Perturbations in* K_1 *due to* O_1.

Time.		Increase in K_1 tidal interval due to O_1.	Increase in semi-range of K_1 tide due to O_1.	Time.		Increase in K_1 tidal interval due to O_1.	Increase in semi-range of K_1 tide due to O_1.
	d. h.	*h. m.*			*d. h.*	*h. m.*	
After maximum declinational tides.	0 00	+0 00 O_1/K_1	+1·00 O_1	After minimum declinational tides.	0 00	0 00 O_1/K_1	—1·00 O_1
	0 06	+0 14 “	+0·99 “		0 06	—1 14 “	—0·97 “
	0 12	+0 28 “	+0·98 “		0 12	—2 16 “	—0·91 “
	0 18	+0 43 “	+0·96 “		0 18	—3 03 “	—0·83 “
	1 00	+0 57 “	+0·94 “		1 00	—3 27 “	—0·73 “
	1 06	+1 11 “	+0·90 “		1 06	—3 44 “	—0·63 “
	1 12	+1 25 “	+0·86 “		1 12	—3 53 “	—0·51 “
	1 18	+1 39 “	+0·81 “		1 18	—3 54 “	—0·40 “
	2 00	+1 53 “	+0·75 “		2 00	—3 52 “	—0·28 “
	2 06	+2 06 “	+0·69 “		2 06	—3 48 “	—0·17 “
	2 12	+2 19 “	+0·62 “		2 12	—3 40 “	—0·06 “
	2 18	+2 32 “	+0·55 “		2 18	—3 32 “	+0·05 “
	3 00	+2 45 “	+0·46 “		3 00	—3 23 “	+0·15 “
	3 06	+2 57 “	+0·37 “		3 06	—3 12 “	+0·25 “
	3 12	+3 09 “	+0·28 “		3 12	—3 01 “	+0·35 “
Before minimum declinational tides.	3 12	+3 01 “	+0·35 “	Before maximum declinational tides.	3 12	—3 09 “	+0·28 “
	3 06	+3 12 “	+0·25 “		3 06	—2 57 “	+0·37 “
	3 00	+3 23 “	+0·15 “		3 00	—2 45 “	+0·46 “
	2 18	+3 32 “	+0·05 “		2 18	—2 32 “	+0·55 “
	2 12	+3 40 “	—0·06 “		2 12	—2 19 “	+0·62 “
	2 06	+3 48 “	—0·17 “		2 06	—2 06 “	+0·69 “
	2 00	+3 52 “	—0·28 “		2 00	—1 53 “	+0·75 “
	1 18	+3 54 “	—0·40 “		1 18	—1 39 “	+0·81 “
	1 12	+3 53 “	—0·51 “		1 12	—1 25 “	+0·86 “
	1 06	+3 44 “	—0·63 “		1 06	—1 11 “	+0·90 “
	1 00	+3 27 “	—0·73 “		1 00	—0 57 “	+0·94 “
	0 18	+3 03 “	—0·83 “		0 18	—0 43 “	+0·96 “
	0 12	+2 16 “	—0·91 “		0 12	—0 28 “	+0·98 “
	0 06	+1 14 “	—0·97 “		0 06	—0 14 “	+0·99 “
	0 00	+0 00 “	—1·00 “		0 00	—0 00 “	+1·00 “

Maximum and minimum declinational tides occur $\frac{K_1^\circ - O_1^\circ}{1·0980}$ hours after extreme and zero declination, provided K_1° is decreased by the acceleration in K_1 due to P_1, Table 31. See § 67.

This table is based upon Tables 1, 15, and 16.

TABLE 28.—*The speed which corresponds to a period of given length.*

Length of half period.			Speed.		Time equal to one degree.
			Per hour.	Per minute.	
h.	*m.*	*m.*	°	°	*m.*
10	00	600	18·000	0·3000	3·333
	10	610	17·705	0·2951	3·389
	20	620	17·419	0·2903	3·445
	30	630	17·143	0·2857	3·500
	40	640	16·875	0·2813	3·556
	50	650	16·615	0·2769	3·611
11	00	660	16·364	0·2727	3·667
	10	670	16·119	0·2687	3·722
	20	680	15·882	0·2647	3·778
	30	690	15·652	0·2609	3·833
	40	700	15·429	0·2571	3·889
	50	710	15·211	0·2535	3·945
12	00	720	15·000	0·2500	4·000
	10	730	14·795	0·2466	4·055
	20	740	14·595	5·2433	4·111
	30	750	14·400	0·2400	4·167
	40	760	14·211	0·2369	4·222
	50	770	14·026	0·2338	4·278
13	00	780	13·846	0·2308	4·333
	10	790	13·671	0·2279	4·389
	20	800	13·500	0·2250	4·444
	30	810	13·333	0·2222	4·500
	40	820	13·171	0·2195	4·555
	50	830	13·012	0·2169	4·611
14	00	840	12·857	0·2143	4·667

TABLE 29.—*The sun's mean longitude at Greenwich mean noon.*

Day.	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
	°	°	°	°	°	°	°	°	°	°	°	°
1	280·6	311·2	338·7	9·3	38·9	69·4	99·0	129·6	160·1	189·7	220·2	249·8
2	281·6	312·1	339·7	10·3	39·9	70·4	100·0	130·6	161·1	190·7	221·2	250·8
3	282·6	313·1	340·7	11·3	40·9	71·4	101·0	131·5	162·1	191·7	222·2	251·8
4	283·6	314·1	341·7	12·3	41·8	72·4	102·0	132·5	163·1	192·6	223·2	252·8
5	284·5	315·1	342·7	13·3	42·8	73·4	102·9	133·5	164·1	193·6	224·2	253·7
6	285·5	316·1	343·7	14·2	43·8	74·4	103·9	134·5	165·0	194·6	225·2	254·7
7	286·5	317·1	344·7	15·2	44·8	75·3	104·9	135·5	166·0	195·6	226·1	255·7
8	287·5	318·0	345·7	16·2	45·8	76·3	105·9	136·5	167·0	196·6	227·1	256·7
9	288·5	319·0	346·6	17·2	46·8	77·3	106·9	137·4	168·0	197·6	228·1	257·7
10	289·5	320·0	347·6	18·2	47·8	78·3	107·9	138·4	169·0	198·5	229·1	258·7
11	290·5	321·0	348·6	19·2	48·7	79·3	108·9	139·4	170·0	199·5	230·1	259·7
12	291·4	322·0	349·6	20·2	49·7	80·3	109·8	140·4	170·9	200·5	231·1	260·7
13	292·4	323·0	350·6	21·1	50·7	81·3	110·8	141·4	171·9	201·5	232·1	261·6
14	293·4	324·0	351·6	22·1	51·7	82·2	111·8	142·4	172·9	202·5	233·1	262·6
15	294·4	324·9	352·5	23·1	52·7	83·2	112·8	143·4	173·9	203·5	234·0	263·6
16	295·4	325·9	353·5	24·1	53·7	84·2	113·8	144·3	174·9	204·5	235·0	264·6
17	296·4	326·9	354·5	25·1	54·7	85·2	114·8	145·3	175·9	205·5	236·0	265·6
18	297·4	327·9	355·5	26·1	55·7	86·2	115·8	146·3	176·9	206·4	237·0	266·6
19	298·3	328·9	356·5	27·1	56·6	87·2	116·7	147·3	177·9	207·4	238·0	267·5
20	299·3	329·9	357·5	28·0	57·6	88·2	117·7	148·3	178·8	208·4	239·0	268·5
21	300·3	330·9	358·5	29·0	58·6	89·1	118·7	149·3	179·8	209·4	240·0	269·5
22	301·3	331·8	359·4	30·0	59·6	90·1	119·7	150·3	180·8	210·4	240·9	270·5
23	302·3	332·8	0·4	31·0	60·6	91·1	120·7	151·2	181·8	211·4	241·9	271·5
24	303·3	333·8	1·4	32·0	61·6	92·1	121·7	152·2	182·8	212·3	242·9	272·5
25	304·3	334·8	2·4	33·0	62·5	93·1	122·7	153·2	183·8	213·3	243·9	273·5
26	305·2	335·8	3·4	33·9	63·5	94·1	123·6	154·2	184·7	214·3	244·9	274·5
27	306·2	336·8	4·4	34·9	64·5	95·1	124·6	155·2	185·7	215·3	245·9	275·4
28	307·2	337·8	5·4	35·9	65·5	96·0	125·6	156·2	186·7	216·3	246·9	276·4
29	308·2	338·7	6·3	36·9	66·5	97·0	126·6	157·2	187·7	217·3	247·8	277·4
30	309·2		7·3	37·9	67·5	98·0	127·6	158·1	188·7	218·3	248·8	278·4
31	310·2		8·3		68·5		128·6	159·1		219·3		279·4

This table assumes that the value for January 1 is 280·6.

TABLE 30.—*Approximate equation of time at Greenwich mean noon.*

[To change apparent to mean time.]

Day.	Jan.	Feb.	Mar.	Apr.	May.	June.	July	Aug.	Sept.	Oct.	Nov.	Dec.
	m.	*m.*	*m.*	*m.*	*m.*	*m.*	*m.*	*m.*	*m.*	*m.*	*m.*	*m.*
1	+ 3.8	+13.8	+12.5	+3.9	−3.0	−2.4	+3.5	+6.1	0.0	−10.3	−16.3	−10.8
2	+ 4.2	+13.9	+12.3	+3.6	−3.1	−2.3	+3.7	+6.0	− 0.4	−10.6	−16.3	−10.5
3	+ 4.7	+14.0	+12.1	+3.3	−3.2	−2.1	+3.9	+6.0	− 0.7	−10.9	−16.3	−10.0
4	+ 5.2	+14.1	+11.9	+3.0	−3.3	−2.0	+4.1	+5.9	− 1.1	−11.2	−16.3	− 9.7
5	+ 5.6	+14.2	+11.7	+2.8	−3.4	−1.8	+4.3	+5.8	− 1.3	−11.5	−16.3	− 9.3
6	+ 6.1	+14.3	+11.4	+2.5	−3.5	−1.6	+4.5	+5.7	− 1.7	−11.8	−16.3	− 8.8
7	+ 6.5	+14.3	+11.2	+2.2	−3.6	−1.4	+4.6	+5.6	− 2.0	−12.1	−16.2	− 8.4
8	+ 6.9	+14.4	+11.0	+1.9	−3.7	−1.3	+4.8	+5.5	− 2.4	−12.4	−16.2	− 8.0
9	+ 7.3	+14.4	+10.7	+1.6	−3.7	−1.0	+4.9	+5.3	− 2.7	−12.7	−16.0	− 7.5
10	+ 7.7	+14.4	+10.4	+1.3	−3.8	−0.9	+5.0	+5.2	− 3.1	−12.9	−16.0	− 7.1
11	+ 8.1	+14.4	+10.2	+1.0	−3.8	−0.7	+5.2	+5.1	− 3.4	−13.2	−15.9	− 6.6
12	+ 8.5	+14.4	+ 9.9	+0.8	−3.8	−0.5	+5.3	+4.9	− 3.7	−13.5	−15.7	− 6.1
13	+ 8.9	+14.4	+ 9.6	+0.5	−3.9	−0.3	+5.5	+4.7	− 4.1	−13.7	−15.6	− 5.7
14	+ 9.3	+14.4	+ 9.3	+0.3	−3.9	−0.1	+5.6	+4.5	− 4.4	−13.9	−15.3	− 5.2
15	+ 9.6	+14.3	+ 9.1	+0.1	−3.9	+0.2	+5.7	+4.3	− 4.8	−14.1	−15.3	− 4.7
16	+10.0	+14.3	+ 8.7	−0.2	−3.9	+0.4	+5.8	+4.1	− 5.2	−14.4	−15.1	− 4.2
17	+10.3	+14.2	+ 8.5	−0.5	−3.8	+0.6	+5.9	+3.9	− 5.5	−14.6	−14.9	− 3.7
18	+10.6	+14.1	+ 8.2	−0.7	−3.8	+0.8	+6.0	+3.7	− 5.9	−14.8	−14.7	− 3.2
19	+10.9	+14.0	+ 7.9	−0.9	−3.8	+1.0	+6.1	+3.5	− 6.2	−14.9	−14.5	− 2.7
20	+11.2	+13.9	+ 7.6	−1.1	−3.7	+1.2	+6.1	+3.3	− 6.6	−15.1	−14.3	− 2.2
21	+11.5	+13.8	+ 7.3	−1.3	−3.6	+1.5	+6.2	+3.0	− 6.9	−15.3	−14.0	− 1.7
22	+11.8	+13.7	+ 7.0	−1.5	−3.6	+1.7	+6.2	+2.8	− 7.3	−15.4	−13.7	− 1.2
23	+12.1	+13.5	+ 6.7	−1.7	−3.5	+1.9	+6.2	+2.5	− 7.6	−15.6	−13.5	− 0.7
24	+12.3	+13.4	+ 6.4	−1.9	−3.4	+2.1	+6.3	+2.3	− 8.0	−15.7	−13.2	− 0.2
25	+12.5	+13.2	+ 6.0	−2.1	−3.3	+2.3	+6.3	+2.0	− 8.3	−15.8	−12.9	+ 0.3
26	+12.8	+13.0	+ 5.8	−2.3	−3.2	+2.5	+6.3	+1.7	− 8.6	−15.9	−12.6	+ 0.7
27	+13.0	+12.9	+ 5.5	−2.4	−3.1	+2.7	+6.3	+1.4	− 9.0	−16.0	−12.2	+ 1.2
28	+13.2	+12.7	+ 5.2	−2.6	−3.0	+3.0	+6.3	+1.2	− 9.3	−16.1	−11.9	+ 1.7
29	+13.3	+12.5	+ 4.9	−2.7	−2.9	+3.2	+6.3	+0.9	− 9.6	−16.2	−11.6	+ 2.2
30	+13.5		+ 4.6	−2.9	−2.7	+3.4	+6.2	+0.6	−10.0	−16.2	−11.2	+ 2.7
31	+13.7		+ 4.3		−2.6		+6.2	+0.3		−16.3		+ 3.2

$$\frac{\text{Sun's mean longitude}}{15} = \text{right ascension of mean sun} = \text{sidereal time of mean noon.}$$

Sun's right ascension = right ascension of mean sun + equation of time.

TABLE 31.—*Perturbations in* K_1, S_2, *due to the components* P_1, K_2, *and* T_2.

Date. Greenwich mean noon.	Acceleration				Resultant amplitude.			
	In K_1 due to P_1.	In S_2 due to K_2.	In S_2 due to solar K_2.	In S_2 due to T_2.	K_1 and P_1 $K_1=1$.*	S_2 and K_2 $S_2=1$.	S_2 and solar K_2 $S_2=1$.	S_2 and T_2 $S_2=1$.
	°	°	°	°				
Jan. 1	−5·2	−7·6	−2·4	+0·1	1·314	·76	0·92	1·06
11	−9·9	−12·5	−3·9	−0·5	1·267	·82	0·94	1·06
21	−13·8	−15·2	−4·8	−1·0	1·197	·90	0·96	1·06
31	−17·0	−15·6	−4·9	−1·5	1·105	·99	0·99	1·05
Feb. 10	−18·8	−14·1	−4·5	−1·9	0·993	1·08	1·02	1·05
20	−18·7	−11·6	−3·7	−2·3	0·885	1·16	1·05	1·04
Mar. 2	−15·8	−8·2	−2·6	−2·7	0·782	1·22	1·07	1·03
12	−9·4	−4·4	−1·4	−3·0	0·706	1·25	1·08	1·02
22	−0·5	−0·3	−0·1	−3·2	0·676	1·27	1·09	1·01
Apr. 1	+8·5	+3·9	+1·2	−3·4	0·701	1·26	1·08	1·00
11	+15·4	+7·8	+2·5	−3·4	0·772	1·23	1·07	0·99
21	+18·6	+11·2	+3·6	−3·3	0·873	1·16	1·04	0·98
May 1	+18·9	+14·0	+4·4	−3·1	0·985	1·09	1·02	0·97
11	+17·3	+15·4	+4·9	−2·9	1·092	1·00	0·99	0·96
21	+14·2	+15·4	+4·9	−2·5	1·188	·91	0·96	0·96
31	+10·2	+13·0	+4·1	−2·0	1·261	·82	0·94	0.95
June 10	+5·7	+8·2	+2·6	−1·4	1·309	·76	0·92	0·95
20	+0·9	+1·3	+0·4	−0·8	1·329	·73	0·91	0·94
30	−4·0	−5·6	−1·8	−0·3	1·321	·74	0·92	0·94
July 10	−8·7	−11·5	−3·7	+0·4	1·282	·80	0·93	0·94
20	−12·8	−14·6	−4·6	+1·0	1·217	·88	0·95	0·94
30	−16·3	−15·6	−5·0	+1·6	1·131	·97	0·98	0·95
Aug. 9	−18·6	−14·6	−4·7	+2·1	1·028	1·06	1·01	0·95
19	−19·1	−12·3	−3·9	+2·6	0·913	1·14	1·04	0·96
29	−17·0	−9·2	−2·9	+2·9	0·807	1·21	1·06	0·97
Sept. 8	−11·7	−5·5	−1·8	+3·2	0·721	1·25	1·08	0·98
18	−3·1	−1·3	−0·4	+3·3	0·677	1·27	1·08	0·99
28	+6·4	+2·8	+0·9	+3·4	0·687	1·26	1·08	1·00
Oct. 8	+14·0	+6·8	+2·2	+3·3	0·748	1·23	1·08	1·01
18	+18·2	+10·4	+3·3	+3·2	0·844	1·18	1·06	1·02
28	+19·2	+13·3	+4·2	+3·0	0·956	1·11	1·03	1·03
Nov. 7	+18·0	+15·2	+4·8	+2·7	1·065	1·03	1·00	1·03
17	+15·1	+15·4	+4·9	+2·3	1·165	·93	0·97	1·04
27	+11·3	+13·7	+4·4	+1·9	1·244	·84	0·95	1·05
Dec. 7	+7·0	+9·7	+3·1	+1·4	1·299	·78	0·93	1·05
17	+2·2	+3·3	+1·0	+0·7	1·327	·73	0·92	1·06
27	−2·7	−4·0	−1·2	+0·3	1·325	·74	0·92	1·06
Jan. 6	−7·4	−9·9	−3·3	−0·2	1·295	·79	0·92	1·06
Modification of tabular value for long. of moon's node.	Tabular value × $F(K_1)$.	Tabular value × $f(K_2)$.	Tabular value.	Tabular value.	(Tab. − 1) × $F(K_1)$ + 1 = c_1 or c_{11}.	(Tab. − 1) × $f(K_2)$ + 1.	Tabular value.	Tabular value.

* i. e. K_1' or the K_1 for the given year.

For the acceleration in S_2 due to lunar K_2, use the tabular value multiplied by $[f(K_2) - 0·317]$.
This table is based upon Tables 1, 15, 16, and 29.

TABLE 32.—*Factor F_1 for clearing D_1 of the effects of the longitude of the moon's node and of P_1.*

Date.		*I*, or inclination of orbit to equator.											
		18½°	19°	20°	21°	22°	23°	24°	25°	26°	27°	28°	28½°
Jan.	1	0·963	0·949	0·923	0·898	0·874	0·853	0·833	0·814	0·796	0·779	0·764	0·757
	11	0·988	0·974	0·946	0·919	0·895	0·872	0·851	0·831	0·813	0·795	0·779	0·772
	21	1·030	1·014	0·983	0·955	0·929	0·905	0·882	0·861	0·841	0·822	0·805	0·797
	30	1·091	1·073	1·038	1·008	0·978	0·951	0·926	0·903	0·881	0·861	0·842	0·833
Feb.	10	1·172	1·151	1·112	1·077	1·043	1·012	0·984	0·958	0·933	0·911	0·889	0·879
	20	1·270	1·245	1·200	1·158	1·119	1·084	1·052	1·022	0·994	0·968	0·945	0·933
Mar.	2	1·373	1·345	1·293	1·245	1·200	1·159	1·122	1·088	1·057	1·028	1·001	0·989
	12	1·463	1·431	1·371	1·317	1·268	1·223	1·182	1·144	1·109	1·077	1·047	1·034
	22	1·499	1·466	1·404	1·347	1·296	1·248	1·205	1·166	1·130	1·097	1·067	1·052
Apr.	1	1·468	1·436	1·376	1·321	1·272	1·226	1·185	1·147	1·112	1·080	1·050	1·037
	11	1·384	1·356	1·302	1·253	1·208	1·167	1·130	1·095	1·063	1·034	1·007	0·994
	21	1·280	1·256	1·209	1·167	1·128	1·093	1·059	1·029	1·000	0·974	0·951	0·939
May	1	1·181	1·161	1·121	1·084	1·051	1·020	0·991	0·964	0·940	0·916	0·895	0·885
	11	1·100	1·082	1·048	1·015	0·986	0·959	0·933	0·909	0·887	0·866	0·848	0·839
	21	1·036	1·019	0·989	0·961	0·934	0·910	0·886	0·865	0·845	0·826	0·809	0·801
	31	0·991	0·977	0·949	0·922	0·898	0·875	0·854	0·833	0·815	0·798	0·782	0·774
June	10	0·965	0·951	0·925	0·900	0·876	0·854	0·834	0·815	0·797	0·780	0·765	0·758
	20	0·954	0·941	0·915	0·890	0·867	0·846	0·826	0·808	0·790	0·773	0·758	0·751
	30	0·958	0·944	0·918	0·893	0·871	0·849	0·829	0·810	0·793	0·776	0·761	0·754
July	10	0·980	0·965	0·938	0·912	0·888	0·866	0·845	0·825	0·807	0·790	0·774	0·767
	20	1·017	1·002	0·972	0·945	0·920	0·895	0·873	0·852	0·833	0·815	0·798	0·790
	30	1·072	1·055	1·022	0·992	0·964	0·937	0·913	0·890	0·869	0·850	0·832	0·823
Aug.	9	1·146	1·127	1·090	1·055	1·024	0·994	0·966	0·941	0·917	0·895	0·875	0·866
	19	1·243	1·219	1·176	1·135	1·099	1·065	1·033	1·005	0·977	0·953	0·930	0·919
	29	1·346	1·319	1·268	1·222	1·179	1·140	1·104	1·071	1·040	1·013	0·987	0·974
Sept.	8	1·443	1·412	1·354	1·301	1·253	1·209	1·169	1·132	1·098	1·066	1·038	1·024
	18	1·499	1·465	1·403	1·346	1·295	1·248	1·205	1·166	1·130	1·097	1·066	1·052
	28	1·487	1·454	1·392	1·336	1·286	1·240	1·197	1·159	1·123	1·091	1·060	1·046
Oct.	8	1·412	1·382	1·326	1·275	1·229	1·187	1·148	1·112	1·079	1·050	1·021	1·008
	18	1·309	1·283	1·235	1·191	1·150	1·113	1·079	1·047	1·018	0·991	0·967	0·955
	28	1·205	1·184	1·142	1·105	1·070	1·037	1·008	0·980	0·954	0·931	0·909	0·898
Nov.	7	1·119	1·100	1·065	1·031	1·001	0·973	0·946	0·922	0·899	0·878	0·858	0·849
	17	1·050	1·033	1·002	0·973	0·946	0·920	0·896	0·874	0·855	0·836	0·818	0·809
	27	1·001	0·986	0·957	0·930	0·906	0·883	0·861	0·841	0·822	0·804	0·788	0·780
Dec.	7	0·970	0·956	0·929	0·904	0·880	0·858	0·838	0·818	0·801	0·784	0·768	0·761
	17	0·955	0·941	0·915	0·891	0·868	0·846	0·826	0·808	0·790	0·774	0·759	0·752
	27	0·956	0·942	0·916	0·892	0·869	0·847	0·827	0·809	0·791	0·775	0·760	0·752
Jan.	6	0·976	0·962	0·934	0·908	0·884	0·862	0·842	0·822	0·804	0·787	0·772	0·764

$$F_1 = \frac{2{\cdot}4066}{c_{11}\ 1{\cdot}4066 f(K_1) + f(O_1)}. \quad F_1 \times \text{observed } D_1 = K_1 + O_1. \quad 1{\cdot}02\, F_1 \times \text{observed } D_1 = D_1.$$

This table is based on Tables 13 and 31.

TABLE 33.—*Factor F_2 for clearing* S_2 *of* K_2 *and* T_2.

Date.	*I*, or inclination of orbit to equator.											
	18½°	19°	20°	21°	22°	23°	24°	25°	26°	27°	28°	28½°
Jan. 1	1·14	1·15	1·16	1·18	1·19	1·20	1·22	1·24	1·27	1·30	1·33	1·34
11	1·08	1·09	1·10	1·10	1·11	1·12	1·14	1·15	1·17	1·19	1·21	1·22
21	1·02	1·02	1·02	1·03	1·03	1·04	1·04	1·05	1·06	1·06	1·07	1·08
31	·96	·96	·96	·96	·96	·96	·96	·96	·96	·96	·96	·96
Feb. 10	·90	·90	·90	·89	·89	·89	·88	·88	·88	·87	·87	·86
20	·86	·86	·85	·85	·84	·84	·83	·83	·82	·81	·81	·80
Mar. 2	·83	·83	·83	·82	·82	·81	·80	·79	·78	·77	·76	·76
12	·83	·82	·82	·81	·80	·79	·78	·77	·76	·75	·74	·74
22	·83	·82	·81	·80	·79	·79	·78	·77	·76	·75	·74	·73
Apr. 1	·83	·83	·83	·82	·81	·80	·79	·78	·77	·76	·75	·75
11	·86	·86	·85	·84	·83	·82	·82	·81	·80	·79	·78	·78
21	·91	·91	·90	·89	·89	·88	·88	·87	·86	·85	·85	·84
May 1	·96	·96	·96	·95	·95	·95	·94	·94	·93	·93	·92	·92
11	1·04	1·04	1·04	1·04	1·04	1·04	1·04	1·04	1·04	1·04	1·04	1·04
21	1·12	1·12	1·13	1·13	1·14	1·14	1·15	1·16	1·16	1·17	1·18	1·19
31	1·23	1·24	1·25	1·26	1·27	1·28	1·30	1·32	1·34	1·37	1·39	1·40
June 10	1·30	1·31	1·33	1·35	1·37	1·39	1·41	1·44	1·47	1·51	1·55	1·57
20	1·35	1·37	1·39	1·41	1·44	1·47	1·50	1·54	1·58	1·63	1·68	1·70
30	1·34	1·35	1·37	1·39	1·42	1·45	1·48	1·51	1·55	1·59	1·64	1·67
July 10	1·26	1·27	1·28	1·30	1·32	1·34	1·36	1·38	1·41	1·43	1·46	1·47
20	1·17	1·18	1·19	1·19	1·20	1·22	1·23	1·24	1·25	1·26	1·27	1·28
30	1·07	1·07	1·08	1·08	1·08	1·09	1·09	1·09	1·09	1·10	1·10	1·10
Aug. 9	1·00	1·00	1·00	1·00	1·00	·99	·99	·99	·98	·98	·97	·97
19	·94	·93	·93	·93	·92	·92	·91	·90	·89	·88	·88	·87
29	·88	·88	·88	·87	·86	·85	·84	·83	·83	·82	·81	·80
Sept. 8	·85	·85	·84	·84	·83	·82	·81	·80	·79	·78	·77	·76
18	·84	·83	·83	·82	·81	·80	·79	·78	·77	·76	·75	·75
28	·83	·83	·83	·82	·81	·80	·79	·78	·77	·76	·75	·74
Oct. 8	·85	·84	·83	·83	·82	·81	·80	·79	·78	·77	·77	·76
18	·86	·86	·86	·85	·85	·84	·83	·82	·81	·80	·79	·79
28	·90	·89	·89	·89	·88	·88	·88	·87	·86	·86	·85	·85
Nov. 7	·95	·95	·95	·95	·95	·94	·94	·94	·94	·94	·93	·93
17	1·01	1·01	1·02	1·02	1·02	1·03	1·03	1·04	1·04	1·05	1·05	1·06
27	1·07	1·08	1·09	1·09	1·10	1·11	1·12	1·14	1·15	1·16	1·18	1·19
Dec. 7	1·13	1·14	1·15	1·16	1·18	1·19	1·21	1·23	1·25	1·28	1·31	1·32
17	1·16	1·18	1·19	1·21	1·23	1·25	1·27	1·30	1·33	1·36	1·39	1·41
27	1·16	1·17	1·18	1·19	1·21	1·23	1·25	1·28	1·31	1·34	1·37	1·39
Jan. 6	1·11	1·11	1·12	1·14	1·15	1·16	1·18	1·20	1·22	1·24	1·27	1·28

This table is based upon Tables 13 and 31.

TABLE 34.—*Effect of* ν_2 *upon the amplitude of* N_2.

Apparent time of moon's upper or lower transit. ☽ in perigee or apogee.			Resultant amplitude N_2 and ν_2. N_2=1.	Apparent time of moon's upper or lower transit. ☽ in perigee or apogee.			Resultant amplitude N_2 and ν_2. N_2=1.
h.	*h.*	*m.*		*h.*	*h.*	*m.*	
0,	12	00	1·19	6,	18	00	0·81
		20	1·19			20	0·81
		40	1·18			40	0·82
1,	13	00	1·17	7,	19	00	0·84
		20	1·16			20	0·86
		40	1·14			40	0·89
2,	14	00	1·11	8,	20	00	0·92
		20	1·08			20	0·95
		40	1·05			40	0·99
3,	15	00	1·02	9,	21	00	1·02
		20	0·99			20	1·05
		40	0·95			40	1·08
4,	16	00	0·92	10,	22	00	1·11
		20	0·89			20	1·14
		40	0·86			40	1·16
5,	17	00	0·84	11,	23	00	1·17
		20	0·82			20	1·18
		40	0·81			40	1·19

This table is based upon Tables 1 and 16.

TABLE 35.—*Group factors.*

For phase reduction.

	Number of tides before springs or neaps.	Number of tides after springs or neaps. 0	2	4	6	8	10	12
$S_2/M_2=0{\cdot}2$	0	1·00	1·01	1·03	1·07	1·13	1·22	1·34
	2	1·01	1·01	1·02	1·06	1·11	1·18	1·29
	4	1·03	1·02	1·03	1·05	1·10	1·16	1·25
	6	1·07	1·06	1·05	1·07	1·11	1·16	1·24
	8	1·13	1·11	1·10	1·11	1·13	1·18	1·25
	10	1·22	1·18	1·16	1·16	1·18	1·22	1·28
	12	1·34	1·29	1·25	1·24	1·25	1·28	1·34
$S_2/M_2=0{\cdot}3$	0	1·00	1·01	1·03	1·07	1·14	1·23	1·35
	2	1·01	1·01	1·03	1·06	1·11	1·19	1·30
	4	1·03	1·03	1·03	1·06	1·10	1·17	1·26
	6	1·07	1·06	1·06	1·08	1·11	1·17	1·25
	8	1·14	1·11	1·10	1·11	1·14	1·19	1·26
	10	1·23	1·19	1·17	1·17	1·19	1·23	1·29
	12	1·35	1·30	1·26	1·25	1·26	1·29	1·35
$S_2/M_2=0{\cdot}4$	0	1·00	1·01	1·03	1·08	1·14	1·24	1·37
	2	1·01	1·01	1·03	1·06	1·12	1·20	1·31
	4	1·03	1·03	1·03	1·06	1·11	1·18	1·27
	6	1·08	1·06	1·06	1·08	1·12	1·17	1·26
	8	1·14	1·12	1·11	1·12	1·15	1·19	1·27
	10	1·24	1·20	1·18	1·17	1·19	1·24	1·30
	12	1·37	1·31	1·27	1·26	1·27	1·30	1·37
$S_2/M_2=0{\cdot}5$	0	1·00	1·01	1·04	1·09	1·16	1·25	1·39
	2	1·01	1·01	1·03	1·07	1·13	1·21	1·33
	4	1·04	1·03	1·04	1·07	1·12	1·19	1·29
	6	1·09	1·07	1·07	1·09	1·13	1·19	1·27
	8	1·16	1·13	1·12	1·13	1·16	1·21	1·29
	10	1·25	1·21	1·19	1·19	1·21	1·25	1·32
	12	1·39	1·33	1·29	1·27	1·29	1·32	1·39

For parallax reduction.

Number of tides before or after greatest and least parallax effects.	Factor.
0	1·00
4	1·01
6	1·02
8	1·04
10	1·06

For declinational reduction.

Number of tides before or after moon's extreme declination.	Factor (for diurnal wave).
0	1·000
2	1·005
4	1·012
6	1·024
8	1·040

This table is based upon Tables 1 and 16.